Ambiguity in a Two-Country World

Irasema Alonso*

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Abstract

The correlation between consumption levels in different countries is much lower than what is suggested by models of efficient risk sharing with common beliefs. Relatedly, observed asset portfolios of consumers in different countries suggest a "bias" toward home-country securities, even for countries where financial markets are quite well developed. This paper examines a mechanism that can generate these observations by considering preferences that allow ambiguity aversion of the sort illustrated by the Ellsberg Paradox. A key assumption is that the home consumer is more ambiguous about the process generating productivity shocks in the foreign country than about that in the home country. This permits formalization of a statement like "I don't hold foreign stocks because I don't know much about them". The specific context here is that of a two-country dynamic general equilibrium model with real business cycles. It is shown that the model generates low consumption correlations, higher output correlations, biased financial portfolios, and biased real investment flows.

*Correspondence: Department of Economics, Harkness Hall, University of Rochester, Rochester NY 14627, USA; e-mail: iral@troi.cc.rochester.edu.

1 Introduction

Following Backus, Kehoe, and Kydland (1992), researchers have been puzzled by the fact that risk-sharing models have difficulty in explaining some aspects of international data. In particular, in the data cross-country output correlations are higher than consumption correlations, while these models imply the opposite. Related to this is a fact associated with the lack of insurance in consumption: the "home bias" in the portfolios of citizens of different countries. It is not easy to rationalize these observations. One possibility is that financial markets are incomplete, so that perfect consumption insurance across countries is made more difficult. However, radical incompleteness of markets would be necessary to explain the facts. For example, access to a foreign stock index would go a long way toward allowing insurance, and such assets have long been available.[1] Therefore one needs to address why domestic agents choose not to hold foreign securities. The present paper discusses a way of thinking about this: we introduce a form of "friction" into the benchmark two-country real business cycle model which has potential to explain the lack of insurance and home bias.

The framework is based on ambiguity aversion. There are two countries and each country knows the probability distribution of its own productivity parameter but is ambiguous about–"does not know precisely "– the probability distribution of the other country's productivity parameter. The motivation for this assumption is that processing costs are higher for information about "far-away places": cultural and language barriers make it more difficult to assess the uncertainty involved in foreign projects (see Grinblatt and Keloharju (2001) for empirical evidence that distance, language, and culture influence stock holdings and trade). We do not explicitly model information processing and its associated costs here; neither do we model learning, which is a potentially important factor in this context. The idea used in this paper is simply that ambiguity need not disappear over time – it can persist asymptotically even though the model is stationary (for simplicity, an Ak model with stochastic A). The model is intended as a short-cut for a complicated world in which growth is achieved by new technologies, new product developments, and so on. The presumption here is that ambiguity will remain in such more realistic contexts. (See Epstein and Schneider (2002) for a model of learning under ambiguity in which ambiguity can persist asymptotically.)

Ambiguity-averse agents prefer gambles where they know the probability distribution over gambles where they do not know it precisely. Such a preference has its origin in the *Ellsberg Paradox* and is confirmed in experimental data. We use the multiple-priors utility specification developed by Gilboa and Schmeidler (1989). They capture ambiguity aversion with a "maxmin" formulation: the consumer behaves *as if* he maximizes expected utility when choosing consumption and an asset portfolio under a worst-case belief that is chosen from a set of conditional probabilities. We thus assume that the domestic consumer uses multiple priors for the returns to foreign equity but a single prior for domestic returns. The trading setup assumes complete markets.

The intuitive mechanism that results is as follows. Maxmin behavior implies a certain "pessimism": consumers at home believe that foreign productivity will not be as high as

[1] In addition, in countries where foreign assets have not been available one needs to explain why they have not been, given that the costs of holding foreign stock for banks and other potential market makers have not been high.

the foreigners themselves believe it to be (since they are not ambiguous about their own productivity). Similarly, foreigners believe that the productivity in the other country will not be as high as home consumers believe it to be. This leads consumers to view investing in foreign countries unfavorably and home consumers "bet" against foreign equity by having a biased portfolio. As a result, if, for example, the foreign productivity turns out to be high, and the domestic one low, there is wealth redistribution ex post due to the different portfolio choices made – foreigners increase their relative world wealth. Because the wealth distribution changes in favor of foreigners, and because consumption in the simple model considered here is proportional to wealth, the ratio of consumption levels of the two countries changes as well. This is the source of the lowering of consumption correlations across countries. Quantitatively, the consumption correlations implied by the model are much lower than those implied by the standard model, and there is a strong portfolio home bias, even when ambiguity is quite limited. Furthermore, we show that consumption correlations can be lower than output correlations.

The paper shows that home bias increases with the amount of ambiguity assumed in the economy. The most extreme home bias which this model can generate is autarky, where there is no insurance/asset trade between countries. It occurs when the parameter measuring ambiguity is large enough. Autarky being the most extreme case of home bias means that short-selling of foreign equity cannot occur as an equilibrium phenomenon under ambiguity aversion. In this sense, home bias in the data conforms with the theory presented here: we do not observe short-selling of foreign equity, but merely that foreign equity is held in small positive amounts. The reason why short-selling is not a possible outcome is that it is not consistent with maxmin behavior. A portfolio where foreign stocks are short-sold makes bad foreign productivity outcomes good from the perspective of ex-post utility. Hence, an agent who minimizes over probabilities would place low probability weight on such outcomes, thus being optimistic about foreign productivity. This, however, contradicts short-selling foreign equity.

Conditional output correlations can be lower than those for consumption correlations in the presence of ambiguity. As n increases, n-period-ahead conditional correlations between outputs of the two countries are more and more dominated by movements in capital and less and less by the productivity correlation across countries, which is constant. Since capital investments in the two countries are highly correlated – unless countries are in autarky, relatively stable shares of total world output are invested in each country's technology every period – output levels move together at long horizons. In particular, small amounts of ambiguity lead to very small movements in investment shares, and thus output correlations are very close to one at long horizons. Consumption correlations across countries, on the other hand, are governed to a large extent by movements in the wealth shares of the two countries, since consumption is proportional to wealth. These wealth shares move significantly here due to consumers taking different portfolio positions. In particular, small amounts of ambiguity lead to large swings in relative wealth levels over long time horizons – many small bets add up to large wealth redistribution – significantly lowering consumption correlations across countries.

The model generates a new propagation mechanism for relative wealths and outputs in different countries which involves persistence in the output of a given country beyond what can be observed in the standard two-country model. When foreign wealth increases as a result of a high productivity shock there, total investment reflects more foreigners' risk assessment: investment moves toward the foreign country as a result of one good shock there, even if productivity shocks are iid.

Related work includes Epstein (2001), who studies a two-period endowment economy with two countries and ambiguity, and Epstein and Miao (2003), who examine an infinite-horizon, continuous-time model. Neither of these papers have real investments, and hence output correlations are given exogenously. Moreover, the present paper focuses significant attention on the equilibrium asset trades that support the planning solution: it characterizes intertemporal trade and insurance across countries, and it derives results that may help explain why international asset markets are not so actively used: (i) autarky occurs naturally in the model when there is a large amount of ambiguity, thus making international financial markets entirely superfluous, and (ii) very aggressive trading in the form of short-selling of foreign stock cannot occur in equilibrium, because even with large amounts of ambiguity the resulting pessimism is limited.

There is a substantial literature addressing the home bias puzzle, for example using arguments of transactions costs, asymmetric information, and the importance of non-traded goods; similar mechanisms have also been emphasized to account for why consumption correlates less across countries than does output. The concluding section of the present paper makes some comments on this literature and suggests that asymmetric ambiguity - more is "known" about the nearby than about the far away - could be a more powerful explanation of these puzzles: it provides a simple and unified explanation of why almost no domestic investors ever hold foreign-biased portfolios (independently of their pattern of consumption and income), why surveys indicate that investors are more optimistic about the returns of domestic stock than about those of foreign stock (thus emphasizing expected return advantages of domestic investment and not how it would be better for risk management reasons), why there is also a local bias in investment within countries (despite no advantage in transactions costs and a disadvantage for risk management), and why local bias almost never goes as far as to short-selling. Existing explanations may be contributing factors behind these facts, but none can account for them all, and though only qualitative so far, asymmetric ambiguity aversion may be able to.

The presentation proceeds as follows. First, in Section 2, the general, infinite-horizon model is described. It is instructive to compare the results here with two alternative, simpler models. Therefore, the economy without ambiguity and with common beliefs is analyzed in Section 3. This is the standard model, for which both countries' residents have the same probability assessments. In that model, the initial relative wealths of the two countries – or, in terms of the planning problem used in the analysis below, the relative weights the planner uses on the foreign country's utility – stay constant over time. Section 4 studies an economy with no ambiguity and different beliefs for the residents of the two countries. Section 5 presents the economy with ambiguity. It discusses both a planning problem and how the planning solution is decentralized with competitive asset trading, and it contains the key insights of the paper. Finally, in Section 6 cross-country consumption and output correlations as well as some features of the propagation of shocks and output determination are derived and discussed. Section 7 concludes and discusses related literature. An appendix contains all the proofs.

2 The Model

Consider an infinite horizon, two-country production economy. Every period each of the two economies is hit by a shock to its productivity.

The residents of each country derive utility from the consumption process $c(s^t)$, where s^t is a history of shocks up to period t. Preferences of the representative agent of a country are described by the recursion

$$V_t(s^t) = u(c(s^t)) + \beta \min_{\pi \in \Pi_{s^t}} E_\pi V_{t+1}(s^{t+1}), \tag{1}$$

where Π_{s^t} is a set of transition probability laws given the history s^t today. We use $u(c) = \log c$.

Aversion to ambiguity is captured by the "minimization" part in the utility formulation above: the consumer behaves with pessimism, i.e., he assumes the worst possible probability distribution. For an axiomatic foundation for this preference formulation see Gilboa and Schmeidler (1989) for the static setting and Epstein and Schneider (2003) for a multiperiod horizon setting. Countries are indexed by $i = 1, 2$. Production in country i is given by

$$y_{it} = A_{it} k_{it}, \tag{2}$$

where A_{it} and k_{it} are productivity and capital, respectively, in country i at time t. Each A_i is stochastic and can take on two values, H and L. The jointly distributed stochastic process for the productivity shocks follows a first-order Markov process with a four-state support: state 1 (HH), state 2 (HL), state 3 (LH), and state 4 (LL). We denote by $\pi_{ss'}$ the true probability of moving to state s' next period when the current state is s and by $\Pi = (\pi_{ss'})$ the true transition matrix.

Agents in country i know the true transition probabilities for domestic shocks, but know only imprecisely the transition probabilities for foreign shocks.[2]

The transition probabilities perceived by country 1 are given by the following *continuum of matrices* parameterized with the variable v:

$$\Pi^1(v) = \begin{pmatrix} \pi_{HH,HH} + v_1 & \pi_{HH,HL} - v_1 & \pi_{HH,LH} + v_1 & \pi_{HH,LL} - v_1 \\ \pi_{HL,HH} + v_2 & \pi_{HL,HL} - v_2 & \pi_{HL,LH} + v_2 & \pi_{HL,LL} - v_2 \\ \pi_{LH,HH} + v_3 & \pi_{LH,HL} - v_3 & \pi_{LH,LH} + v_3 & \pi_{LH,LL} - v_3 \\ \pi_{LL,HH} + v_4 & \pi_{LL,HL} - v_4 & \pi_{LL,LH} + v_4 & \pi_{LL,LL} - v_4 \end{pmatrix},$$

where $v_i \in [-a, a]$, with restrictions on a such that all probabilities are in [0,1]. The parameter a measures the amount of ambiguity in the economy. The larger is a the larger is the set of transition probabilities over which the consumer is minimizing.

[2] However, each country has ambiguity about the probability of its own productivity shock 2 periods from now as long as the shocks are serially correlated and there are spillovers. To illustrate this point suppose that the current state is HH. Then the probability that country 1's productivity shock is H two periods from now is $(\pi_{11} + v_1)(\pi_{11} + \pi_{12}) + (\pi_{12} - v_2)(\pi_{21} + \pi_{22}) + (\pi_{13} - v_3)(\pi_{31} + \pi_{32}) + \pi(\pi_{14} - v_4)(\pi_{41} + \pi_{42})$, a number which depends on v.

This matrix captures the fact that there is no ambiguity with respect to the own productivity. The probability that country 1 next period gets a high shock (H) if today's state is HH is given by:

$$(\pi_{HH,HH} + v_1) + (\pi_{HH,HL} - v_1) = \pi_{HH,HH} + \pi_{HH,HL},$$

a number that does not depend on v. Similarly, the transition probability matrix perceived by country 2 is:

$$\Pi^2(v) = \begin{pmatrix} \pi_{HH,HH} + v_1 & \pi_{HH,HL} + v_1 & \pi_{HH,LH} - v_1 & \pi_{HH,LL} - v_1 \\ \pi_{HL,HH} + v_2 & \pi_{HL,HL} + v_2 & \pi_{HL,LH} - v_2 & \pi_{HL,LL} - v_2 \\ \pi_{LH,HH} + v_3 & \pi_{LH,HL} + v_3 & \pi_{LH,LH} - v_3 & \pi_{LH,LL} - v_3 \\ \pi_{LL,HH} + v_4 & \pi_{LL,HL} + v_4 & \pi_{LL,LH} - v_4 & \pi_{LL,LL} - v_4 \end{pmatrix}.$$

The country's consumers find any of these matrices "possible", and will – assuming ambiguity aversion – behave as if they choose to believe in the matrix which is the *worst* one for them in the sense specified by the utility function described above.

