Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER Journal of Economic Dynamics & Control 27 (2003) 1253–1288
JOURNAL OF Economic Dynamics & Control
www.elsevier.com/locate/econbase

# A two-person dynamic equilibrium under ambiguity

Larry G. Epstein*, Jianjun Miao

*Department of Economics, University of Rochester, Rochester, 14627 New York, USA*

**Abstract**

This paper describes a pure-exchange, continuous-time economy with two heterogeneous agents and complete markets. A novel feature of the economy is that agents perceive some security returns as ambiguous in the sense often attributed to Frank Knight. The equilibrium is described completely in closed-form. After identifying agents as countries, the model is applied to address the consumption home bias and equity home-bias puzzles. © 2002 Elsevier Science B.V. All rights reserved.

*JEL classification:* D81; G11; G12; G15

*Keywords:* Ambiguity; Home bias; Continuous-time; Dynamic general equilibrium; Ellsberg Paradox; Knightian uncertainty; Asset returns; Heterogenous-agent economies

## 1. Introduction

### 1.1. Outline

This paper analyses a pure-exchange, continuous-time economy with two agents and complete markets. A novel feature of the economy is that agents do not view all consumption processes or security returns as purely risky (probabilistic). Rather, they perceive some as ambiguous in the sense often attributed to Frank Knight and illustrated by the Ellsberg Paradox.[1] Agents differ not only in endowments, but also in where they perceive ambiguity and in their aversion to ambiguity. The equilibrium is described completely in closed-form. In particular, closed-form solutions are obtained

* Corresponding author. Tel.: +14627-585-275-4320; fax: +585-256-2309.
*E-mail addresses:* lepn@troi.cc.rochester.edu (L.G. Epstein), mias@troi.cc.rochester.edu (J. Miao).

[1] We deviate from Knight's terminology in using 'uncertainty' as a comprehensive term. In our terminology, every process or event is uncertain, and each is either risky or ambiguous (but not both).


PII: S0165-1889(02)00059-3

for the equilibrium processes describing individual consumption, the riskless rate, the market price of uncertainty, security prices and trading strategies.

In order to accommodate a concern with ambiguity, we model agents as having *recursive multiple-priors utility*. This model of utility is a multi-period extension of Gilboa and Schmeidler (1989) whereby the single probability measure of the standard Savage model is replaced by a set of probabilities or priors. Chen and Epstein (2002) formulate recursive multiple-priors utility in a continuous-time setting and show that it affords the Knightian distinction between risk and ambiguity. The corresponding discrete-time model dates back to Epstein and Wang (1994) and has recently been axiomatized in Epstein and Schneider (2001).

By interpreting the agents in our model as representative consumers in each of two countries, we use it to address two well-known home-bias puzzles. The first is the 'equity home-bias' puzzle whereby individuals invest too little in foreign securities (see Lewis, 1999). Naturally, 'too little' is from the perspective of a model where securities are differentiated only via their risk characteristics. However, if foreign securities are more ambiguous than domestic ones, as in our model, then seemingly biased portfolios may be optimal.

Thus our model can be viewed as a formalization of the suggestion by French and Poterba (1991) that equity home bias may be due to differences in beliefs. They speculate (p. 225) that investors 'may impute extra "risk" to foreign investments because they know less about foreign markets, institutions and firms'. They also cite evidence in Heath and Tversky (1991) that 'households behave as though unfamiliar gambles are riskier than familiar gambles, even when they assign identical probability distributions to the two gambles'. The widespread tendency to invest in the familiar has been documented recently in Huberman (2001), with the home country bias being just one instance; see also Grinblatt and Keloharju (2001). We formalize the difference between the familiar and less familiar as a difference in ambiguity.

There exists survey evidence supporting systematic differences in returns expectations between domestic and foreign investors—investors tend to be more optimistic about domestic securities (see Shiller et al., 1996; Kilka and Weber, 1997; Strong and Xu, 1999). While these surveys do not address ambiguity and they elicit at best a single probability measure from each subject (rather than a set of priors), their findings are consistent with our model if we interpret the elicited measures as including an adjustment for ambiguity. (See the discussion in Section 4.4 for elaboration.) Thus we take these studies as providing further indirect support for our modeling approach. [2]

A second puzzle, 'consumption home bias', is the high correlation between country-specific consumption growth and country-specific output growth (Lewis, pp. 574–575). In the standard model where individuals maximize additive expected utility functions, where utility indices may differ but the probability measure is common, efficiency

[2] Further supporting arguments may be found in Brennan and Cao (1997, p. 1853) and the references cited therein. These arguments are offered to support the hypothesis of information asymmetry between domestic and foreign investors and thus ultimately to motivate a noisy rational expectations model where individuals have common single priors but observe different signals. However, they serve just as well to motivate a model with heterogeneous sets of priors.

implies that every individual's consumption level is a deterministic function of aggregate consumption. We show, however, that the presence of ambiguity moves predictions in the direction indicated by data.

It may not be surprising to some that the assumption that foreign securities are more ambiguous than domestic ones leads to a bias towards domestic securities and to consumption growth that is sensitive to domestic shocks. However, it merits emphasis that these results are achieved as part of a dynamic general equilibrium and along with other (overidentifying) predictions that can be used to evaluate the model. These include: (i) positive correlation between security returns in the two countries and between returns and consumption growth rates within either country; and (ii) the country with the larger instantaneous mean growth rate of consumption has (under a suitable assumption on parameters) the higher instantaneous variance for consumption growth. A final prediction concerns an added piece of the equity home-bias puzzle whereby while foreign equity *holdings* by domestic residents are small, foreign equity *flows* are large and volatile (Lewis, pp. 585–590).

The paper proceeds as follows: Related literature is discussed next. Recursive multiple-priors utility is described in the ensuing section. The economy and equilibrium are described in Section 3. The nature of equilibrium and the model's application to the home-bias puzzles in equities and consumption are discussed in Section 4. Proofs are relegated to an appendix.

### *1.2. Related literature*

While our primary interpretation is in terms of an international setting and the home bias puzzles, we view the paper as contributing more broadly to the literature on dynamic heterogeneous-agent economies. From that perspective, note that while there are numerous papers dealing with existence and characterization of equilibrium in heterogeneous-agent economies, there are fewer that derive qualitative or quantitative predictions in a continuous-time setting.