Capital cannot flow between countries but final goods, including investment, can flow instantaneously. The resource constraint is

$$c_{1t} + c_{2t} + k_{1,t+1} + k_{2,t+1} = y_{1t} + y_{2t}. \tag{3}$$

We will assume that there are complete markets and solve for allocations mostly by studying planning problems; explicit decentralizations are, however, discussed below as well.

3 Common Beliefs without Ambiguity

This is the standard model in which the subjective probability transition process used by each country coincides with the true one. We now show, in order to establish a benchmark, that this model delivers perfect consumption risk sharing. The use of particular functional forms for utility and production is helpful here because it delivers closed-form solutions for the equilibrium and thus enables precise comparisons across different setups.

We solve the infinite-horizon planning problem where the weights assigned to the agents of country 1 and 2 are θ and $1-\theta$ respectively, with $0 \le \theta \le 1$.

The net-present-value weighted utility in state s and when the capital stocks are k_1 and k_2, is solved recursively as follows:

$$V_s(k_1, k_2) = \max_{c_1, c_2, k_1', k_2'} \theta \log c_1 + (1-\theta) \log c_2 + \beta \sum_{s'=1}^{4} \pi_{ss'} V_{s'}(k_1', k_2')$$

subject to

$$c_1 + c_2 + k_1' + k_2' = A_{1s}k_1 + A_{2s}k_2.$$

We define $y_s(k_1, k_2) \equiv A_{1s}k_1 + A_{2s}k_2$ and let $\bar{A}$ denote the high realization of A_s and $\underline{A}$ denote its low realization.

The solution to the functional equation above is the value function

$$V_s(k_1, k_2) = \frac{1}{1-\beta} \log y_s(k_1, k_2) + C_s, \qquad s = 1, 2, 3, 4.$$

The solution to the constant terms C_s are given implicitly by the solution to four equations displayed in the Appendix (see Section 9.1).

The optimal consumption levels are

$$c_{1s} = \theta(1-\beta)y$$

$$c_{2s} = (1-\theta)(1-\beta)y.$$

The ratio of consumption to output does not depend on the state: consumptions in the two countries are perfectly correlated with total output and thus also with each other.

Optimal investments are

$$k'_{1s} = \beta \frac{(\bar{A}\pi_{s2} - \underline{A}\pi_{s3})}{(\bar{A} - \underline{A})(\pi_{s2} + \pi_{s3})} y$$

$$k'_{2s} = \beta y - k'_{1s}.$$

Investment shares in each country depend on the current state. Investment in country 1 and 2 does not depend on π_{s1} and π_{s4} because in states 1 and 4 each country gets the same productivity shock. How total investment is split between the two countries depends on how different π_{s2} is from π_{s3} since the planner wants to invest more in the country with the higher probability of a high productivity shock next period.

Under common beliefs, consumption, capital next period, and the value function depend on the value of total production and not on how capital is distributed between the two countries.

4 Heterogeneous Beliefs without Ambiguity

We now assume that countries have heterogeneous probabilistic beliefs. This is a simple intermediate step that will be convenient to analyze before we adopt the more complex setup where agents display ambiguity aversion. As will be shown below there is a form of observational equivalence between this model and the model with ambiguity. Moreover, solution of the simpler model with heterogeneous probabilistic beliefs illustrates the approach adopted later to solving the planner's problem in an economy with ambiguity.

To solve the planner's problem in an economy where countries' beliefs are probabilistic and heterogeneous, we adapt the methodology from Lucas and Stokey (1984) to a stochastic environment. Lucas and Stokey permit discount factors to vary endogenously and to differ across countries and show that the planning problem can be written recursively using an additional state variable: the relative weight the planner attaches to a given agent. This weight evolves endogenously over time (and favorably toward the patient agent). Here, discount factors are constant and common across agents, but beliefs differ, making a similar planning formulation necessary. We thus have

$$V_s(k_1, k_2, \theta) = \max_{c_1, c_2, k_1', k_2', z_1(s'), z_2(s')} \theta \log c_1 + (1-\theta) \log c_2$$

$$+\beta \left(\theta \sum_{s'=1}^{4} \pi^1_{ss'} z_1(s') + (1-\theta) \sum_{s'=1}^{4} \pi^2_{ss'} z_2(s') \right)$$

subject to

$$\min_{\theta'(s')} V_{s'}(k_1', k_2', \theta'(s')) - \theta'(s') z_1(s') - (1-\theta'(s')) z_2(s') \geq 0 \qquad s' = 1, 2, 3, 4 \tag{4}$$

and

$$c_1 + c_2 + k_1' + k_2' = A_{1s} k_1 + A_{2s} k_2. \tag{5}$$

The variable $z_i(s')$ represents the continuation utility (for next period and on) for country i. In this formulation θ is endogenous and is an additional state variable; constraint (4) makes the problem recursive and determines the next-period weight $\theta'(s')$ that implements the plan for future utilities. The variables k_1' and k_2' define a utility feasibility frontier, a convex set in $z_1(s')$ and $z_2(s')$ for each s'.

It is straightforward to show that the minimization in constraint (4) together with the maximization over continuation utilities imply, first, that

$$V_{s'}(k_1', k_2', \theta'(s')) = \theta'(s') z_1(s') + (1-\theta'(s')) z_2(s') \tag{6}$$

holds for all s' and, second, that the weights evolve so that

$$\frac{1-\theta'(s')}{\theta'(s')} = \frac{(1-\theta)\pi^2_{ss'}}{\theta \pi^1_{ss'}}. \tag{7}$$

In other words, weights are simply adjusted in accordance with the different utility weights (probabilities) placed by the two agents on the respective states of nature next period. This equation can be written as

$$\theta'(s') = \frac{\theta \pi^1_{ss'}}{\theta \pi^1_{ss'} + (1-\theta)\pi^2_{ss'}}.$$

In the special case where $\pi^1_{ss'} = \pi^2_{ss'}$, $\theta'(s') = \theta$. If, on the other hand, $\pi^2_{ss'} > \pi^1_{ss'}$, country 2 believes more in state s' tomorrow than country 1 does, then the planner increases country 2's relative weight in that state. From an equilibrium perspective, as we shall see below, θ represents the relative wealth levels of countries. If country 2 believes more in state s', it will invest more in the s'-contingent asset – bet against the other country – and thus increase its relative wealth if state s' occurs: relative wealth will drift.[3] Finally, note that the result on the evolution of the weights, and hence on the evolution of equilibrium wealth, does

[3] With symmetric beliefs around the truth, the drift is symmetric and nonstationary. If country 1's beliefs are closer to the truth, country 1 will be right more often than country 2 and over time (as time goes to infinity), θ goes to one, i.e., country 2's relative wealth goes to zero.

not require that utility be logarithmic nor place restrictions on technology: it is a general implication of belief heterogeneity.

One can show that the following value function satisfies the functional equation above:

$$V_s(k_1, k_2, \theta) = \frac{1}{1-\beta} \log y_s(k_1, k_2) + C_s(\theta). \tag{8}$$

Here, C_s is a *function* of θ; it does not always admit a closed-form solution, but the Appendix shows how it is uniquely constructed.

The optimal policies for consumption and investment plans can again be solved in closed form. In particular, the movements in θ govern consumption and investment; as in the common-beliefs case we have, for $s = 1, 2, 3, 4$,

$$c_{1s} = \theta(1-\beta)y$$

$$c_{2s} = (1-\theta)(1-\beta)y,$$

but consumption levels in the two countries are now not perfectly correlated with total output since the consumption shares depend on θ which is now stochastic. In particular, since countries bet against each other, θ will move in some of the states, thus leading to a mechanism which is not present in the economy with common beliefs.

The optimal investments are, for $s = 1, 2, 3, 4$,

$$k'_{1s} = \beta \frac{\bar{A}(\pi^1_{s2}\theta + \pi^2_{s2}(1-\theta)) - \underline{A}(\pi^1_{s3}\theta + \pi^2_{s3}(1-\theta))}{(\bar{A} - \underline{A})(\pi^1_{s2}\theta + \pi^2_{s2}(1-\theta) + \pi^1_{s3}\theta + \pi^2_{s3}(1-\theta))} y$$

$$k'_{2s} = \beta y - k'_{1s}.$$

These are the same investment expressions as those in the economy with common beliefs except for the fact that now the probabilities are written as a weighted average of the probabilities perceived by country 1 and 2. The higher is θ, the closer is the probability distribution used by the planner to allocate investment to that of country 1.

5 Ambiguity

We now consider agents who know the probability distribution of the productivity shock at home but who are ambiguity-averse about the probability distribution of the productivity shock abroad. Given the transition probability matrices for country 1 and 2 shown in Section 2, countries select the worst-case scenario among the continuum of these matrices. To formulate the planner's problem we extend the approach from Section 4 to incorporate multiple-priors utility.

The planning problem becomes

$$V_s(k_1, k_2, \theta) = \max_{c_1, c_2, k'_1, k'_2, z_1(s'), z_2(s')} \theta \log c_1 + (1-\theta) \log c_2 +$$

$$\beta \left(\theta \min_{v^1 \in [-a,a]} \sum_{s'=1}^{4} \pi^1_{ss'}(v^1) z_1(s') + (1-\theta) \min_{v^2 \in [-a,a]} \sum_{j=1}^{4} \pi^2_{ss'}(v^2) z_2(s') \right)$$

subject to

$$\min_{\theta'(s')} V_{s'}(k_1', k_2', \theta'(s')) - (\theta'(s')z_1(s') + (1-\theta'(s'))z_2(s')) \geq 0, \quad s' = 1,2,3,4, \tag{9}$$

and

$$c_1 + c_2 + k_1' + k_2' = A_{1s}k_1 + A_{2s}k_2. \tag{10}$$

Here $\pi^1_{ss'}(v) \equiv \pi_{ss'} + v$ for $s' = 1, 3$ and $\pi^1_{ss'}(v) \equiv \pi_{ss'} - v$ for $s' = 2, 4$, and $\pi^2_{ss'}(v) \equiv \pi_{ss'} + v$ for $s' = 1, 2$ and $\pi^2_{ss'}(v) \equiv \pi_{ss'} - v$ for $s' = 3, 4$, which are the conditional probabilities of going from state s to state s' specified in the matrices $\Pi^i(v)$, $i = 1, 2$, from Section 2.

The value function still can be shown to satisfy

$$V_s(k_1, k_2, \theta) = \frac{1}{1-\beta} \log y_s(k_1, k_2) + C_s(\theta)$$

for some function $C_s(\theta)$. The optimal values for c_{1s}, c_{2s}, k_{1s}' and k_{2s}' and the optimal motion for θ will be the same as for the heterogeneous beliefs economy, keeping in mind that probabilities are endogenous and a function of optimal values for the vs. We denote the latter $v_s^{1*}(\theta)$ and $v_s^{2*}(\theta)$. The optimal values for v^1 and v^2 do not depend on the scale of the economy; this is obvious from inspecting the first-order conditions for the v variables, where the level of production (y) does not appear.

Thus, it is possible to find a function $C_s(\theta)$ and optimal choices for the vs such that the stated value function and the associated decision rules

$$c_{1s} = \theta(1-\beta)y \tag{11}$$

$$c_{2s} = (1-\theta)(1-\beta)y \tag{12}$$

$$\theta'(s') = \frac{\theta\pi^1_{ss'}(v_s^{1*})}{\theta\pi^1_{ss'}(v_s^{1*}) + (1-\theta)\pi^2_{ss'}(v_s^{2*})} \tag{13}$$

$$k_{1s}'(\theta) = \beta \frac{\bar{A}\tilde{\pi}_{s2} - \underline{A}\tilde{\pi}_{s3}}{(\bar{A} - \underline{A})(\pi_{s2} + \pi_{s3})} y \tag{14}$$

$$k_{2s}'(\theta) = \beta \frac{\bar{A}\tilde{\pi}_{s3} - \underline{A}\tilde{\pi}_{s2}}{(\bar{A} - \underline{A})(\pi_{s2} + \pi_{s3})} y \tag{15}$$

and

$$k_{2s}'(\theta) + k_{1s}'(\theta) = \beta y, \tag{16}$$

where $\tilde{\pi}_{s2} \equiv (\pi_{s2} - v_s^{1*}(\theta))\theta + (\pi_{s2} + v_s^{2*}(\theta))(1-\theta)$ and $\tilde{\pi}_{s3} \equiv (\pi_{s3} + v_s^{1*}(\theta))\theta + (\pi_{s3} - v_s^{2*}(\theta))(1-\theta)$, satisfy the functional equation of the planner. In general, this kind of model needs to be solved numerically since the $C_s(\theta)$ functions which are needed for finding the optimal decision rules for the vs may not be possible to find in closed form.[4] In an interesting special case, however, where the productivity shocks are uncorrelated across countries and over time and symmetric (so that all states are equally likely), it is possible to solve the model in closed form using a guess-and-verify strategy and thus to go beyond the characterization in this

[4]For numerical solution, it is straightforward to apply a standard contraction-mapping algorithm.

section. This analysis is contained in Section 5.2; the results are summarized in Proposition 1. As a background for that analysis, it is helpful to study the decentralized version of this economy and define a recursive competitive equilibrium; this motivates Section 5.1 below.

In addition to beliefs being scale-independent here, there are economies for which the optimal choices of the vs also do not depend on s or θ. This will especially occur if a is small, i.e., if there is only a small amount of ambiguity; then one can show that $v_s^{1*}(\theta) = v_s^{2*}(\theta) = -a$ for all s and θ: both domestic and foreign beliefs are corner solutions. This outcome, therefore, has beliefs that differ across countries, but these differences are symmetric relative to the objective probability distribution.

As argued above in the heterogeneous beliefs section, countries 1 and 2 are "betting against each other" in some of the states. In this case they agree on the probabilities they assign to states 1 and 4 – these are $\pi_{s1} + v$ and $\pi_{s4} - v$, respectively, assuming a symmetric allocation. However, they disagree on the probabilities assigned to states 2 and 3: these are $\pi_{s2}^1 = \pi_{s3}^2 = \pi - v$ and $\pi_{s3}^1 = \pi_{s2}^2 = \pi + v$. Countries then are "betting against each other" in states 2 and 3. As will be detailed in section 5.2.2, this has the implication that country 1 buys a larger amount of the technology 1 asset than does country 2 and vice versa. It is in those states that the two countries disagree and where we now have a deviation from pure risk sharing: if state 2 occurs, there is a gain for country 1 but a loss for country 2.

This implication of differences in beliefs in turn means that the world wealth distribution evolves endogenously over time. What are the implications of ambiguity for the long-run characteristics of this distribution? In the case just discussed – where ambiguity is limited so that countries have symmetric and constant biases of an amount given by the ambiguity parameter a – the wealth distribution will display large long-run swings and not be stationary: in this case, one can show that the (log of the) planner's weight will follow a random walk. In cases where there is asymmetry, the country "closest to the truth" will tend to end up with all the wealth asymptotically. Whether the endogenous determination of the vs under ambiguity could lead to wealth-stabilizing changes in beliefs, so that relatively rich countries tend to adapt more extreme, further-from-objective beliefs or, alternatively, to wealth-destabilizing changes in beliefs, is an open question. In the special case of the model studied below, there is full symmetry, and thus wealth changes are entirely symmetric and there is no asymptotic wealth concentration in one country.

Investment, and thus the endogenous output process, is different under ambiguity than under common beliefs. With common beliefs, investment in a given country depends on the current state – in case shocks are serially correlated – via the probabilities of different productivity outcomes next period. Note in particular that investment in country 1 is higher than that in country 2 if $\pi_{s2} > \pi_{s3}$, i.e., if it is more likely that country 1's productivity is high and country 2's productivity is low than vice versa. The presence of ambiguity can change this conclusion. In particular, investment in country 1 depends negatively on v^1 and positively on v^2: to the extent this country is pessimistic about the other country's output, so that $v^1 < 0$, there is a force toward investing in country 1 (and not in country 2), but also a parallel force against country 1 investment if country 2 is pessimistic about country 1 ($v^2 < 0$). Which of these forces is stronger depends on θ: the larger it is, the more the "view" of country 1 matters.

A general feature here is that optimal behavior under ambiguity is observationally equivalent to optimal behavior where the beliefs are exogenous and equal to the minimizing beliefs under ambiguity. The model with ambiguity is attractive due to its intuitive appeal and to the experimental evidence on the Ellsberg Paradox. Assuming agents who are ambiguity-averse seems more appropriate than simply assuming that they have heterogeneous beliefs that are incorrect in a specific way.

Of course, when one changes the setting of study, one cannot rely on observational equivalence. Thus, as the setting (say, technology, or other aspects of preferences) changes, a setup with ambiguity will embody predictions for how beliefs will change, whereas the setup with exogenous belief differences will not or will mean that no changes in beliefs will occur. This point is well illustrated in the present context. As will be shown in section 5.2, the amount of pessimism implied under ambiguity is limited: autarky is the most extreme outcome and shortselling of foreign equity cannot ever occur in equilibrium. Under heterogeneous beliefs, however, shortselling is an equilibrium outcome when the disagreement in the countries' (exogenous) beliefs with regard to states 2 and 3 is strong enough. Thus, the hypothesis of ambiguity has a general implication for portfolio behavior that the assumption of exogenous differences in beliefs does not.

5.1 A decentralized version of the model

Much of the focus in this paper is on a competitive-equilibrium interpretation of the results; using standard welfare theorems, any planning allocation corresponds to a competitive equilibrium with complete markets. In the decentralized setting, the problem of the representative consumer in country i is

$$V_s(w,\theta,y) = \max_{c,a'_{s'},g'_1,g'_2} \log c + \beta \min_{v\in[-a,a]} \sum_{s'} \pi^i_{ss'}(v) V_{s'}(w'_{s'},\theta'_{ss'},y'_{ss'})$$

subject to

$$c + \sum_{s'} q_{ss'}(\theta,y)a'_{s'} + g'_1 + g'_2 = w,$$

$$w'_{s'} = a'_{s'} + g'_1 A_{1s'} + g'_2 A_{2s'},$$

$$y'_{ss'} = H_{ss'}(\theta,y),$$

and

$$\theta'_{ss'} = h_{ss'}(\theta,y),$$

where $a_{s'}$ is the agent's holding of contingent claim s'; its associated price in state s is $q_{ss'}(\theta,y)$, where θ is the economy-wide state variable representing the wealth distribution: the relative wealth of consumers in country 1. The variable w is the beginning-of-period wealth. We also let the agent hold capital separately: g'_1 and g'_2. The function $h_{ss'}$ describes the law of motion for the relative wealth of country 1 and $H_{ss'}$ describes the law of motion of total wealth. The solutions for V, c, $a'_{s'}$, g'_1, g'_2, and v for the country i consumer are denoted $V_{is}(w,\theta,y)$, $c_{is}(w,\theta,y)$, $a'_{iss'}(w,\theta,y)$, $g_{i1s}(w,\theta,y)$, $g_{i2s}(w,\theta,y)$, and $v_{is}(w,\theta,y)$.

A **recursive competitive equilibrium** is a set of functions (where we suppress the dependence on the state vector) V_i, c_i, $a'_{is'}$, v_i, g_{i1}, g_{i2}, $h_{s'}$, $H_{s'}$, and $q_{s'}$ with the following properties:

1. *Consumer maximization*: for each country i, V_i, c_i, $a'_{is'}$, v_i, g_{i1}, and g_{i2}, for all s', solve the dynamic programming problem.

2. *Market clearing in contingent claims*: $a'_{1ss'}(\theta y, \theta, y) + a'_{2ss'}((1-\theta)y, \theta, y) = 0$ for all θ, y, s, and s'.

3. *Relative wealth dynamics*: $h_{s'}$ satisfies, for all s', and all values of the aggregate state (θ, s, y),
$$h_{ss'}(\theta, y) = \frac{w'_{1ss'}(\theta, y)}{w'_{1ss'}(\theta, y) + w'_{2ss'}(\theta, y)},$$
where
$$w'_{1ss'}(\theta, y) \equiv a'_{1ss'}(\theta y, \theta, y) + g_{11s}(\theta y, \theta, y)A_{1s'} + g_{12s}(\theta y, \theta, y)A_{2s'}$$
and
$$w'_{2ss'}(\theta, y) \equiv a'_{2ss'}((1-\theta)y, \theta, y) + g_{21s}((1-\theta)y, \theta, y)A_{1s'} + g_{22s}((1-\theta)y, \theta, y)A_{2s'}.$$

4. *Total wealth dynamics*: $H_{s'}$ satisfies, for all values of the aggregate state,
$$H_{ss'}(\theta, y) = \left(g_{11s}(\theta y, \theta, y) + g_{21s}((1-\theta)y, \theta, y)\right) A_{1s'}$$
$$+ \left(g_{12s}(\theta y, \theta, y) + g_{22s}((1-\theta)y, \theta, y)\right) A_{2s'}.$$

Given that we assume Ak technologies and logarithmic utility, it is possible to characterize most of the equilibrium in closed form, and for the special iid case below all equilibrium functions can in fact be derived explicitly. A key simplification, as for the planning solution, is that none of the equilibrium functions, except $H_{ss'}$, will have a dependence on y – the scale of the economy.

5.2 A benchmark case: symmetric, independent, iid transitions

We now investigate the case where the productivity shocks are uncorrelated across countries and over time and symmetric (all states are equally likely).

Under this assumption one can prove that the objective of the planner (written recursively) and the decision rules can be solved in closed form; they are stated in Proposition 1 below.

A key insight behind the characterization in Proposition 1, stated in Lemma 1, is that the consumption insurance in the optimal allocation is very special. More precisely, put in terms of the decentralized solution, each agent's optimal portfolio can be expressed completely in terms of holding just two assets: capital in country 1 and capital in country 2. In other words, additional trade in contingent claims is not necessary. Thus, we have

Lemma 1 *Assuming that the optimal vs are symmetric for both countries ($v^{1*} = v^{2*}$) and constant (independent of s and θ), then each agent's consumption allocation is supported by a competitive equilibrium portfolio consisting of just two assets: capital in country 1 and capital in country 2.*

Given this lemma, the idea behind the construction in the main proposition is then first to conjecture that the optimal vs are indeed symmetric in the two countries ($v^{1*} = v^{2*}$) and independent of s and θ: the pessimism of country 1 toward country 2 technology is equal to the pessimism of country 2 toward country 1 technology and constant. Because the complete-markets outcome can be achieved with the two capital assets only (Lemma 1) and the returns on these assets are exogenously given, each agent's equilibrium utility can be calculated without knowing what the other agent is doing: there are no nontrivial equilibrium price determinations. Hence, each agent's value function can be computed in closed form – it is linear in log individual wealth – and this is all that is needed for finding the C functions. Finally, we verify by appealing to the first-order conditions for the vs (see the Appendix, equations (37) and (38)) that our guess on the decision rules for the vs was correct.

The main proposition is thus

Proposition 1 *When transitions are iid, symmetric, and independent across countries, so that $\pi_{ss'} = 1/4$ for all (s, s'), the value function and the decision rules can be solved in closed form. The value function is given by:*

$$V(\theta, y) = B + \frac{1}{1-\beta}(\theta \log \theta + (1-\theta)\log(1-\theta)) + \frac{1}{1-\beta}\log y,$$

where the constant B is shown in the Appendix. Moreover, the beliefs are given by

$$v_s^* = \begin{cases} -a & \text{for } a \leq \frac{\bar{A}-\underline{A}}{4(\bar{A}+\underline{A})} \\ -\frac{\bar{A}-\underline{A}}{4(\bar{A}+\underline{A})} & \text{for } a > \frac{\bar{A}-\underline{A}}{4(\bar{A}+\underline{A})} \end{cases}$$

for all s.

Thus, the solution for the unknown function $C(\theta)$ satisfies

$$C(\theta) = B + \frac{1}{1-\beta}(\theta \log \theta + (1-\theta)\log(1-\theta)).$$

As the amount of ambiguity in the economy increases (i.e., a increases) and since $v = -a$ as long as $v \leq \frac{\bar{A}-\underline{A}}{4(\bar{A}+\underline{A})}$, the consumer of country 1 puts more weight on (believes more in) state 2 relative to state 1, thus shifting probability mass from state 1 ($\pi^1_{s1} = \pi_{s1} + v$) towards state 2 ($\pi^1_{s2} = \pi_{s2} - v$). The same argument applies regarding states 3 and 4; here probability mass is shifted from state 3 towards state 4. In addition, since the consumer of country 1 buys a portfolio which is biased towards the technology of country 1, as ambiguity increases so does the portfolio bias, and the same occurs for country 2.

In what follows we assume the symmetric iid matrix for the productivity shocks.

5.2.1 The decentralized solution

We now describe in more detail the recursive competitive equilibrium for the special iid transition matrix. The value function is given by

$$V_{i,s}(w, \theta, y) = B + \frac{1}{1-\beta}\log w,$$

for $i = 1, 2$, and all states (s, w, θ, y); the constant B is identical to that in Proposition 1. Notice that only the individual state, the country's own wealth w, appears here.

The consumption function of country i satisfies

$$c_{is}(w, \theta, y) = (1 - \beta)w,$$

for $i = 1, 2$, and all states (s, w, θ, y). Individual consumption is a linear function of agents' own wealth and does not depend on the aggregate state.