Dumas (1989) and Wang (1996) consider two-agent economies with complete markets; the former has linear production while Wang considers a pure exchange economy. Both authors assume expected additive utility maximization and permit some differences in utilities.[3] They refer to the heterogeneity in utilities as modeling differences in risk aversion. However, because risk aversion and intertemporal substitutability are confounded in the standard utility specification, the interpretation of their results is problematic. A degree of disentangling is permitted by the recursive utility (or stochastic differential utility) model of Duffie and Epstein (1992). That model is applied in Dumas et al. (2000), where analytical solutions are provided for a specification in which there is heterogeneity in substitutability, but not in risk aversion. Our specification of utility also confounds risk aversion and substitution, but it disentangles these two aspects of

[3] Dumas relies completely on numerical techniques to analyze his model. Wang provides closed-form solutions for equilibrium consumption, interest rates and the market price of risk and a PDE determining security prices, but only by assuming a very special relation between the elasticity parameters of the felicities of the two agents.

preference from ambiguity aversion. Because we focus on ambiguity and heterogeneity in attitudes towards ambiguity, this degree of separation permits the interpretations suggested below.

Another difference is that both cited models admit a representative agent. As a result, implications for aggregate variables and prices are similar to those delivered by representative agent models, given a suitable specification of utility for the representative agent. Further, the standard characterization of efficient consumption allocations (individual consumption is a deterministic function of aggregate consumption) is valid for their models. In contrast, our model does not admit a representative agent and, as noted earlier, delivers a qualitatively different characterization of efficiency.

We show that all equilibrium variables (state prices, securities prices, trading strategies and consumption allocations) are driven by the aggregate endowment process and an endogenously determined 'disagreement process'. A similar structure is present in Detemple and Murthy (1994), Zapatero (1998) and Basak (2000), where individuals are heterogeneous in their (single) priors, and in Basak and Cuoco (1998), where individuals are heterogeneous in their financial investment opportunities. The primary difference between these models and ours is that here the key disagreement process arises endogenously due to ambiguity, while the analogous process in the first three cited papers arises due to differences in Bayesian estimates of the drift of aggregate consumption; and the corresponding process in Basak and Cuoco (1998) arises due to differences in investment opportunity sets, summarized by endogenous individual specific state prices.

Finally, an alternative approach to modeling concern with ambiguity, based on robust control theory, is proposed in Hansen et al. (1999), Anderson et al. (2000) and Hansen and Sargent (2000), for example. The reader is referred to Epstein and Schneider (2001) for a detailed comparison of the robust control and recursive multiple-priors approaches. [4]

## 2. Recursive multiple-priors utility

In this section we outline a special case of recursive multiple-priors for a single individual that will be used later in the equilibrium model. The reader is referred to Chen and Epstein (2002) for further details and for justification for asserted interpretations.

### 2.1. Consumption processes

Time varies over $[0,T]$ and uncertainty is represented by a probability space $(\Omega,\mathscr{F},P)$. Here, unlike in standard models, $P$ represents neither the true objective measure nor the subjective measure used by the individual being described. Its role is to define null events; there will be no disagreement or ambiguity about which events are null. Let $W=(W_t)$ be a standard $d$-dimensional Brownian motion defined on

[4] One difference is that the updating rule underlying the robust control model implies that conditional preference at time $t>0$ depends on what might have happened in other unrealized parts of the event tree.

$(\Omega, \mathscr{F}, P)$ and $\mathbb{F} = \{\mathscr{F}_t\}_{0 \leqslant t \leqslant T}$ the (augmented) filtration that it generates, representing the information available to the individual. Assume $\mathscr{F} = \mathscr{F}_T$ and that $\mathscr{F}_0$ is trivial. All processes in the sequel are progressively measurable with respect to $\mathbb{F}$ and all equalities involving random variables (processes) are understood to hold $\mathrm{d}P$ a.s. ($\mathrm{d}t \otimes \mathrm{d}P$ a.s.).

There is a single consumption good at each instant. Consumption processes lie in $\mathscr{C}$, a subset of the set of positive progressively measurable processes that are also square integrable ($\mathrm{E}_P[\int_0^T c_s^2 \,\mathrm{d}s] < \infty$).

### 2.2. Utility

Define a utility process $(V_t(c))$ for each consumption process $c$ in $\mathscr{C}$ as follows:

$$V_t(c) = \min_{Q \in \mathscr{P}} \mathrm{E}_Q \left[ \int_t^T \mathrm{e}^{-\beta(s-t)} u(c_s) \,\mathrm{d}s \,\middle|\, \mathscr{F}_t \right], \tag{1}$$

where $\mathscr{P}$ is a set of priors on state space $(\Omega, \mathscr{F}_T)$ that is to be specified. Abbreviate $V_0(\cdot)$ by $V(\cdot)$ and refer to it as *recursive multiple-priors utility*.

As for the specification of $\mathscr{P}$, because all priors in $\mathscr{P}$ are taken to be mutually absolutely continuous with respect to $P$, they can be defined via their densities. These, in turn, may be defined by use of *density generators*, which is how we refer to any $\mathbb{R}^d$-valued process $\theta = (\theta_t)$ satisfying

$$\sup_t |\theta_t^i| \leqslant \kappa_i, \quad i = 1, \ldots, d,$$

where $\kappa = (\kappa_1, \ldots, \kappa_d)^\top$ is a vector of non-negative parameters. A density generator $\theta$ is a process that delivers the continuous-time counterpart of the (logarithm of) conditional 1-step-ahead density. Any process of 1-step-ahead densities can be used to construct, by the usual probability calculus, a measure on the state space. In continuous-time, this construction takes the following form: Each $\theta$ generates a $P$-martingale $(z_t^\theta)$ via the equation

$$\mathrm{d}z_t^\theta = -z_t^\theta \theta_t \cdot \mathrm{d}W_t, \quad z_0^\theta = 1 \tag{2}$$

or equivalently,

$$z_t^\theta = \exp\left\{ -\frac{1}{2} \int_0^t \|\theta_s\|^2 \mathrm{d}s - \int_0^t \theta_s \cdot \mathrm{d}W_s \right\}, \quad 0 \leqslant t \leqslant T. \tag{3}$$

Because $1 = z_0^\theta = \mathrm{E}[z_T^\theta]$, $z_T^\theta$ is a $P$-density and thus determines a probability measure $Q^\theta$ on $(\Omega, \mathscr{F}_T)$ via

$$\frac{\mathrm{d}Q^\theta}{\mathrm{d}P} = z_T^\theta. \tag{4}$$

Denote by $\Theta$ the set of all density generators. Then, the set of priors is

$$\mathscr{P} = \{Q^\theta \colon \theta \in \Theta \text{ and } Q^\theta \text{ is defined by (4)}\}. \tag{5}$$

When $\kappa = 0$, $\mathscr{P}$ collapses to the single measure $P$ as in a model without ambiguity. More generally, $\mathscr{P}$ is a non-singleton that expands as any component of $\kappa$ is increased. The natural interpretation is that ambiguity increases with $\kappa$, or alternatively, that ambiguity aversion increases with $\kappa$.