The demand for technologies 1 and 2 by country 1 and 2 obey

$$g_{11s}(w, \theta, y) = \left(\frac{1}{2} - 2v\frac{\bar{A} + \underline{A}}{\bar{A} - \underline{A}}\right)\beta w,$$

$$g_{12s}(w, \theta, y) = \left(\frac{1}{2} + 2v\frac{\bar{A} + \underline{A}}{\bar{A} - \underline{A}}\right)\beta w,$$

$$g_{21s}(w, \theta, y) = \left(\frac{1}{2} + 2v\frac{\bar{A} + \underline{A}}{\bar{A} - \underline{A}}\right)\beta w,$$

and

$$g_{22s}(w, \theta, y) = \left(\frac{1}{2} - 2v\frac{\bar{A} + \underline{A}}{\bar{A} - \underline{A}}\right)\beta w.$$

Note that there is symmetry: both countries invest the same fraction of their savings in the domestic technology. These fractions do not depend on θ because the returns to technologies 1 and 2 are exogenous, and they do not depend on s because the shocks are iid.

The demand for contingent claims are

$$a'_{iss'}(w, \theta, y) = 0,$$

for $i = 1, 2$, all s', and all states (w, s, θ, y). As demonstrated in Lemma 1, there is no residual need for contingent claims so the demand for these claims is zero and independent of the aggregate and individual states.

The choice of the variable v satisfies

$$v_{is}(w, \theta, y) = \begin{cases} -a & \text{for } a \leq \frac{\bar{A} - \underline{A}}{4(\bar{A} + \underline{A})} \equiv \bar{a} \\ \\ -\bar{a} & \text{for } a > \bar{a} \end{cases}$$

for all $i = 1, 2$, and all states (w, θ, y). As shown in Proposition 1, v is independent of aggregate and individual states. For a small amount of ambiguity, v is a corner solution but when ambiguity is large enough, v becomes interior.

The prices of contingent claims are

$$q_{1s}(\theta, y) = \frac{\pi + v}{\bar{A}},$$

$$q_{2s}(\theta, y) = q_{3s}(\theta, y) = \frac{2\pi}{\bar{A} + \underline{A}},$$

and

$$q_{4s}(\theta, y) = \frac{\pi - v}{\underline{A}}$$

for all values of the aggregate state (θ, y). The price of contingent claim 4 is higher than that of contingent claim 1 when there is no ambiguity because countries' marginal utilities are higher in the state where the economy is poorer. Ambiguity, in addition, increases the difference in prices because both countries perceive state 1 as a less likely than state 4. The demand for contingent claims does not play a role in the determination of the prices in the sense that agents do not "need" contingent claims: the two technologies, which have fixed rates of return, suffice. Hence, the prices of the contingent claims do not depend on the wealth distribution (θ).

The functions governing the relative wealth dynamics satisfy

$$h_1(\theta, y) = h_4(\theta, y) = \theta,$$

$$h_2(\theta, y) = \frac{\theta(\pi - v^*)}{\theta(\pi - v^*) + (1-\theta)(\pi + v^*)},$$

and

$$h_3(\theta, y) = \frac{\theta(\pi + v^*)}{\theta(\pi + v^*) + (1-\theta)(\pi - v^*)}$$

for all values of the aggregate state (θ, y): in the states where countries' subjective probabilities agree, the relative wealth does not change; it only changes in the states where the two countries disagree.

Finally, the functions governing aggregate wealth dynamics are described by

$$H_{s'}(\theta, y) = \frac{\beta y(2\pi(\bar{A}A_{1s'} - \underline{A}A_{2s'}) + (\bar{A} + \underline{A})(\pi + 2v\theta - v)(A_{2s'} - A_{1s'}))}{2\pi(\bar{A} - \underline{A})}.$$

for all s' and all values of the aggregate state (θ, y). Total output depends on ambiguity to the extent that both (i) countries' wealth is not identical ($\theta \neq 1/2$) so that total investment in each of the two technologies is unequal, and (ii) productivity outcomes differ in the two countries ($A_{1s'} \neq A_{2s'}$) since in that case how much is invested in each technology matters.

5.2.2 Interpretation

Here we will explain the underlying qualitative mechanisms in the model with independent, symmetric iid shocks across countries. First, we will point to the main implications of the proposition and informally discuss the underlying intuition. After that, we defend the intuition by deriving some key aspects of agents' behavior formally.

Portfolio choice: home bias

Country 1 is ambiguity-averse toward the production technology in country 2. Thus, country 1 minimizes expected utility with respect to the joint probability distribution: given that country 1 holds the foreign asset, it attaches less weight to the states where that asset has a high payoff – states 1 and 3 – and more weight to the states where it has a low payoff – states 2 and 4. Country 1 then ends up being pessimistic in terms of its probability beliefs about country 2's productivity shock: it attaches higher probability to country 2 doing badly than if it were not ambiguity-averse. This makes country 1 invest more in its own technology than in the technology of country 2. That is, it chooses a home-biased portfolio.

Autarky as a special case of the model

One interesting feature of the model is that international portfolio autarky – a situation where domestic residents buy capital in the home country only – occurs naturally as a special case. In other words, this version of the model can be viewed as a potential reason for why (international) asset markets would be incomplete: with enough ambiguity aversion, consumers simply do not need access to foreign equity. The argument goes as follows: as ambiguity in the economy increases, so does the pessimism toward the other country's productivity, leading to a more pronounced home bias. When the home bias increases, foreign productivity becomes increasingly irrelevant for the performance of a country's portfolio. Further, there will be a point at which there is enough ambiguity that the domestic resident chooses to own no foreign stock at all – autarky – and, hence, is entirely indifferent as to how the other country's technology is doing. Moreover, when ambiguity exceeds this cutoff point, autarky will still result: autarky is an outcome for all values of a above the cutoff level $\bar{a}$. In other words, pessimism toward the other country reaches a maximum level, which is our third main finding: countries never become so pessimistic so as to want to short-sell foreign equity. What lies behind this finding?

Short-selling the other country's stock cannot occur

A feature of the equilibrium portfolio outcomes in this model is that countries never short-sell foreign stock, which is broadly consistent with the data. This is not an obvious outcome, because if one believes in the home over the foreign country, it would seem that a strong enough such belief would lead to short-selling foreign assets. This is correct, but such beliefs cannot occur in equilibrium. Intuitively, if short-selling did occur, we would have a qualitatively different situation than in the case discussed above: a country which short-sells foreign stock would by definition *do well* when the other country does badly. This would mean, however, that short-selling as investment behavior could not be consistent with minimizing expected utility with respect to the probability distribution: minimizing expected utility conditionally on short-selling would mean choosing a belief with a high probability weight on the other country doing well and a low weight on the other country doing badly, because then the portfolio does badly. But this probability choice contradicts the assumption that short-selling the foreign security represents optimal portfolio behavior, since for short-selling foreign equity to be optimal, one would need to believe that foreign equity is unlikely to do well. In sum, short-selling foreign stock requires pessimism about foreign stock but such pessimism contradicts minmax behavior under short-selling.

It is worthwhile reminding the reader here that the implications for shortselling is a case which illustrates the power of the hypothesis of (aversion to) ambiguity, as compared to simply relying on exogenous heterogeneity in beliefs. In particular, shortselling can be an equilibrium outcome in the model with heterogeneous beliefs. As an example, assuming the subjective probabilities $\pi^1_{i1} = \pi^1_{i3} = \pi^2_{i1} = \pi^2_{i2} = 0.25 - a$ and $\pi^1_{i2} = \pi^1_{i4} = \pi^2_{i3} = \pi^2_{i4} = 0.25 + a$ ($i = 1, 2, 3, 4$) with $\bar{a} < a < 0.25$ delivers shortselling. More generally, as the economy's primitives change, the model with exogenous differences in beliefs can move into a region with shortselling. This cannot occur under ambiguity.

Turning now to a more formal discussion of the above points, one can show that the first-order condition for the optimal v is $v \geq -\frac{\bar{A}-\underline{A}}{4(\bar{A}+\underline{A})} \equiv -\bar{a}$; this is shown in the Appendix. This implies that v is a corner solution for values of a smaller than $\bar{a}$ and an interior solution for values of a larger than $\bar{a}$. However, it is helpful to show the minimization problem for the v explicitly in order to explain when and why an interior solution for probabilities is chosen. Country 1 solves the minimization problem given by

$$\min_{v\in[-a,a]} \sum_{s'} \pi^1_{ss'}(v) V_{s'}(w'_{s'})$$

where $w'_{s'}$ depends on v. Equivalently, it solves

$$\min_{v\in[-a,a]} (\pi_{s1}+v)(B+\frac{1}{1-\beta}\log \bar{A}(x^1+y^1)\beta w) + (\pi_{s2}-v)(B+\frac{1}{1-\beta}\log(\bar{A}x^1\beta w+\underline{A}y^1\beta w))+$$

$$(\pi_{s3}+v)(B+\frac{1}{1-\beta}\log(\underline{A}x^1\beta w+\bar{A}y^1\beta w)) + (\pi_{s4}-v))(B+\frac{1}{1-\beta}\log \underline{A}(x^1+y^1)\beta w),$$

where x^1 and y^1 are the optimal fractions of savings invested in the technologies of countries 1 and 2; these also depend on v. Simplifying, one obtains

$$\min_{v\in[-a,a]} v(\log \bar{A}(x^1+y^1) - \log(\bar{A}x^1+\underline{A}y^1) + \log(\underline{A}x^1+\bar{A}y^1) - \log \underline{A}(x^1+y^1)). \tag{17}$$

In equilibrium the portfolio fractions are given by

$$x^1 = y^2 = \frac{1}{2} - 2v\frac{\bar{A}+\underline{A}}{\bar{A}-\underline{A}} \tag{18}$$

and

$$y^1 = x^2 = \frac{1}{2} + 2v\frac{\bar{A}+\underline{A}}{\bar{A}-\underline{A}}. \tag{19}$$

Note that the objective function in the minimization problem for v is not linear in v, because x^1 and y^1 depend on v.

When there is no ambiguity ($a = 0$), equilibrium portfolios are given by $x^1 = y^1 = 1/2$ and the term in parenthesis multiplying v in (17) is a strictly positive number since the consumer of country 1 (and the world in fact) is richer in state 1 than in state 2 and richer in state 3 than in state 4. When a small amount of ambiguity is introduced (a is a positive but a small number), continuity in the solution for x^1 and y^1 in equations (18)-(19) guarantees that the term in parenthesis in (17) is still strictly positive and, thus, that the minimum in (17) is obtained at $v = -a$. As long as a is not too large, $v = -a$ remains the solution to the minimization problem since the optimal portfolio satisfies $x^1 < 1$ and $y^1 > 0$, and thus the term in parenthesis in (17) is strictly positive.

If the disagreement in beliefs is significant enough (when the ambiguity parameter a is large) the residents of the two countries make large bets against each other and country 1 will gain significantly if state 2 occurs. There is a limit to this betting against each other, however, since at some point the optimal value of v becomes interior. In other words, if the parameter a is very high, so that the consumers consider a very large range of probabilities possible, the consumers still "choose beliefs" that are moderate, i.e., such that the vs are less than a (in absolute value). Thus, when a reaches a high enough level, $a = \bar{a}$, the optimal v satisfies $v = -\bar{a}$. We see from equations (18)–(19) that in this case, $x^1 = 1$, $y^1 = 0$, $x^2 = 0$, and $y^2 = 1$, i.e., country 1 and 2 only invest in their own technologies. At this point the term in parenthesis in equation (17) is zero. For values of a larger or equal than $\bar{a}$ the solution to v is interior and equal to $-\bar{a}$. Optimal portfolios again imply autarky and the term in parenthesis in equation (17) continues to be zero. Short-selling cannot occur because then the term in parenthesis in (17) would be negative, contradicting that $v < 0$ is a solution to the minimization problem.

6 Cross-country consumption and output correlations

Here we study the implications of ambiguity for conditional cross-country consumption and output correlations as well as for the serial correlation properties of investment. Some key model dynamics are summarized in the following figure.[5]

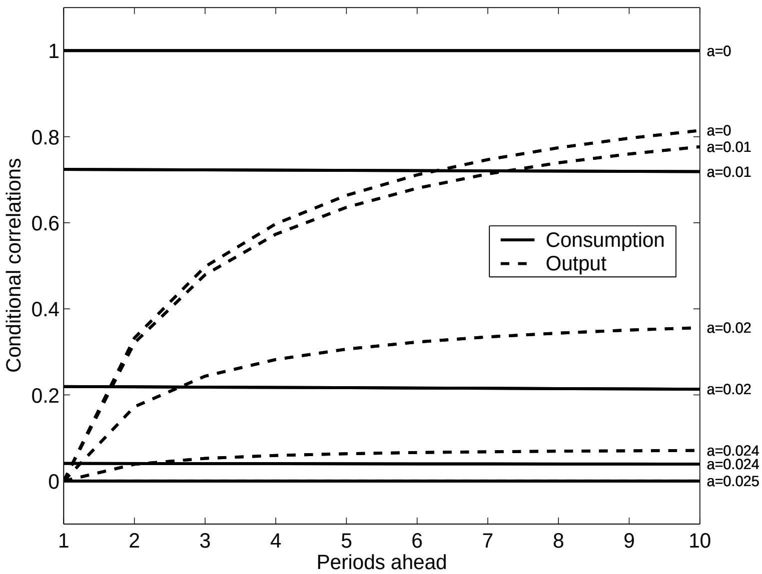

Figure 1: n-step-ahead conditional consumption and output correlations

The figure considers the parameter values $\beta = 0.98$, $\bar{A} = 1.1$, and $\underline{A} = 0.9$ and the benchmark transition matrix for the productivity shocks, i.e., $\pi_{ss'} = 1/4$. In addition, the results from the n-step ahead correlations shown in the figure assume an initial value for θ of 0.5. What the initial value for output is does not affect the values of the correlations since the optimal vs do not depend on output.

[5]The figures in this section are based on analytically calculated second moments and not on numerical simulations.

Cross-country consumption correlations

Consumption correlations are 1 when there is no ambiguity and they decrease as the amount of ambiguity measured by a increases. This is shown by the solid lines for the different values for a, i.e., $a = 0.1$, $a = 0.2$, and $a = 0.24$.[6] For values of a larger than or equal to 0.025, however, the optimal value for v goes from a corner to an interior solution so it remains at the constant value of 0.025 and therefore consumption correlations do not change as a increases beyond that threshold value. In fact, at $a \geq 0.025$, countries are in autarky. They invest solely in their own technology and since the shocks between the two countries' technologies are uncorrelated, consumption between countries are uncorrelated too.

The intuition for why autarky results at a rather low level of ambiguity is that even though diversification by investing in both countries is a less risky prospect for any given country, in the tradeoff between expected value (which is maximized under the subjective prior by choosing a portfolio with only home assets) and risk (which is minimized with a more balanced portfolio) the former wins with the logarithmic preferences assumed here: the individual countries choose autarky.

Cross-country output correlations

Output correlations are given by the dotted lines. Except when $a \geq 0.025$, where output correlations are equal to consumption correlations (since there is autarky), the conditional output correlations increase with the horizon, n, and the entire curves are higher up as a decreases. The 1-step ahead output correlation is zero since output in each country is solely driven by the productivity shocks (which are uncorrelated). As the time horizon increases the conditional correlation increases since output in a country is given by the product of the productivity shock and investment and the latter depends on last period's world output: with a longer horizon, the fluctuations in total output dominate the fluctuations in the productivity shocks, which are stationary, so output levels become more and more correlated over time. As ambiguity increases towards 0.025, countries invest more and more in their own technologies, implying a lower correlation of investment in the two countries, thus reducing output correlations.

Output correlations can be higher than consumption correlations

It is clear from the figure how this model allows output correlations between countries to exceed consumption correlations. In the no-ambiguity case, the output correlation graph never reaches that of consumption, but in all other cases we see that, except for very short-horizon conditional correlations (e.g., 1 or two periods ahead for $a = 0.02$), output correlations exceed consumption correlations. In the opposite corner case – autarky – output and consumption correlations are equal at zero.

The mechanism behind why cross-country consumption correlations can be lower than cross-country output correlations is the following: both output and consumption levels depend on *total output* (except in the case of autarky) creating a cause for co-movement for

[6] Consumption correlations do not depend on θ. This is a consequence of the proof of Proposition 1. Since optimal portfolios only contain the two countries' technologies and the returns on these are exogenously given, countries can solve their maximization problems without regard to the aggregate wealth given by θ. Therefore, optimal consumption levels do not depend on θ and neither do cross-country consumption correlations.

both consumption and output across countries. In addition, consumption to total output ratios depend on θ and $1-\theta$ which are perfectly negatively correlated. Over the long run, the variability of θ is large – it goes from zero to one (even with little ambiguity). This force lowers long-run consumption correlations significantly. The variability of the investment to total output ratios is lower than the variability in θ, since countries diversify by investing in both technologies, With very little ambiguity, these ratios are still always close to 1/2: even if country 1 has almost all the wealth so that world investment is allocated according to the views of country 1 residents, close to 50% of world investment still is allocated to the country 2 technology because country 1 residents have "balanced views" due to there being very little ambiguity. Therefore, at long enough horizons, consumption levels are less correlated than output levels.

Judging from the figure, the quantitative effect of a small amount of ambiguity (a) on consumption correlations, and therefore also for equilibrium portfolio allocations, seems large. A possible reason for the quantitatively large effect is the well-known result that standard levels of risk aversion (as with logarithmic utility here) lead agents to trade based on expected values more than on risk. Therefore, small differences in beliefs due to a small amount of ambiguity can lead to large differences in portfolios – a large home bias – and, consequently, to much less consumption insurance. In the concluding section, I discuss how the effects of ambiguity can be larger or smaller with other utility functions.

Autocorrelations in investment shares

In the next figure we see that investment shares in the two countries are (conditionally) positively serially correlated under ambiguity. Under common beliefs, these correlations are zero: since half is invested in each country in each period, the ratio between investment in country 1 and output in country 1 at time t is $k_{1,t+1}/y_{1t} = (\beta y_t/2)/y_{1t} = (\beta(\beta A_{1t}/2 + \beta A_{2t}/2)y_{t-1}/2)/(\beta y_{t-1}A_{1t}/2) = \beta(1 + (A_{2t}/A_{1t}))/2$, which is uncorrelated over time since the individual technology shocks are. Under ambiguity, in contrast, how much is invested in country 1 depends on the relative wealth of the two countries, which is serially correlated. These induced autocorrelations are part of the model's nontrivial propagation mechanisms. In the example in the figure, however, the induced correlation is quite weak (below 0.01).

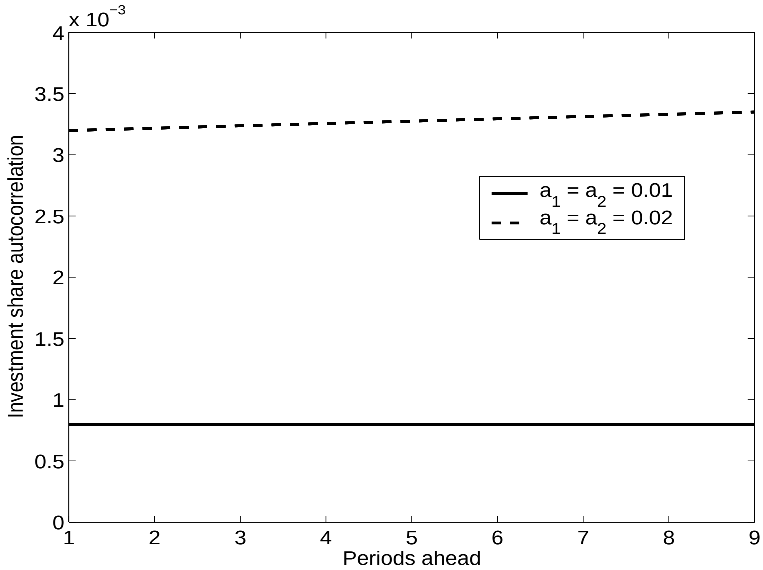

Figure 2: n-step-ahead conditional autocorrelations in a country's investment share

7 Conclusions

A two-country model with ambiguity and ambiguity aversion was formulated. The model can be solved in closed form for the special case of symmetric, independent, and iid shocks. The effect of small amounts of ambiguity for consumption correlations and portfolios can be large. Conditional T-period-ahead consumption correlations across countries are below output correlations, as long as T is high enough. Portfolios are home-biased; if there is enough ambiguity, they are so biased that there is portfolio autarky, but not so biased that short-selling of foreign stock occurs.

The present work indicates that what seems like a small amount of ambiguity aversion can influence outcomes substantially. Whether this conclusion remains also when the amount of ambiguity aversion is subjected to some form of independent calibration is an open issue. The model here is highly stylized, and the closed-form solutions come at the expense of realism. Quantitative evaluation of the importance of ambiguity aversion for international risk sharing and portfolio behavior ought to allow for endogenous labor supply as well as neoclassical production, as in Backus, Kehoe, and Kydland (1992). However, a reasonable conjecture seems to be that the large quantitative effects of ambiguity remain in such frameworks, because their influence on outcomes are not second-order effects, as are risk concerns: ambiguity aversion leads to differences in perceived expected returns and therefore first-order effects on utility, whereas risk aversion operates via second moments.

Some extensions of the present work are straightforward. For example, the use of logarithmic utility here is merely a convenient one. With more curvature – with more aversion to risk – behavior is less influenced by the presence of ambiguity. It is illustrative to consider an extreme case: linear utility, i.e., the case of no aversion to risk. In that case, the result – given that the remaining assumptions of the model are maintained – is that there will be autarky no matter how much ambiguity (as described by the parameter a) is present, but that there will, nevertheless, be common beliefs in the world. In other words, the presence of ambiguity actually does not lead to heterogeneous beliefs but it does generate the strongest form of home bias. The intuitive argument for this result is that without risk aversion there is no advantage to diversification, but there is an advantage to specialization given ambiguity aversion: the tension between diversification and specialization is resolved in favor of the latter. Less loosely, the logic is the following. Suppose that v is negative, i.e., that beliefs are biased toward the home technology. Because of linear utility, the agent would then want to go short abroad. If he went short, however, he would not be minimizing by choosing a negative v which is a contradiction. Thus the logic behind why short-selling cannot occur in equilibrium applies also when utility is not logarithmic. Similarly, a positive v cannot occur either, and the only remaining possibility is that $v = 0$: common, unbiased beliefs. In this case, note that the consumer will be indifferent regarding his portfolio composition. However, only autarky is possible as an outcome, because if the consumer holds a positive share of the portfolio in foreign assets, ambiguity aversion would imply a negative choice for v, which has been ruled out. In conclusion, only the combination of autarky and $v = 0$ is an equilibrium.

Does the theory of home bias based on ambiguity aversion improve on alternative explanations for the home bias?[7] Its closest relative is the hypothesis that domestic investors simply have less information about foreign equity than about home equity. This is formalized by assuming that consumers receive a signal on the future performance of the home stock that comes from a return distribution that has a lower variance than does the return distribution of the foreign stock. Such a hypothesis, however, faces the challenge that better information, through a better signal, could equally well imply that domestic investors should *sell* home assets. In particular, because the superior information about home stock would often suggest that the prospects of home stock are worse than what the rest of the world perceives they are, the time series for domestic consumers' portfolios would have significant spells of foreign bias.[8] This is not observed in the data. Moreover, the hypothesis about differential information suggests short-selling of foreign stock, which is also not common in the data. In contrast, as has been demonstrated in the present paper, ambiguity aversion naturally delivers both the absence of short-selling *and* a portfolio time series where home bias is always present and foreign bias is never present.[9]

Evidence on the theory presented here is supported by empirical work on the patterns of asset holdings which argue that consumers invest in the familiar and, especially, by surveys

[7]For an excellent survey, see Lewis (1999).

[8]One device for modeling differential information is the assumption that the domestic investor perceives the same mean return home and abroad but larger variance abroad; (see Gehrig (1993)). Realistically, however, differential information presumably involves also differences in mean returns perceived by home and foreign investors.

[9]See Hatchondo (2004) for an interesting recent explanation based on asymmetric information that does not suffer from these problems. The argument is that the superior information about home equity is not about average returns but about ranking of different stock. This information therefore allows "stock-picking", leading domestic consumers to hold diversified foreign portfolios and less diversified home portfolios. This prediction is borne out in the empirical study by Albuquerque et al. discussed below. It is also consistent with findings of the sort reported in Coval and Moskowitz (2001), who show that fund managers who invest in selected local companies, and who tend to stock-pick, earn substantial abnormal returns: 2.67% per year for the average fund manager relative to nonlocal holdings.

which show that consumers exhibit optimism toward nearby stocks. For example, Huberman (2001) documents that employees tend to choose their employers's stock for their retirement accounts, and therefore do not diversify optimally. Shiller et al. (1996) present survey data from American and Japanese investors showing that American investors feel more optimistic about the U.S. stock index than about that of Japan, and vice versa. These authors suggest that consumers have irrational expectations. However, as was shown in this paper, these kinds of expectations can be rationalized. The assumption of asymmetric ambiguity, which seems natural, means that consumers perceive that the range of possible probabilities of the return distribution of the foreign stock is larger than that for the home stock, and the assumption of ambiguity aversion, which also seems natural, implies that consumers "choose" stronger pessimism toward the foreign and therefore hold a larger proportion of the home asset.

In a recent study, Albuquerque, Bris, and Schneider (2004) find that most of the home bias is accounted for by a large group of investors holding zero foreign equity, as opposed to by widespread low but positive holdings of foreign equity. In fact, investors that participate in foreign stocks show significantly less home bias than the aggregate population. This indicates that most investors are ambiguous about foreign stock, apart from a few people who feel fairly "familiar" with foreign equity.

Another kind of explanation for the home bias is based on transactions costs in acquiring and selling assets. This explanation faces the challenge that transactions costs are very small in practice. Moreover, they do not explain why there is also a local bias in investment within countries. Transactions costs in acquiring and processing information seem more important, and indeed they motivate the approach taken in this paper.