By the Girsanov Theorem $(W_t + \int_0^t \theta_s \, ds)$, is a Brownian motion relative to $Q^\theta$. Thus the multiplicity of measures in $\mathscr{P}$ can be interpreted as modeling ambiguity about the drift of the driving process. The drift may be zero ($\theta = 0$); but another possibility according to $\mathscr{P}$ is that the 'true' measure is such that $(W_t^1 + \kappa_1 t, W_t^2 + \kappa_2 t)$ is a Brownian motion, corresponding to $\theta_t = (\kappa_1, \kappa_2)^\top$ for all $t$.

An important feature of this specification of $\mathscr{P}$ is that it delivers recursivity of utility (and hence also dynamic consistency) in the sense that

$$V_t = \min_{Q \in \mathscr{P}} \mathrm{E}_Q \left[ \int_t^\tau \mathrm{e}^{-\beta(s-t)} u(c_s) \, \mathrm{d}s + \mathrm{e}^{-\beta(\tau - t)} V_\tau \middle| \mathscr{F}_t \right], \quad 0 \leqslant t < \tau \leqslant T.$$

Recursivity follows from the fact that the utility process solves (uniquely) a backward stochastic differential equation (BSDE), that is, for each $c$, there exists a unique process $(V_t(c), \sigma_t(c))$ satisfying, for $0 \leqslant t \leqslant T$,

$$\mathrm{d}V_t = [-u(c_t) + \beta V_t + \max_{\theta \in \Theta} \theta_t \cdot \sigma_t] \, \mathrm{d}t + \sigma_t \cdot \mathrm{d}W_t, \quad V_T = 0 \tag{6}$$

or equivalently, [5]

$$\mathrm{d}V_t = [-u(c_t) + \beta V_t + \kappa \cdot |\sigma_t|] \, \mathrm{d}t + \sigma_t \cdot \mathrm{d}W_t, \quad V_T = 0. \tag{7}$$

Note that the volatility of utility $\sigma_t(c)$ is determined as part of the solution to the BSDE; it plays a key role in the sequel.

Additional conditions deliver a range of natural properties for utility. For example, if $u$ is increasing and (strictly) concave, then so is each $V_t(\cdot)$.

Finally, a natural question concerns learning. Constancy of $\kappa$ reflects the fact that we do not model learning. Rather we focus on a state where individuals have learned all they can about their environment and yet ambiguity persists. See Section 5 for further discussion of learning.

### 2.3. Supergradients

The supergradients of utility are important for characterizing security prices and equilibrium more generally. A *supergradient* for $V$ at the consumption process $c$ is a process $(\pi_t)$ satisfying

$$V(c') - V(c) \leqslant \mathrm{E}_P \left[ \int_0^T \pi_t (c_t' - c_t) \, \mathrm{d}t \right], \text{ for all } c' \text{ in } \mathscr{C}. \tag{8}$$

Because $V$ is a lower envelope of expected additive utility functions (1), an envelope theorem and the well-known structure of supergradients of the standard utility function

[5] For any $d$-dimensional vector $x$, $|x|$ denotes the vector with $i$th component $|x_i|$.

immediately deliver supergradients for $V$. In particular, any process $(\pi_t(c))$ of the following form is a supergradient for $V$ at $c$:

$$\pi_t(c) = \mathrm{e}^{-\beta t}\, u'(c_t)\, z_t^{\theta^*}, \quad \theta^* \in \Theta_c, \tag{9}$$

where $\theta^*$ solves (for every $t$) the instantaneous maximization appearing in (6), that is,[6]

$$\theta^* \in \Theta_c \equiv \{(\theta_t)\colon \theta_t = \kappa \otimes \mathrm{sgn}(\sigma_t) \text{ all } t\}. \tag{10}$$

To express these supergradients in another convenient way, define $\mathscr{P}_c$ to be the set of measures in $\mathscr{P}$ such that

$$V(c) = \mathrm{E}_{Q^*}\left[\int_0^T \mathrm{e}^{-\beta t}\, u(c_t)\, \mathrm{d}t\right]. \tag{11}$$

Note that $Q^* \in \mathscr{P}_c$ *if and only if* $Q^* = Q^{\theta^*}$ for some $\theta^*$ in $\Theta_c$. Conclude, using (4), that

$$\pi_t(c) = \mathrm{e}^{-\beta t}\, u'(c_t)\, \left.\frac{\mathrm{d}Q^*}{\mathrm{d}P}\right|_{\mathscr{F}_t}, \quad Q^* \in \mathscr{P}_c \tag{12}$$

is a supergradient at $c$. Though there may be other supergradients at some processes $c$, and these would lead to different equilibria below, we restrict attention to equilibria corresponding to the above supergradients.

### 2.4. Example

We can compute utility explicitly for consumption processes $c$ of the form

$$\mathrm{d}c_t/c_t = \mu^c\, \mathrm{d}t + s^c \cdot \mathrm{d}W_t,$$

where $\mu^c$ and $s^c$ are constant. Suppose that $u(c_t) = (c_t^\alpha - 1)/\alpha$, for $\alpha \leqslant 1$, where $\alpha = 0$ corresponds to the log specification. Then

$$V_t(c) = A_t\,(c_t^\alpha - 1)/\alpha - \frac{1}{\beta\rho}\,\frac{(\rho-\beta)}{\alpha} + \mathrm{e}^{\beta(t-T)}\,\frac{1}{\beta\rho}\,\frac{[\rho - \beta \mathrm{e}^{(\rho-\beta)(t-T)}]}{\alpha},$$

where

$$A_t = \rho^{-1}[1 - \exp(\rho(t-T))] \quad \text{and}$$
$$(\rho-\beta)/\alpha = -(\mu^c - (1-\alpha)s^c \cdot s^c/2 - \kappa \cdot |s^c|).$$

The associated volatility is

$$\sigma_t = A_t c_t^\alpha s^c.$$

[6] For any $d$-dimensional vector $x$, $\mathrm{sgn}(x)$ is the $d$-dimensional vector with $i$th component equal to $\mathrm{sgn}(x_i) = |x_i|/x_i$ if $x_i \neq 0$ and $= 0$ if $x_i = 0$. For any $y \in \mathbb{R}^d$, $y \otimes \mathrm{sgn}(x)$ denotes the vector in $\mathbb{R}^d$ with $i$th component $y_i\, \mathrm{sgn}(x_i)$.

Evidently the utility of the given consumption process depends on the initial level of consumption and on the adjusted mean growth rate $\mu^c - (1-\alpha)s^c \cdot s^c/2 - \kappa \cdot |s^c|$, where the adjustment is both for risk (via the second term) and ambiguity (via the third term).