In the literature on international macroeconomics, several additional mechanisms have been proposed in order to explain the home bias and the lack of consumption correlation across countries. One of these is based on introducing nontradable goods. As shown in Stockman and Dellas (1989), when utility in traded and nontraded goods is separable, domestic agents hold all the equity in the nontradable industry at the same time as there is perfect international diversification in the tradable industries.[10] The intuition is that the returns on the nontraded industries is perfectly correlated with the expenditures on nontraded goods, thus providing perfect consumption insurance in the consumption of nontraded goods. This theory does not, however, explain why there is a home bias also for corporations producing traded goods. A related explanation is that put forth in DeMarzo, Kaniel, and Kremer (2004). Here, price fluctuations for "local goods" make consumers want to hold the same portfolios as their neighbors, who compete with them for the local goods. This preserves the relative wealth ranking of households and reduces risk in the consumption of local goods, provided that asset markets are incomplete in certain ways. Here, diversification would be in everybody's interest, but the market incompleteness generates an externality that prevents it.

Other explanations have also been suggested in the macroeconomic literature, but most of these rely on a representative-agent construct: the idea is to show that a home-biased portfolio represents optimal risk management for this agent under some specific assumptions on preferences and income processes, which differ across countries (for recent work not sur-

[10]The result also requires homotheticity in the different traded goods, or perfect equality in initial wealth.

veyed in Lewis, 1999, see, e.g., Heathcote and Perri, 2004, and Juillard, 2004). The challenge then, it seems, is to explain why the home bias in the data goes far beyond the average portfolio behavior of countries' citizens: it appears pervasively for so many consumers within a country with different tastes and different income processes. The ambiguity hypothesis, in contrast, perhaps allows a joint explanation for these phenomena.

Finally, an additional test of the hypothesis proposed here comes from the result that the wealth shares of countries drift over time. In particular, at least in the simple benchmark model, the wealth and consumption levels of different countries are not cointegrated.[11] This implication is likely shared with other explanations of the home bias, since these other explanations also must imply wealth shares that at least are not constant over time.

8 References

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[11]This result requires symmetry of primitives across countries. Whether it carries over to asymmetric setups is unclear; for example, it is possible that the wealth share of one country would converge to 1 in equilibrium if country 1 residents were less averse to ambiguity than country 2 residents. On the other hand, the implication is robust to extensions to neoclassical growth setups, because the result that the planner weights drift neither relies on specific features of preferences (apart from beliefs) nor of technology.

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9 Appendix

9.1 Common Beliefs Without Ambiguity

Verification of the solution to the economy with common beliefs:

Claim 1 *The value function*

$$V_s(y) = \frac{1}{1-\beta}\log y + C_s,$$

satisfies the functional equation in the economy with common beliefs and the policy functions for consumption and investment in each country are given by

$$c_{1s} = \theta(1-\beta)y_s$$

$$c_{2s} = (1-\theta)(1-\beta)y_s$$

$$k'_{1s} = \beta\frac{(\underline{A}\pi_{s3} - \bar{A}\pi_{s2})}{(\underline{A} - \bar{A})(\pi_{s2} + \pi_{s3})}y_s$$

$$k'_{2s} = \beta y_s - k'_{1s}.$$

To verify the claim we take first-order conditions for consumption levels c_1 and c_2 and investment levels k'_1 and k'_2 in the planning problem where the guess on the value function above has been used. These are

$$\frac{\theta}{c_1} = \frac{1-\theta}{c_2} = \mu,$$

$$\mu = \frac{\beta}{1-\beta}\sum_{s'=1}^{4}\pi_{ss'}\frac{A_{1s'}}{y'_s(k'_1, k'_2)}$$

and

$$\mu = \frac{\beta}{1-\beta}\sum_{s'=1}^{4}\pi_{ss'}\frac{A_{2s'}}{y'_s(k'_1, k'_2)},$$

where μ is the multiplier for the resource constraint.

Next we show that the solution to these first-order conditions are the policy functions

$$k'_{1s} = \kappa_{1s}y_s(k_1, k_2)$$

$$k'_{2s} = \kappa_{2s}y_s(k_1, k_2)$$

$$c_{1s} = \tilde{c}_{1s}y_s(k_1, k_2)$$

$$c_{2s} = \tilde{c}_{2s}y_s(k_1, k_2),$$

where $y_s(k_1, k_2) \equiv A_{1s}k_1 + A_{2s}k_2$. The last two equations and the first-order conditions for consumption imply that $\tilde{c}_{1s}/\tilde{c}_{2s} = \frac{\theta}{1-\theta}$. Defining $\tilde{c}_s$ as $\tilde{c}_{1s} + \tilde{c}_{2s}$, we can write $\tilde{c}_{1s} = \theta\tilde{c}_s$ and $\tilde{c}_{2s} = (1-\theta)\tilde{c}_s$.

The resource constraint now delivers

$$\tilde{c}_s + \kappa_{1s} + \kappa_{2s} = 1$$

for $s = 1, 2, 3, 4$. The first-order conditions for capital, when μ is replaced and $y_s(k_1, k_2)$ is cancelled from each side, yield

$$\frac{1}{\tilde{c}_s} = \frac{\beta}{1-\beta} \sum_{s'=1}^{4} \pi_{ss'} \frac{A_{1s'}}{A_{1s'}\kappa_{1s} + A_{2s'}\kappa_{2s}}$$

and

$$\frac{1}{\tilde{c}_s} = \frac{\beta}{1-\beta} \sum_{s'=1}^{4} \pi_{ss'} \frac{A_{2s'}}{A_{1s'}\kappa_{1s} + A_{2s'}\kappa_{2s}};$$

each of these have to hold for $s = 1, 2, 3, 4$.[12] Thus, we have $3 \cdot 4$ equations in the $3 \cdot 4$ unknown policy function parameters.

To implement the 2-shock case that we have in this model means to assume that $A_{11} = A_{12} = A_{21} = A_{23} \equiv \bar{A}$ and that $A_{13} = A_{14} = A_{22} = A_{24} \equiv \underline{A}$.

The solution for $\tilde{c}_s$, κ_{1s}, and κ_{2s} is

$$\tilde{c}_s = 1 - \beta$$

$$\kappa_{1s} = \beta \frac{(\bar{A}\pi_{s2} - \underline{A}\pi_{s3})}{(\bar{A} - \underline{A})(\pi_{s2} + \pi_{s3})}$$

$$\kappa_{2s} = \beta - \kappa_{1s}.$$

To verify our guess on the value function, we now need to find the value function parameters. Thus, we have

$$\frac{1}{1-\beta} \log y_s(k_1, k_2) + C_s = \theta \log \theta \tilde{c}_s y_s(k_1, k_2) + (1-\theta) \log(1-\theta)\tilde{c}_s y_s(k_1, k_2) + \beta \sum_{s'=1}^{4} \pi_{ss'} \left\{ \frac{1}{1-\beta} \log\left(A_{1s'}\kappa_{1s} y_s(k_1, k_2) + A_{2s'}\kappa_{2s} y_s(k_1, k_2)\right) + C_{s'} \right\}.$$

We see that the coefficient on $\log y_s$ on the left hand side, $\frac{1}{1-\beta}$, equals the coefficient on $\log y_s$ on the right hand side. Moreover, equating the intercepts allows us to solve for the C_ss: for $s = 1, 2, 3, 4$,

$$C_s = \theta \log \theta + (1-\theta) \log(1-\theta) + \log(1-\beta) + \beta \sum_{s'=1}^{4} \pi_{ss'} \left\{ \frac{1}{1-\beta} \log\left(A_{1s'}\kappa_{1s'} + A_{2s'}\kappa_{2s'}\right) + C_{s'} \right\}.$$

These last equations allow us to solve for the C_ss linearly and in closed form as a function of the policy function parameters.

[12] We focus on the cases where the parameters of the economy are such that k'_1 and k'_2 are non-negative.

9.2 Heterogeneous Beliefs without Ambiguity

Our main result for this section is

Claim 2 *There exist functions $C_s(\theta)$ for $s = 1, 2, 3, 4$ such that the value function*

$$V_s(y, \theta) = \frac{1}{1-\beta} \log y + C_s(\theta) \tag{20}$$

satisfies the functional equation in the economy with heterogeneous beliefs and the policy functions for consumption and investment in each country are given by

$$c_{1s} = \theta(1-\beta)y_s$$

$$c_{2s} = (1-\theta)(1-\beta)y_s$$

$$k'_{1s} = \beta \frac{\bar{A}(\pi^1_{s2}\theta + \pi^2_{s2}(1-\theta)) - \underline{A}(\pi^1_{s3}\theta + \pi^2_{s3}(1-\theta))}{(\underline{A} - \bar{A})(\pi^1_{s2}\theta + \pi^2_{s2}(1-\theta) + \pi^1_{s3}\theta + \pi^2_{s3}(1-\theta))} y_s$$

$$k'_{2s} = \beta y_s - k'_{1s}.$$

Below, we prove this claim in two steps. First, we derive the solution for consumption and investment implied by the properties of the value function stated in the claim. Second, we prove that the specified value function satisfies the functional equation implied by the planning problem by showing that the functions $C_s(\theta)$ exist and are unique, bounded and continuous.

Verification of the solution for consumption and investment:

We derive necessary first-order conditions for interior solutions for consumption levels c_1 and c_2 and investment levels k'_1, and k'_2 and we obtain

$$\theta/c_1 = (1-\theta)/c_2 = \mu \tag{21}$$

$$\sum_{s'=1}^{4} \delta_{s'} \partial V_{s'}/\partial k'_1 = \sum_{s'=1}^{4} \delta_{s'} \partial V_{s'}/\partial k'_2 = \mu, \tag{22}$$

where μ is the multiplier of the resource constraint and $\delta_{s'}$ is the multiplier on constraint (4). Taking derivatives with respect to $z_1(s')$ and $z_2(s')$ we arrive at

$$\frac{\beta}{\delta_{s'}} \theta \pi^1_{ss'} = \theta'(s')$$

$$\frac{\beta}{\delta_{s'}} (1-\theta) \pi^2_{ss'} = 1 - \theta'(s').$$

Rewriting this, we obtain

$$\frac{\theta'(s')}{1-\theta'(s')} = \frac{\theta}{(1-\theta)} \frac{\pi^1_{ss'}}{\pi^2_{ss'}}.$$

Finally, $z_1(s')$ and $z_2(s')$ must satisfy

$$\partial V_{s'}/\partial\theta'(s') = z_1(s') - z_2(s').$$

We can simplify the planning problem by using the policy rule for θ' in equation (7) and substituting $z_1(s')$ from (4) with equality into the objective.[13] This gives

$$V_s(y,\theta) = \max_{c_1,c_2,k_1',k_2'} \theta \log c_1 + (1-\theta)\log c_2 + \beta \sum_{s'=1}^{4} \tilde{\pi}_{ss'} V_{s'}(y_{s'}(k_1',k_2'),\theta'(s')) \tag{23}$$

subject to the resource constraint, where $\tilde{\pi}_{ss'} \equiv \pi^1_{ss'}\theta + \pi^2_{ss'}(1-\theta)$. This is a more compact problem.

Using the guess on the value function from the claim above, the right-hand side of the Bellman equation (23) can be written as

$$\max_{c_1,c_2,k_1',k_2'} \theta \log c_1 + (1-\theta)\log c_2 + \beta \sum_{s'=1}^{4} \tilde{\pi}_{ss'} \left\{ \frac{1}{1-\beta} \log(A_{1s'}k_1' + A_{2s'}k_2') + C_{s'}(\theta'(s')) \right\}$$

subject to

$$c_1 + c_2 + k_1' + k_2' = y. \tag{24}$$

The first-order conditions with respect to c_1, c_2, k_1', and k_2' are, after elimination of the multipliers,

$$c_1 = \frac{\theta}{1-\theta} c_2 \tag{25}$$

$$\frac{\theta}{c_1} = \frac{\beta}{1-\beta} \sum_{s'=1}^{4} \tilde{\pi}_{ss'} \frac{A_{1s'}}{A_{1s'}k_1' + A_{2s'}k_2'} \tag{26}$$

and

$$\frac{\theta}{c_1} = \frac{\beta}{1-\beta} \sum_{s'=1}^{4} \tilde{\pi}_{ss'} \frac{A_{2s'}}{A_{1s'}k_1' + A_{2s'}k_2'}. \tag{27}$$

Given the states y and θ, (24)-(27) determine the choices c_1, c_2, k_1', and k_2'.

We guess that the policy functions have the form: $c_1 = \tilde{c}_{1s}(\theta)y$, $c_2 = \tilde{c}_{2s}(\theta)y$, $k_1' = \kappa_{1s}(\theta)y$, and $k_2' = \kappa_{2s}(\theta)y$.

Using these guesses it can easily be verified that the coefficient on $\log y$ on the left- and right-hand sides of the Bellman equation is indeed $\frac{1}{1-\beta}$.