From (9) and (3), a supergradient at $c$ is given by

$$\pi_t(c) = \mathrm{e}^{-\beta t} c_t^{\alpha-1} \exp\left\{ -\frac{1}{2}\|\kappa\|^2 t - \kappa \otimes \mathrm{sgn}(s^c) \cdot W_t \right\}.$$

## 3. Two-person equilibrium

### 3.1. The economy

*Information structure and preferences.* The primitive probability space is $(\Omega, \mathscr{F}_T, P)$. Suppose the associated Brownian motion is two-dimensional, $W_t = (W_t^1, W_t^2)$. We assume a population of two individuals. They have the common information structure represented by the augmented Brownian filtration $\mathbb{F} = \{\mathscr{F}_t\}_{0 \leqslant t \leqslant T}$. In particular, the differing beliefs of the two individuals described below are *not* due to asymmetric information; they reflect differing prior views about the environment. [7] Note that the assumption that consumer $i$ observes realizations of both $W^i$ and $W^j$ does not contradict the intuition described in the introduction whereby $i$ is less familiar with securities that are driven primarily by $W^j$ than with those driven by $W^i$. For example, Canadian sports fans have access to scores and satellite telecasts of soccer matches. However, typically they do not pay much attention to them with the result that many feel much more familiar with hockey and prefer to bet on hockey rather than on soccer matches.

There is a single perishable good (the numeraire), leading to the consumption set $\mathscr{C}$. Each individual has a recursive multiple-priors utility function on $\mathscr{C}$ and for $i = 1, 2, c^i$ and $(V_t^i(\cdot))$ denote $i$'s consumption and utility processes. Each utility function has the form (1) with common rate of time preference $\beta$ and felicity function [8]

$$u(c_t) = \log c_t. \tag{13}$$

Preferences differ, however, because individuals have different sets of priors, that is, different ambiguity parameters $\kappa^i$. We assume that

$$\kappa^1 = (0, \kappa_1)^\top \quad \text{and} \quad \kappa^2 = (\kappa_2, 0)^\top. \tag{14}$$

The interpretation is that $i$ is more familiar with 'her own' component process $W^i$ than with the other component $W^j$. [9] In extreme form this leads to no ambiguity for $i$

[7] See Morris (1995) for a discussion, in a Bayesian setting, of the merits of differing priors, rather than asymmetric information, as a basis for differing beliefs.

[8] From Chen–Epstein, $V$ is well-defined if $\mathrm{E}[\int_0^T [\log c_t]^2 \, \mathrm{d}t] < \infty$ for all $c$ in $\mathscr{C}$. This square integrability condition determines the domain $\mathscr{C}$. Because the aggregate endowment process specified below lies in $\mathscr{C}$, we can safely proceed in the equilibrium analysis under the assumption that for all intents and purposes the foundations for utility provided in Chen–Epstein (Theorem 2.3) apply to our model.

[9] We write the second component of $\kappa^1$ as $\kappa_1$ to indicate that it is the ambiguity parameter for individual 1, though it relates to ambiguity about $W^2$. When referring to individuals $i$ and $j$, it is understood that $i \neq j$.

about $W^i$, though $W^j$ is ambiguous for her. A concrete setting where this specification seems natural is where $i$ is a representative consumer in country $i$ in which $W^i$ is the driving state process. Henceforth we adopt this interpretation and refer to individuals alternatively as countries.

*Securities markets.* Investment opportunities are represented by a locally riskless bond earning the instantaneous interest rate $r$ and by two securities, with respective (non-negative) dividend streams $Y_t^1$ and $Y_t^2$. Thus cumulative dividends are described by

$$D_t^\top = \begin{cases} (0, \int_0^t Y_s^1 \, \mathrm{d}s, \int_0^t Y_s^2 \, \mathrm{d}s) & \text{if } 0 \leqslant t < T, \\ (1, \int_0^T Y_s^1 \, \mathrm{d}s, \int_0^T Y_s^2 \, \mathrm{d}s) & \text{if } t = T. \end{cases} \tag{15}$$

Because some of our results do not require that we specify further the nature of the individual processes $(Y_t^i)$, we defer further assumptions until they are needed (Section 4.4). In anticipation of the more detailed specification, the reader might think of $(Y_t^i)$ being driven 'primarily' by the state process $(W_t^i)$ associated with country $i$.

At each $t$, securities are traded in a competitive market at prices $S_t = (S_t^0, S_t^1, S_t^2)^\top$ denominated in units of consumption. In equilibrium, $S$ is an Ito process so that the gain process $S + D$ is also an Ito process,

$$\mathrm{d}(S_t + D_t) = \mu_t^G \, \mathrm{d}t + s_t^G \, \mathrm{d}W_t,$$

where $\mu_t^G$ is $\mathbb{R}^3$-valued and $s_t^G$ is $\mathbb{R}^{3\times 2}$-valued. A *trading strategy* is an $\mathbb{R}^3$-valued process $\gamma = (\gamma_t)$, satisfying

$$\int_0^T |\gamma_t \cdot \mu_t^G| \mathrm{d}t + \int_0^T \gamma_t^\top s_t^G (s_t^G)^\top \gamma_t \, \mathrm{d}t < \infty.$$

This condition insures that the stochastic integral $\int \gamma_t \cdot \mathrm{d}(S_t + D_t)$ is well defined. Note that $\gamma_t = (\gamma_{0,t}, \gamma_{1,t}, \gamma_{2,t})^\top$, where $\gamma_{n,t}$ represents the number of shares of the bond $(n=0)$ and securities 1 and 2. The set of all trading strategies is denoted $\Gamma$.

*Endowments and objectives.* The aggregate endowment or output process $(Y_t)$ is assumed to follow the geometric law

$$\mathrm{d}Y_t/Y_t = \mu^Y \, \mathrm{d}t + s^Y \cdot \mathrm{d}W_t, \tag{16}$$

where $\mu^Y$ and $s^Y = (s_1^Y, s_2^Y)^\top$ are constants. Aggregate dividends do not exhaust output. Rather we assume that

$$Y_t = Y_t^1 + Y_t^2 + \Phi_t,$$

where $(\Phi_t)$ is the part of aggregate output that is not traded.[10]

[10] The presence of a non-traded endowment complicates the model somewhat. We include it not for greater generality but primarily because, as explained in Section 4.4, it is unavoidable given the intuition we are trying to capture with our model and given our desire to obtain closed-form solutions.

Each country owns $\frac{1}{2}$ of the non-traded endowment ($\Phi_t$). Initial share holdings are given by

$$\gamma^1 = (0, \gamma^1_{1,0}, \gamma^1_{2,0})^\top \quad \text{and} \quad \gamma^2 = (0, 1 - \gamma^1_{1,0}, 1 - \gamma^1_{2,0})^\top$$

the assumption of zero initial bond holdings is made purely for simplicity.