With minor manipulation of the first-order conditions for the choice variables, we obtain

$$\frac{1}{1-\kappa_{1s}(\theta)-\kappa_{2s}(\theta)} = \frac{\beta}{1-\beta} \sum_{s'=1}^{4} \tilde{\pi}_{ss'} \frac{A_{1s'}}{A_{1s'}\kappa_{1s}(\theta) + A_{2s'}\kappa_{2s}(\theta)}$$

and

$$\frac{1}{1-\kappa_{1s}(\theta)-\kappa_{2s}(\theta)} = \frac{\beta}{1-\beta} \sum_{s'=1}^{4} \tilde{\pi}_{ss'} \frac{A_{2s'}}{A_{1s'}\kappa_{1s}(\theta) + A_{2s'}\kappa_{2s}(\theta)},$$

[13]Constraint (4) is binding since its multiplier $\delta_{s'} = \beta(\theta\pi^1_{ss'} + (1-\theta)\pi^2_{ss'})$ is strictly greater than zero.

which have to hold for all s and all θ. Thus, these are 8 functional equations in the 8 unknown functions $\{\kappa_{1s}(\theta), \kappa_{2s}(\theta)\}_{s=1}^4$.

Solving for $\kappa_{1s}(\theta)$ and $\kappa_{2s}(\theta)$ we obtain

$$\kappa_{1s}(\theta) = \beta \frac{\bar{A}(\pi_{s2}^1\theta + \pi_{s2}^2(1-\theta)) - \underline{A}(\pi_{s3}^1\theta + \pi_{s3}^2(1-\theta))}{(\bar{A} - \underline{A})(\pi_{s2}^1\theta + \pi_{s2}^2(1-\theta) + \pi_{s3}^1\theta + \pi_{s3}^2(1-\theta))} \tag{28}$$

$$\kappa_{2s}(\theta) = \beta - \kappa_{1s}(\theta). \tag{29}$$

Given the κ functions, we find $\tilde{c}_{1s}(\theta)$ from

$$\tilde{c}_{1s}(\theta) = \theta(1 - \kappa_{1s}(\theta) - \kappa_{2s}(\theta))$$

and

$$\tilde{c}_{2s}(\theta) = (1-\theta)(1 - \kappa_{1s}(\theta) - \kappa_{2s}(\theta))$$

and we therefore obtain

$$\tilde{c}_{1s}(\theta) = \theta(1-\beta)$$
$$\tilde{c}_{2s}(\theta) = (1-\theta)(1-\beta).$$

Given a solution for the κ and $\tilde{c}$ functions, we can determine the C functions as follows:

$$C_s(\theta) = \theta\log\theta + (1-\theta)\log(1-\theta) + \log(1-\beta) +$$
$$\beta \sum_{s'=1}^4 \tilde{\pi}_{ss'}\left[\frac{1}{1-\beta}\log(A_{1s'}\kappa_{1s'}(\theta) + A_{2s'}\kappa_{2s'}(\theta)) + C_{s'}(\theta'(s'))\right]. \tag{30}$$

Below we prove that this functional equation has a unique continuous and bounded solution and, therefore, that our guess on the value function (20) satisfies the functional equation (23) by construction.

Proof that the solution for C is unique and bounded:

Recall that $(TC)(\theta)$ is defined as

$$\theta\log\theta + (1-\theta)\log(1-\theta) + \log(1-\beta) + \beta\sum_{s'=1}^4 \tilde{\pi}_{ss'}\left[\frac{1}{1-\beta}\log(A_{1s'}\kappa_{1s'}(\theta) + A_{2s'}\kappa_{2s'}(\theta)) + C_{s'}(\theta'(s'))\right].$$

If T is defined in the space of continuous and bounded functions, it is easy to see that TC is also continuous and bounded; this follows from observing that $\lim_{\theta\to 0}\theta\log\theta = 0$. Therefore T maps the set of continuous and bounded functions into itself. If we show that T satisfies monotonicity and discounting then T is a contraction and by the contraction mapping theorem we can therefore establish that T has a unique fixed point. Hence, the functional equation (30) has a unique continuous and bounded solution.

We first show discounting, i.e., that $[T(C+a)](\theta) \leq (TC)(\theta) + \delta a$ for $\delta \in (0,1)$ and $a \in R$. Simply observe that

$$[T(C+a)](\theta) =$$
$$\theta\log\theta + (1-\theta)\log(1-\theta) + \log(1-\beta) + \beta\sum_{s'=1}^4 \tilde{\pi}_{ss'}\left[\frac{1}{1-\beta}\log(A_{1s'}\kappa_{1s'}(\theta) + A_{2s'}\kappa_{2s'}(\theta)) + C_{s'}(\theta'(s')) + a\right] =$$

$$(TC)(\theta) + \beta a \leq (TC)(\theta) + \delta a$$

for $\delta \in (\beta, 1)$.

Monotonicity, i.e., $\bar{C}(\theta) \leq \hat{C}(\theta)$ implies that $T\bar{C}(\theta) \leq T\hat{C}(\theta)$ for all θ, is verified by noting that

$$\theta \log \theta + (1-\theta) \log(1-\theta) + \log(1-\beta) + \beta \sum_{s'=1}^{4} \tilde{\pi}_{ss'} \left[\frac{1}{1-\beta} \log(A_{1s'}\kappa_{1s'}(\theta) + A_{2s'}\kappa_{2s'}(\theta)) + \bar{C}_{s'}(\theta'(s')) \right] \leq$$

$$\theta \log \theta + (1-\theta) \log(1-\theta) + \log(1-\beta) + \beta \sum_{s'=1}^{4} \tilde{\pi}_{ss'} \left[\frac{1}{1-\beta} \log(A_{1s'}\kappa_{1s'}(\theta) + A_{2s'}\kappa_{2s'}(\theta)) + \hat{C}_{s'}(\theta'(s')) \right]$$

since

$$\sum_{s'=1}^{4} \tilde{\pi}_{ss'} \bar{C}_{s'}(\theta'(s')) \leq \sum_{s'=1}^{4} \tilde{\pi}_{ss'} \hat{C}_{s'}(\theta'(s')).$$

9.3 Ambiguity

The first result discussed in this section is

Claim 3 *There exist functions $C_s(\theta)$ for $s = 1, 2, 3, 4$ such that the value function*

$$V_s(y, \theta) = \frac{1}{1-\beta} \log y + C_s(\theta) \tag{31}$$

satisfies the functional equation in the economy with ambiguity and the policy functions for consumption and investment in each country are given by

$$c_{1s} = \theta(1-\beta)y_s$$

$$c_{2s} = (1-\theta)(1-\beta)y_s$$

$$k'_{1s}(\theta) = \beta \frac{\bar{A}\tilde{\pi}_{s2} - \underline{A}\tilde{\pi}_{s3}}{(\bar{A} - \underline{A})(\pi_{s2} + \pi_{s3})} y_s$$

$$k'_{2s}(\theta) = \beta y_s - k'_{1s}(\theta),$$

with $\tilde{\pi}_{s2} \equiv (\pi_{s2} - v_s^{1*})\theta + (\pi_{s2} + v_s^{2*})(1-\theta)$ and $\tilde{\pi}_{s3} \equiv (\pi_{s3} + v_s^{1*})\theta + (\pi_{s3} - v_s^{2*})(1-\theta)$, and where v_s^{1*} and v_s^{2*}, $s = 1, 2, 3, 4$, determine the optimal beliefs for countries 1 and 2, respectively.

In this section, we prove three results. First, we prove claim 3 above for an economy with ambiguity and a general true transition probability matrix. Then we prove Lemma 1 and Proposition 1 of the main text for an economy with ambiguity where the productivity shocks are independent across countries and over time and are symmetric.

Proof of Claim 3:

The planner's problem is

$$V_s(y, \theta) = \max_{c_1, c_2, k'_1, k'_2, z_1(s'), z_2(s')} \theta \log c_1 + (1-\theta) \log c_2 +$$

$$\beta(\min_{v^1\in[-a,a]}\theta\sum_{s'=1}^{4}\pi^1_{ss'}(v^1)z_1(s')+(1-\theta)\min_{v^2\in[-a,a]}\sum_{s'=1}^{4}\pi^2_{ss'}(v^2)z_2(s')) \tag{32}$$

subject to

$$\min_{\theta'(s')}V_{s'}(y_{s'}(k_1',k_2'),\theta'(s'))-(\theta'(s')z_1(s')+(1-\theta'(s'))z_2(s'))\geq 0 \tag{33}$$

for all s' and

$$c_1+c_2+k_1'+k_2'=A_{1s}k_1+A_{2s}k_2 \tag{34}$$

for $s=1,2,3,4$ and where the $\pi(v)$s are given by the transition matrices defined in Section 2.

Using the minimax theorem to reverse the min and max operations, we rewrite the right-hand side problem as

$$\min_{v^1,v^2\in[-a,a]}\max_{c_1,c_2,k_1',k_2',z_1(s'),z_2(s')}\theta\log c_1+(1-\theta)\log c_2+\beta\sum_{s'=1}^{4}(\pi^1_{ss'}(v^1)\theta z_1(s')+\pi^2_{ss'}(v^2)(1-\theta)z_2(s'))$$

and note that maximization with respect to $z_1(s')$ and $z_2(s')$ for all s' delivers the law of motion for θ:

$$\theta'(s')=\frac{\pi^1_{ss'}(v^1_s)\theta}{\pi^1_{ss'}(v^1_s)\theta+\pi^2_{ss'}(v^2_s)(1-\theta)}.$$

Substituting this into the minimization over $\theta'(s')$ and requiring equality, rearranging, and then substituting into the objective, we obtain

$$V_s(y,\theta)=\min_{v^1,v^2\in[-a,a]}\max_{c_1,c_2,k_1',k_2'}\theta\log c_1+(1-\theta)\log c_2+\beta\sum_{s'=1}^{4}\tilde{\pi}_{ss'}V_{s'}(y_{s'}(k_1',k_2'),\theta'(s')),$$

where $\tilde{\pi}_{ss'}\equiv\pi^1_{ss'}(v^1)\theta+\pi^2_{ss'}(v^2)(1-\theta)$, subject to the resource constraint.

Inserting the guess on the value function, we obtain

$$\frac{1}{1-\beta}\log y_s(k_1,k_2)+C_s(\theta)=$$

$$\min_{v^1,v^2\in[-a,a]}\max_{c_1,c_2,k_1',k_2'}\theta\log c_1+(1-\theta)\log c_2+\beta\sum_{s'=1}^{4}\tilde{\pi}_{ss'}\left(\frac{1}{1-\beta}\log y_{s'}(k_1',k_2')+C_{s'}(\theta'(s'))\right).$$

As was proved in the economy with heterogeneous beliefs above, the maximization problem

$$\max_{c_1,c_2,k_1',k_2'}\theta\log c_1+(1-\theta)\log c_2+\beta\sum_{s'=1}^{4}\tilde{\pi}_{ss'}\left(\frac{1}{1-\beta}\log y_{s'}(k_1',k_2')+C_{s'}(\theta'(s'))\right)$$

implies, for any value of the countries' beliefs v^1 and v^2 in $[-a,a]$, the following decision rules for the consumption and investment levels:

$$c_1=\theta(1-\beta)y_s$$

$$c_2 = (1-\theta)(1-\beta)y_s$$
$$k'_{1s}(\theta) = \beta \frac{\bar{A}\tilde{\pi}_{s2} - \underline{A}\tilde{\pi}_{s3}}{(\bar{A}-\underline{A})(\pi_{s2}+\pi_{s3})} y_s$$
$$k'_{2s}(\theta) = \beta y_s - k'_{1s}(\theta).$$

Substituting these decision rules into the right-hand side of the Bellman equation, we have

$$\min_{v^1, v^2 \in [-a,a]} \theta \log \theta(1-\beta)y_s + (1-\theta)\log(1-\theta)(1-\beta)y_s +$$
$$\beta \sum_{s'=1}^{4} \tilde{\pi}_{ss'} \left(\frac{1}{1-\beta} \log \left(A_{1s'}\beta \frac{(\bar{A}\tilde{\pi}_{s2} - \underline{A}\tilde{\pi}_{s3})}{(\bar{A}-\underline{A})(\pi_{s2}+\pi_{s3})} y_s + A_{2s'}\beta \frac{(\bar{A}\tilde{\pi}_{s3} - \underline{A}\tilde{\pi}_{s2})}{(\bar{A}-\underline{A})(\pi_{s2}+\pi_{s3})} y_s \right) + C_{s'}(\theta'(s')) \right).$$

It is straightforward to verify that the coefficient on $\log y$ on the right- and left-hand sides of the value function is $\frac{1}{1-\beta}$. Solving for the intercept coefficient of the value function, we arrive at

$$C_s(\theta) = \min_{v^1, v^2} \theta \log \theta(1-\beta) + (1-\theta)\log(1-\theta)(1-\beta) +$$
$$\beta \sum_{s'=1}^{4} \tilde{\pi}_{ss'} \left(\frac{1}{1-\beta} \log \beta \frac{A_{1s'}(\bar{A}\tilde{\pi}_{s2} - \underline{A}\tilde{\pi}_{s3}) + A_{2s'}(\bar{A}\tilde{\pi}_{s3} - \underline{A}\tilde{\pi}_{s2})}{(\bar{A}-\underline{A})(\pi_{s2}+\pi_{s3})} + C_{s'}(\theta'(s')) \right).$$

Again, this is a functional equation and, by the same arguments as in the last section, it can be shown that the right-hand side defines an operator from a set of continuous and bounded functions into itself. Notice that the minimization is well defined since v_1 and v_2 lie in compact sets and C is continuous. As in the previous section, it is easy to show that this operator is a contraction and by the contraction mapping theorem, there exists a unique fixed point. Therefore, the guess on the value function satisfies the functional equation (32) by construction.