Given a security price process $S$, individual $i$ solves

$$\sup_{(e^i,\gamma^i)\in\mathscr{C}\times\Gamma} V^i(e^i) \quad \text{subject to} \tag{17}$$

$$S_t \cdot \gamma^i_t = S_0 \cdot \gamma^i_0 + \int_0^t \gamma^i_s \cdot \mathrm{d}(S_s + D_s) - \int_0^t (e^i_s - \tfrac{1}{2}\Phi_s)\,\mathrm{d}s, \quad t \in [0, T] \tag{18}$$

and a credit constraint that is specified in Appendix A (see (A.15)).

The preceding defines the economy

$$\mathscr{E} = ((\Omega, \mathscr{F}, \mathbb{F}, P), (W_t), (u, \beta, \kappa^i, \gamma^i_0)_{i=1,2},\ (D_t), (Y_t)). \tag{19}$$

### 3.2. Equilibrium

We define two notions of equilibrium. An *Arrow–Debreu equilibrium* for the economy $\mathscr{E}$ is a tuple $((c^i)_{i=1,2}, p)$ where $p$ is a non-negative real-valued (state) price process, $c^i$ solves (for $i = 1, 2$)

$$\sup_{e^i\in\mathscr{C}} V^i(e^i) \quad \text{subject to}$$

$$\mathrm{E}\left[\int_0^T p_s(e^i_s - \tfrac{1}{2}\,\Phi_s)\,\mathrm{d}s\right] \leqslant \gamma^i_{i,0}\mathrm{E}\left[\int_0^T p_s Y^i_s\,\mathrm{d}s\right] + \gamma^i_{j,0}\mathrm{E}\left[\int_0^T p_s Y^j_s\,\mathrm{d}s\right] \tag{20}$$

and where markets for contingent consumption clear, that is,

$$c^1 + c^2 = Y.$$

A *Radner equilibrium* for the economy $\mathscr{E}$ is a tuple $((c^i, \gamma^i)_{i=1,2}, S)$ such that given the security price process $S, (c^i, \gamma^i)$ solves problem (17) for $i = 1, 2$ and markets clear:

$$\gamma^1_t + \gamma^2_t = (0, 1, 1)^\top \text{ for all } t \text{ and } c^1 + c^2 = Y. \tag{21}$$

According to this definition, individuals make consumption and portfolio plans for the entire horizon at $t = 0$. Recursivity of utility ensures that plans will be carried out.

The riskless rate and the bond price are related by

$$r_t\,\mathrm{d}t = \mathrm{d}S^0_t / S^0_t.$$

Let the returns process for risky securities be

$$\mathrm{d}R^n_t = \frac{\mathrm{d}S^n_t + Y^n_t\,\mathrm{d}t}{S^n_t}, \quad n = 1, 2 \tag{22}$$

and write $R_t = (R_t^1, R_t^2)^\top$,

$$\mathrm{d}R_t = b_t \mathrm{d}t + s_t \, \mathrm{d}W_t, \tag{23}$$

where $b_t$ is $\mathbb{R}^2$-valued and each $s_t$ is a $2 \times 2$ matrix. The equilibrium has complete markets (in the usual sense) if $s_t$ is invertible. In that case, the state price process $(p_t)$ satisfies

$$-\,\mathrm{d}p_t/p_t = r_t \mathrm{d}t + \eta_t \cdot \mathrm{d}W_t, \quad p_0 = 1, \tag{24}$$

where $\eta_t \equiv s_t^{-1}(b_t - r_t \mathbf{1})$. To permit later use of the martingale approach, assume that $r_t$ and $\eta_t$ are uniformly bounded.[11] Typically, $\eta_t$ is referred to as the market price of risk. We refer to it as the *market price of uncertainty* to reflect the fact that $(W_t)$ and hence also security returns, embody both risk and ambiguity.

We establish existence of a complete markets equilibrium and characterize it 'almost' completely in closed form, under the assumption that

$$0 \leqslant \kappa_1 < s_2^Y \quad \text{and} \quad 0 \leqslant \kappa_2 < s_1^Y. \tag{25}$$

Because these restrictions limit the ambiguity parameters to be 'small', they seem uncontentious. The derivation of even a limited analytical solution may seem surprising, (it was to us), because supergradients for recursive multiple-priors utility depend on the volatility of the utility process (see (10) and (9)), about which one might expect typically to know very little. However, under (25), we show that the density generators that support equilibrium consumption processes are

$$\theta_t^{*1} = (0, \kappa_1)^\top \quad \text{and} \quad \theta_t^{*2} = (\kappa_2, 0)^\top \quad \text{for all } t, \tag{26}$$

which explicit expressions are the key to the availability of an analytical solution.

The description of equilibrium makes use of the process

$$\varsigma_t \equiv \exp\left\{ \frac{1}{2}((\kappa_1)^2 - (\kappa_2)^2)t + \kappa_1 W_t^2 - \kappa_2 W_t^1 \right\} \tag{27}$$

called later the *disagreement process*, and $\lambda$, the relative Pareto utility weight for country 2, which is given by (26), (3) and

$$\lambda = \frac{\mathrm{E}[\int_0^T \mathrm{e}^{-\beta t} z_t^{\theta^{*1}} (1 - \gamma_{1,0}^1 Y_t^1/Y_t - \gamma_{2,0}^1 Y_t^2/Y_t - \frac{1}{2}\,\Phi_t/Y_t)\,\mathrm{d}t]}{\mathrm{E}[\int_0^T \mathrm{e}^{-\beta t} z_t^{\theta^{*2}} (\gamma_{1,0}^1 Y_t^1/Y_t + \gamma_{2,0}^1 Y_t^2/Y_t + \frac{1}{2}\Phi_t/Y_t)\,\mathrm{d}t]}. \tag{28}$$

It is useful also to introduce the (shadow) price of the non-traded endowment given by

$$\bar{S}_t = \frac{1}{p_t} \mathrm{E}\left[ \int_t^T p_s \Phi_s \, \mathrm{d}s \mid \mathscr{F}_t \right].$$

Write

$$\mathrm{d}\bar{S}_t = \bar{\mu}_t \mathrm{d}t + \bar{s}_t \cdot \mathrm{d}W_t. \tag{29}$$

[11] As shown in Theorem 1, they are uniformly bounded in equilibrium.

Define the (total) wealth process for $i$ by

$$\bar{X}_t^i = S_t \cdot \gamma_t^i + \tfrac{1}{2}\,\bar{S}_t. \tag{30}$$

**Theorem 1.** *Assume* (25) *and define* $\varsigma_t$ *and* $\lambda$ *as above.*

(i) *There exists an Arrow–Debreu equilibrium* $((c^i)_{i=1,2}, p)$ *where*

$$c_t^1 = \frac{1}{1+\lambda\varsigma_t}\,Y_t, \quad c_t^2 = \frac{\lambda\varsigma_t}{1+\lambda\varsigma_t}\,Y_t \tag{31}$$

*and*

$$p_t = \frac{\mathrm{e}^{-\beta t} z_t^{\theta^{*i}}}{(c_t^i/c_0^i)}, \quad i = 1 \text{ or } 2. \tag{32}$$

*Here* $\theta^{*i}$ *and* $z_t^{\theta^* i}$ *are defined by* (26) *and* (3).

(ii) *Define prices of the two risky securities by*

$$S_t^n = \frac{1}{p_t}\,\mathrm{E}\left[\int_t^T p_\tau Y_\tau^n \,\mathrm{d}\tau \,\middle|\, \mathscr{F}_t\right], \quad n = 1,2 \tag{33}$$

*and define the bond price by*

$$S_t^0 = \frac{1}{p_t}\,\mathrm{E}[p_T|\mathscr{F}_t]. \tag{34}$$

*Let* $s_t$ *be the returns volatility matrix as in* (23). *If* $s_t$ *is invertible*, *then the Arrow–Debreu equilibrium* $((c^i)_{i=1,2}, p)$ *can be implemented by the Radner equilibrium* $((c^i,\gamma^i)_{i=1,2}, S)$ *described as follows*:

(a) *The interest rate* $r_t$ *satisfies*

$$r_t = \beta + \mu^Y - s^Y \cdot s^Y - \left[\kappa_2 s_1^Y - \frac{c_t^1}{Y_t}(\kappa_2 s_1^Y - \kappa_1 s_2^Y)\right] \tag{35}$$

*the market price of uncertainty* $\eta_t$ *is*

$$\eta_t = s^Y + \begin{bmatrix} \kappa_2 c_t^2/Y_t \\ \kappa_1 c_t^1/Y_t \end{bmatrix} \tag{36}$$

*and the state price process* $(p_t)$ *satisfies* (24).

(b) *Excess returns for the two risky assets are*

$$b_t^1 - r_t = s_t^1 \cdot s^Y + \left(\kappa_2\,\frac{c_t^2}{Y_t}\,s_t^{11} + \kappa_1\,\frac{c_t^1}{Y_t}\,s_t^{12}\right),$$

$$b_t^2 - r_t = s_t^2 \cdot s^Y + \left(\kappa_1\,\frac{c_t^1}{Y_t}\,s_t^{22} + \kappa_2\,\frac{c_t^2}{Y_t}\,s_t^{21}\right), \tag{37}$$

*where* $s_t^n$ *is the* $n$th *row of* $s_t$, *and* $s_t^{nm}$ *is the* $(n,m)$ *element of* $s_t$.

(c) *Wealth processes satisfy*

$$\bar{X}_t^i = \beta^{-1}(1 - e^{-\beta(T-t)})c_t^i. \tag{38}$$

(d) *Trading strategies for the risky securities are given by*[12]

$$\begin{bmatrix} S_t^1\gamma_{1,t}^1 \\ S_t^2\gamma_{2,t}^1 \end{bmatrix} = \bar{X}_t^1(s_t s_t^\top)^{-1}(b_t - r_t\mathbf{1}) - \tfrac{1}{2}(s_t^\top)^{-1}\bar{s}_t + \bar{X}_t^1 \frac{\kappa_1}{\det(s_t)}\begin{bmatrix} s_t^{21} \\ -s_t^{11} \end{bmatrix}$$

$$\begin{bmatrix} S_t^1\gamma_{1,t}^2 \\ S_t^2\gamma_{2,t}^2 \end{bmatrix} = \bar{X}_t^2(s_t s_t^\top)^{-1}(b_t - r_t\mathbf{1}) - \tfrac{1}{2}(s_t^\top)^{-1}\bar{s}_t + \bar{X}_t^2 \frac{\kappa_2}{\det(s_t)}\begin{bmatrix} -s_t^{22} \\ s_t^{12} \end{bmatrix}, \tag{39}$$

*where* $\bar{s}_t$, *the volatility defined in* (29), *is given by*

$$\bar{s}_t = \frac{1 - e^{-\beta(T-t)}}{\beta} Y_t s^Y - (s_t^1)^\top S_t^1 - (s_t^2)^\top S_t^2. \tag{40}$$

Given security prices as in (33), Eqs. (22) and (23) determine the equilibrium drift and volatility of returns $b_t$ and $s_t$. Thus the characterization of equilibrium provided by the theorem is complete, apart from the gap regarding the invertibility of $s_t$. For the particular specification of the individual processes $(Y_t^i)$ described below (Section 4.4), we derive explicit solutions for $S_t$ and confirm the invertibility of $s_t$, providing thereby a complete characterization of equilibrium.

Finally, we comment briefly on the robustness of the theorem to relaxation of our assumptions that each country: (i) has log felicity; and (ii) has no ambiguity about its own Brownian motion. If we replace log felicity by a common power felicity, then the induced hedging demands make it difficult to describe trading strategies in closed-form.[13]

However, (ii) can be relaxed. For example, generalize (14) and (25) to

$$\kappa^1 = (\kappa_1^1, \kappa_2^1)^\top, \quad \kappa^2 = (\kappa_1^2, \kappa_2^2)^\top$$

and

$$0 < \kappa_1^2 - \kappa_1^1 < s_1^Y, \quad 0 < \kappa_2^1 - \kappa_2^2 < s_2^Y. \tag{41}$$

Here $\kappa_j^i$ measures $i$'s ambiguity about the shocks in country $j$. Thus (41) imposes that the driving process in country $i$ is more ambiguous for the foreigner than for the resident in $i$, and also that differences in ambiguity parameters are not too large. Then one can show, proceeding as in the proof of the stated theorem, that an equilibrium exists and that it admits a similar characterization in terms of a disagreement process paralleling (27). The new process takes the form

$$\varsigma_t \equiv \exp\left\{\frac{1}{2}((\kappa_1^2)^2 + (\kappa_2^2)^2 - (\kappa_1^1)^2 - (\kappa_2^1)^2)t - (\kappa_1^2 - \kappa_1^1)W_t^1 - (\kappa_2^2 - \kappa_2^1)W_t^2\right\},$$

[12] Trading strategies for the bond are described in the proof of the theorem.

[13] We do not know if Malliavin calculus can be applied to deliver closed-form solutions.

corresponding to the equilibrium density generators

$$\theta_t^{*1} = (\kappa_1^1, \kappa_2^1)^\top, \quad \theta_t^{*2} = (\kappa_1^2, \kappa_2^2)^\top.$$

It is evident that the qualitative nature of the disagreement process is unchanged from our base model and hence, that qualitative properties of the equilibrium are robust.