Since the function C cannot be solved in closed form in general, neither can the solution for the optimal vs.

We will also make some remarks regarding the characterization of the solution for the beliefs. To find the optimal vs we take first-order conditions with respect to v^1 and v^2 in the problem given by (32). They are

$$z_1(1) - z_1(2) + z_1(3) - z_1(4) \geq 0$$

and

$$z_2(1) + z_2(2) - z_2(3) - z_2(4) \geq 0,$$

respectively; here, the zs are the optimal choices from the problem above. The v is interior if the associated expression holds with equality and equal to $-a$ otherwise. To find the z choices, one needs to find the first-order condition for the minimization problem for $\theta'(s')$, which gives

$$\frac{\partial V_{s'}(y_{s'}(k'_1, k'_2)), \theta'(s'))}{\partial \theta'(s')} = \frac{dC_{s'}(\theta'(s'))}{d\theta'(s')} = z_1(s') - z_2(s').$$

Solving for the zs using (33) with equality and substituting back into the first-order conditions for the vs, we have

$$\frac{1}{1-\beta}(\log y_1(k_1', k_2') - \log y_2(k_1', k_2') + \log y_3(k_1', k_2') - \log y_4(k_1', k_2'))+$$

$$+C_1(\theta_1') - C_2(\theta_2') + C_3(\theta_3') - C_4(\theta_4')+$$

$$+(1-\theta_1')\frac{dC_1(\theta_1')}{d\theta_1'} - (1-\theta_2')\frac{dC_2(\theta_2')}{d\theta_2'} + (1-\theta_3')\frac{dC_3(\theta_3')}{d\theta_3'} - (1-\theta_4')\frac{dC_4(\theta_4')}{d\theta_4'} \geq 0 \qquad (35)$$

and

$$\frac{1}{1-\beta}(\log y_1(k_1', k_2') + \log y_2(k_1', k_2') - \log y_3(k_1', k_2') - \log y_4(k_1', k_2'))+$$

$$+C_1(\theta_1') + C_2(\theta_2') - C_3(\theta_3') - C_4(\theta_4')-$$

$$-\theta_1'\frac{dC_1(\theta_1')}{d\theta_1'} - \theta_2'\frac{dC_2(\theta_2')}{d\theta_2'} + \theta_3'\frac{dC_3(\theta_3')}{d\theta_3'} + \theta_4'\frac{dC_4(\theta_4')}{d\theta_4'} \geq 0. \qquad (36)$$

Incomplete markets is not restrictive (proof of Lemma 1 in the main text):

We assume that the vs are constant (independent of s and θ). First, we study an agent's problem – that of a resident of country i – under complete markets in order to find the prices of the contingent claims. An agent's typical budget is

$$c + \sum_{s'} q_{ss'}(\theta)a_{ss'} = w,$$

where $a_{ss'}$ is the agent's holding of contingent claim s' when the current state is s; the associated price is $q_{ss'}(\theta)$, where θ is the economy-wide state variable representing the wealth distribution (it is the fraction of wealth held by residents in country 1 or, equivalently, the weight the planner places on these agents in the associate planning problem). The variable w is the beginning-of-period wealth. Next period, the agent's wealth is $w'_{s'} = a_{ss'}$. The first-order condition for contingent claim $a_{ss'}$ is

$$q_{ss'}(\theta)u'(c) = \beta\pi^i_{ss'}u'(c'_{s'}).$$

We know that in this equilibrium, the agent will save a fraction β of his wealth and consume a fraction $1-\beta$. Thus, $c = (1-\beta)\omega$ and $c'_{s'} = (1-\beta)w'_{s'}$. In equilibrium, $w = \theta y$: the agent is a representative i resident, with y being world output. Thus, the first-order condition can be written

$$q_{ss'}(\theta) = \beta\pi^i_{ss'}\frac{\theta y}{\theta'_{s'}y'_{s'}}.$$

Using the updating formula for $\theta'_{s'}$, we have

$$q_{ss'}(\theta) = \beta\tilde{\pi}_{ss'}(\theta)\frac{y}{y'_{s'}}.$$

We also know that $y'_{s'} = (A_{1s'}\kappa_{1s}(\theta) + A_{2s'}\kappa_{2s}(\theta))y$, where the κs are the fractions of total output invested in each country's technology (equations (28), (29)). This gives the closed form for the price as a function of the aggregate state:

$$q_{ss'}(\theta) = \beta \frac{\tilde{\pi}_{ss'}(\theta)}{A_{1s'}\kappa_{1s}(\theta) + A_{2s'}\kappa_{2s}(\theta)}.$$

Second, one can show by analyzing the agent's problem that only two assets – the two technologies – suffice for attaining the equilibrium allocation. The agents's decision rule for contingent claims satisfies $a_{ss'}(w, \theta) = \beta\alpha_{ss'}(\theta)w$, where $\alpha_{ss'}(\theta)$ are constants to be determined (i.e., the constant does not depend on the agent's own wealth but on the distribution of wealth). The first-order condition for the i agent now becomes, given these features of optimal behavior,

$$q_{ss'}(\theta) = \frac{\pi^i_{ss'}}{\alpha_{ss'}(\theta)}.$$

That is, we have that

$$\alpha_{ss'}(\theta) = \frac{\pi^i_{ss'}}{q_{ss'}(\theta)} = \frac{\pi^i_{ss'}}{\beta\tilde{\pi}_{ss'}(\theta)}(A_{1s'}\kappa_{1s}(\theta) + A_{2s'}\kappa_{2s}(\theta)).$$

Buying one unit of the country one technology requires the portfolio of contingent claims $(\bar{A}, \bar{A}, \underline{A}, \underline{A})$ and for the country two technology it requires $(\bar{A}, \underline{A}, \bar{A}, \underline{A})$. Therefore, if there exist $x^i_s(\theta)$ and $y^i_s(\theta)$, the fraction of savings spent on each of the two technologies, such that $\alpha_{s1}(\theta) = x^i_s(\theta)\bar{A} + y^i_s(\theta)\bar{A}$, $\alpha_{s2}(\theta) = x^i_s(\theta)\bar{A} + y^i_s(\theta)\underline{A}$, $\alpha_{s3}(\theta) = x^i_s(\theta)\underline{A} + y^i_s(\theta)\bar{A}$, and $\alpha_{s4}(\theta) = x^i_s(\theta)\underline{A} + y^i_s(\theta)\underline{A}$, then complete markets are not necessary: the two assets suffice. Notice that these are four equations and there are only two unknowns.

We now use the formulas for the κs. After some algebra, the expressions for the αs become

$$\alpha_{s1}(\theta) = \frac{\pi^i_{s1}}{\tilde{\pi}_{s1}(\theta)}\bar{A},$$

$$\alpha_{s2}(\theta) = \frac{\pi^i_{s2}}{\tilde{\pi}_{s2}(\theta) + \tilde{\pi}_{s3}(\theta)}(\bar{A} + \underline{A}),$$

$$\alpha_{s3}(\theta) = \frac{\pi^i_{s3}}{\tilde{\pi}_{s2}(\theta) + \tilde{\pi}_{s3}(\theta)}(\bar{A} + \underline{A}),$$

and

$$\alpha_{s4}(\theta) = \frac{\pi^i_{s4}}{\tilde{\pi}_{s4}(\theta)}\underline{A}.$$

Using the expressions for the αs above and the assumption that the vs are the same and symmetric – in which case agents' probabilities agree in states 1 and 4 – and doing a little more algebra, we see that $\alpha_{s1}(\theta) = \bar{A}$ and $\alpha_{s4}(\theta) = \underline{A}$. This implies that $x^k_s(\theta) + y^k_s(\theta) = 1$, and that one of the four above equations (either the first or the fourth) is redundant. Notice now that $\pi^k_{s2} + \pi^k_{s3} = \pi_{s2} + \pi_{s3}$, which holds generally. Furthermore, we see that

$\tilde{\pi}_{s2}(\theta) + \tilde{\pi}_{s3}(\theta) = \pi_{s2} + \pi_{s3}$; this holds generally, whether or not the outcome is symmetric, interior, or a corner for the vs. Given these facts, adding the two middle equations we obtain

$$\alpha_{s2}(\theta) + \alpha_{s3}(\theta) = \bar{A} + \underline{A},$$

again delivering $x_s^k(\theta) + y_s^k(\theta) = 1$. Thus, one the two middle equations is also redundant: it is indeed possible to remove two equations and solve for the two unknowns x and y. That is, complete markets are not necessary.

Proof of Proposition 1:

Assume, first, that the vs are as conjectured, i.e., constant and equal to v^*. Given that we know that the agent can solve his problem without regard to the distribution of wealth, we can write his dynamic problem as

$$V_s^i(w) = \max_{k_1',k_2'} \log(w - k_1' - k_2') + \beta \sum_{s'} \pi_{ss'}^i V_{s'}^i(k_1' A_{1s} + k_2' A_{2s'}).$$

It is straightforward to verify that the solution must be

$$V_s^i(w) = B_s^i + \frac{1}{1-\beta} \log w,$$

where B_s^i is a constant that in the iid case becomes independent of both i and s but in general depends on these variables.

Finally, recall that the value function of the planner is

$$V_s(\theta, y) = \theta PVU_s^1(\theta y) + (1-\theta) PVU_s^2((1-\theta)y)$$

so that we have, after substituting the present-value utilities from above,

$$V_s(\theta, y) = \theta B_s^1 + (1-\theta) B_s^2 + \frac{1}{1-\beta}(\theta \log \theta + (1-\theta)\log(1-\theta)) + \frac{1}{1-\beta} \log y,$$

from which we learn that

$$C_s(\theta) = \theta B_s^1 + (1-\theta) B_s^2 + \frac{1}{1-\beta}(\theta \log \theta + (1-\theta)\log(1-\theta)).$$

In the special iid case we have

$$C_s(\theta) = C(\theta) = B + \frac{1}{1-\beta}\left(\theta \log \theta + (1-\theta)\log(1-\theta)\right),$$

where

$$B \equiv \frac{\log(1-\beta)}{1-\beta} +$$

$$\frac{\beta}{(1-\beta)^2}(\log \beta + 1/2 \log(\bar{A} + \underline{A}) + (1/4 + v^*) \log \bar{A}(1/2 + 2v^*) + (1/4 - v^*) \log \underline{A}(1/2 - 2v^*)).$$

Turning to the determination of v^1 and v^2, using the first-order conditions (35) and (36), we obtain

$$\frac{1}{1-\beta}(\log y_1(k_1', k_2') - \log y_2(k_1', k_2') + \log y_3(k_1', k_2') - \log y_4(k_1', k_2'))+$$
$$+C_1(\theta_1') - C_2(\theta_2') + C_3(\theta_3') - C_4(\theta_4')+$$
$$+(1-\theta_1')\frac{dC_1(\theta_1')}{d\theta_1'} - (1-\theta_2')\frac{dC_2(\theta_2')}{d\theta_2'} + (1-\theta_3')\frac{dC_3(\theta_3')}{d\theta_3'} - (1-\theta_4')\frac{dC_4(\theta_4')}{d\theta_4'} \geq 0 \tag{37}$$

and

$$\frac{1}{1-\beta}(\log y_1(k_1', k_2') + \log y_2(k_1', k_2') - \log y_3(k_1', k_2') - \log y_4(k_1', k_2'))+$$
$$+C_1(\theta_1') + C_2(\theta_2') - C_3(\theta_3') - C_4(\theta_4')-$$
$$-\theta_1'\frac{dC_1(\theta_1')}{d\theta_1'} - \theta_2'\frac{dC_2(\theta_2')}{d\theta_2'} + \theta_3'\frac{dC_3(\theta_3')}{d\theta_3'} + \theta_4'\frac{dC_4(\theta_4')}{d\theta_4'} \geq 0. \tag{38}$$

These expressions simplify to

$$v^1 \geq \frac{-\pi(\underline{A} - \bar{A})}{\bar{A} + \underline{A}} \equiv -\bar{a} \tag{39}$$

$$v^2 \geq \frac{-\pi(\underline{A} - \bar{A})}{\bar{A} + \underline{A}} \equiv -\bar{a}. \tag{40}$$

These conditions verify our conjecture on the vs: they do not depend on θ or s. Moreover, when $a > \bar{a}$, the optimal vs are interior solutions – (39)-(40)are satisfied with equality – and when $a \leq \bar{a}$, the optimal vs are a corner solutions – (39)-(40) are satisfied with strict inequality.