## 4. The nature of equilibrium

Consider briefly equilibrium in the benchmark model $\kappa_1 = \kappa_2 = 0$. Because there is no ambiguity and (representative individuals in) both countries use the single and common probability measure $P$, equilibrium has the familiar form. For example, each country consumes a fixed proportion of the world output, implying equal growth rates of consumption. The riskless rate and market price of uncertainty are constant and depend in the familiar fashion on properties of the aggregate endowment process and excess returns for the risky securities are determined as in the representative-agent C-CAPM. Finally, each country's (value) portfolio of risky securities consists of two components, the mean-variance-efficient portfolio $\bar{X}_t^i(s_t s_t^\top)^{-1}(b_t - r_t\mathbf{1})$ and a component $-\frac{1}{2}(s_t^\top)^{-1}\bar{s}_t$ that hedges the risk due to the non-traded endowment.

Next we discuss equilibrium in the presence of ambiguity. In the sequel, references to 'mean excess returns', covariances or other moments of distributions induced by stochastic processes are intended relative to the measure $P$. The reader may wish to think of $P$ as being the true measure.

### *4.1. Which country faces more ambiguity?*

Naturally, our interpretation of the equilibrium described in the theorem centers on the presence of ambiguity. One aspect of the presence of ambiguity is the question "which country faces more ambiguity?" We will see that the answer influences several properties of equilibrium. [14]

Our answer is that *country 2 faces more ambiguity than does country* 1 if

$$\kappa_2 s_1^Y - \kappa_1 s_2^Y > 0. \tag{42}$$

We use the aggregate output process $(Y_t)$ to measure ambiguity. Thus an informal justification for the suggested interpretation of (42) is that it is true if $s_1^Y$ is sufficiently large relative to $s_2^Y$ and in that case, aggregate output is driven mostly by $W^1$, which is unambiguous for country 1 but ambiguous for 2.

For a more formal argument, let $(Y_t^*)$ be the 'reference' process satisfying

$$\mathrm{d}Y_t^*/Y_t^* = \mu^*\mathrm{d}t + (s_1^Y + s_2^Y)\mathrm{d}W_t^1, \;\; Y_0^* = Y_0,$$

[14] Put another way, the sign of $\kappa_2 s_1^Y - \kappa_1 s_2^Y$ affects the qualitative properties of equilibrium and our goal here is to suggest an interpretation for this sign.

where the drift $\mu^*$ is chosen so that country 1 is indifferent between $(Y_t^*)$ and the aggregate output process $(Y_t)$. Because $(Y_t^*)$ involves no ambiguity for 1, it serves as a 'risky equivalent' process for $(Y_t)$ from the perspective of country 1 and $\mu^Y - \mu^*$ measures the 'cost' of ambiguity in $(Y_t)$ for country 1. Because both processes are geometric, we can apply the illustrative calculation in Section 2.4 to compute (from the hypothesis that the two processes imply the same utility at time 0) that

$$\mu^Y - s^Y \cdot s^Y/2 - \kappa_1 s_2^Y = \mu^* - (s_1^Y + s_2^Y)^2/2.$$

Similarly for country 2 use the 'risky equivalent' process $(Y_t^{**})$, where

$$\mathrm{d}Y_t^{**}/Y_t^{**} = \mu^{**}\mathrm{d}t + (s_1^Y + s_2^Y)\,\mathrm{d}W_t^2, \quad Y_0^{**} = Y_0$$

and calculate that

$$\mu^Y - s^Y \cdot s^Y/2 - \kappa_2 s_1^Y = \mu^{**} - (s_1^Y + s_2^Y)^2/2.$$

Conclude that (42) is equivalent to $\mu^* > \mu^{**}$. Because the two reference risky processes $(Y_t^*)$ and $(Y_t^{**})$ involve the same risk for both countries (the measure $P$ applies in both cases and the identical probability distributions are induced), we are justified in interpreting $\mu^* > \mu^{**}$, or equivalently,

$$\mu^Y - \mu^* < \mu^Y - \mu^{**},$$

as expressing that the cost of ambiguity for 1 is smaller than that for 2.

### 4.2. Consumption

Eq. (31) makes explicit the implications of ambiguity for the equilibrium (or efficient) allocation of consumption. Individual consumption levels depend not only on the aggregate endowment but also on country-specific shocks $W_t^1$ and $W_t^2$. This dependence is readily understood as we now show.

Let $\theta^{*1}$ and $\theta^{*2}$ be the density generators given in (26) and $Q^{\theta^{*1}}$ and $Q^{\theta^{*2}}$ the corresponding measures as in (4). Given (11), it is natural to refer to $Q^{\theta^{*1}}$ and $Q^{\theta^{*2}}$ as *ambiguity-adjusted probabilistic beliefs* of the two individuals. We noted above that $(W_t^1, W_t^2 + \kappa_1 t)$ is a Brownian motion under $Q^{\theta^{*1}}$. In particular, under $Q^{\theta^{*1}}$ the unconditional distribution for $W_t^2$ is $\mathrm{N}(-\kappa_1 t, t)$, while it is $\mathrm{N}(0, t)$ under $P$. This leftward shift as a result of country 1's ambiguity about $W^2$ is intuitive. Roughly, the assumption that aggregate output covaries with $W^2$ ($s_2^Y > 0$) implies that higher values of $W_t^2$ are better for both countries and particularly for 1.[15] As a multiple-priors decision-maker, country 1 evaluates prospects through the worst-case scenario. Thus she is led to attach relatively less weight (than under $P$) to good realizations of $W_t^2$, which explains the

[15] In precise terms, the claim is that (under the parameter assumptions in the theorem) the equilibrium utility process for country 1 has positive volatility with respect to $W_t^2$. This is equivalent to (26), which is the key to the theorem.

leftward shift and the related fact that the restricted density $\mathrm{d}Q^{\theta^{*1}}/\mathrm{d}P|_{\mathscr{F}_t}$ is decreasing in $W_t^2$.

The ambiguity-adjusted probabilities affect consumption because, as in (12), $i$'s marginal rate of substitution between time 0 and time $t$ consumption is

$$MRS_{0,t}^i = \frac{\mathrm{e}^{-\beta t}u'(c_t^i)}{u'(c_0^i)} \left.\frac{\mathrm{d}Q^{\theta^{*i}}}{\mathrm{d}P}\right|_{\mathscr{F}_t}$$

with the density term acting like a preference shock that redistributes weight away from states where $W_t^j$ is large. Thus, if we define $(\varsigma_t)$ as in (27), or equivalently by [16]

$$\varsigma_t = \left.\frac{\mathrm{d}Q^{\theta^{*2}}/\mathrm{d}P}{\mathrm{d}Q^{\theta^{*1}}/\mathrm{d}P}\right|_{\mathscr{F}_t}$$

then (i) $\varsigma_t$ is increasing in $W_t^2$ and decreasing in $W_t^1$; and (ii) a larger value for $\varsigma_t$ increases $MRS_{0,t}^2$ relative to $MRS_{0,t}^1$, inducing a shift in time $t$ consumption towards individual 2. Given the log utility specification, the latter effect takes the precise form

$$\varsigma_t = \frac{c_t^2/c_0^2}{c_t^1/c_0^1},$$

that is, $\varsigma_t$ equals the relative average consumption growth rates of the two countries. Because $\varsigma_t$ measures the difference in ambiguity-adjusted beliefs (restricted to $\mathscr{F}_t$), we refer to $(\varsigma_t)$ as the *disagreement process* (of 2 relative to 1). [17]

The above simple intuition explains also other non-standard features of equilibrium consumption processes. First, the presence of disagreement leads to the ‘crossing’ of individual consumption paths in some realizations; that is, even if $\lambda < 1$ and thus $c_0^1 > c_0^2$, country 2 consumes more than country 1 at times and states where $\varsigma_t$ is sufficiently large. Assuming that $P$ is the true measure, then, conditional on $\mathscr{F}_\tau$, the (log) consumption ratio $\log(c_t^2/c_t^1)$ is normally distributed with mean $[\log(c_\tau^2/c_\tau^1) + \frac{1}{2}((\kappa_1)^2 - (\kappa_2)^2)(t-\tau)]$ and variance $[((\kappa_1)^2 + (\kappa_2)^2)(t-\tau)]$; it is a $P$-martingale if $\kappa_1 = \kappa_2$. [18]

Unlike the case in the standard model, the consumption share $c_t^i/Y_t$ of each country is stochastic. The behavior of these shares is readily deduced from Ito's Lemma—$\mathrm{d}(c_t^1/Y_t)$ is positively correlated with $\mathrm{d}Y_t$ if and only if 1 faces less ambiguity than does 2 in the sense of (42). This is true, for example, if $s_1^Y$ is sufficiently larger than $s_2^Y$. Then aggregate output is driven mostly by $W^1$, which situation is favorable for country 1 because her ambiguity concerns only the other process $W^2$ and country 1's consumption increases more than proportionately with total output. The mechanics underlying this effect stem from the following relation between output growth and the

[16] Equivalence follows from (3) and (4).

[17] Recall the discussion in the introduction of the role of such processes both here and in related literature.

[18] Because each individual consumes a deterministic and common fraction of wealth in equilibrium (see (38)), the log wealth ratio has similar properties.

disagreement process:

$$cov_t\left(\frac{\mathrm{d}\varsigma_t}{\varsigma_t}, \frac{\mathrm{d}Y_t}{Y_t}\right) = -\kappa_2 s_1^Y + \kappa_1 s_2^Y < 0.$$

In light of the connection described above between $(\varsigma_t)$ and marginal rates of substitution, the instantaneous change in the ratio $MRS_{0,t}^1/MRS_{0,t}^2$ covaries with $\mathrm{d}Y_t$; thus $\mathrm{d}Y_t$ being positive leads to an increase in the share of consumption going to country 1.

Turn to instantaneous mean growth rates. Ito's Lemma applied to (31) shows that $\mathrm{d}c_t^i/c_t^i, i = 1, 2$, have drifts

$$\mu_t^{c,1} = \mu^Y + \frac{c_t^2}{Y_t}\left[(s_1^Y\kappa_2 - s_2^Y\kappa_1) + \left(\frac{c_t^2}{Y_t}(\kappa_2)^2 - \frac{c_t^1}{Y_t}(\kappa_1)^2\right)\right],$$

$$\mu_t^{c,2} = \mu^Y - \frac{c_t^1}{Y_t}\left[(s_1^Y\kappa_2 - s_2^Y\kappa_1) + \left(\frac{c_t^2}{Y_t}(\kappa_2)^2 - \frac{c_t^1}{Y_t}(\kappa_1)^2\right)\right]. \tag{43}$$

Evidently, mean growth rates differ from one another and from the rate for aggregate output, with one country growing faster and the other slower than aggregate output. To identify the faster growing country, assume for simplicity that

$$\kappa_1 = \kappa_2. \tag{44}$$

Then

$$\mu_t^{c,1} - \mu_t^{c,2} = \kappa_1(s_1^Y - s_2^Y) + (\kappa_1)^2\left(\frac{c_t^2 - c_t^1}{Y_t}\right). \tag{45}$$

Thus $s_1^Y > s_2^Y$ (which is here equivalent to (42), that is, 2 faces more ambiguity than 1) contributes to a larger mean growth rate in country 1 and this effect is larger the larger is the common degree of ambiguity aversion. The second component on the right is time varying and stabilizing in that it raises the relative mean growth rate of the country with lower consumption. The difference in mean growth rates is an increasing function of $\lambda$ (for given realizations of $W$). Consequently, the noted difference increases if initial endowments are redistributed in favor of country 2. [19]

Second-order moments of consumption processes are also non-standard. Once again, by Ito's Lemma, the volatilities of $\mathrm{d}c_t^i/c_t^i, i = 1, 2$, are given by

$$s_t^{c,1} = s^Y + \frac{c_t^2}{Y_t}\begin{bmatrix}\kappa_2 \\ -\kappa_1\end{bmatrix},$$

$$s_t^{c,2} = s^Y + \frac{c_t^1}{Y_t}\begin{bmatrix}-\kappa_2 \\ \kappa_1\end{bmatrix}. \tag{46}$$

[19] It is straightforward to show that $\lambda$ increases in response to such a redistribution of initial endowments. We use this fact frequently in the sequel.

Thus, from (25), consumption growth rates in the two countries are positively correlated, as in the standard risk-based model. However, unlike the standard model, the country-specific growth rate $\mathrm{d}c_t^1/c_t^1 - \mathrm{d}Y_t/Y_t$ is positively correlated with shocks in country 1, that is,

$$cov_t(\mathrm{d}c_t^1/c_t^1 - \mathrm{d}Y_t/Y_t, \mathrm{d}W_t^1) = \kappa_2 s_1^Y c_t^2/Y_t > 0.$$

Such positive correlation is essentially what Lewis (p. 574) defines as *consumption home bias*. (See Section 4.4 for more on her definition and the predictions of our model.)

Assuming (44), then (46) implies that

$$s_t^{c,1} \cdot s_t^{c,1} = s^Y \cdot s^Y + 2(\kappa_1)^2 \left(\frac{c_t^2}{Y_t}\right)^2 + 2\kappa_1 \left(\frac{c_t^2}{Y_t}\right)(s_1^Y - s_2^Y)$$

$$s_t^{c,2} \cdot s_t^{c,2} = s^Y \cdot s^Y + 2(\kappa_1)^2 \left(\frac{c_t^1}{Y_t}\right)^2 - 2\kappa_1 \left(\frac{c_t^1}{Y_t}\right)(s_1^Y - s_2^Y)$$

and hence that the difference in variances is

$$\begin{aligned} s_t^{c,1} \cdot s_t^{c,1} - s_t^{c,2} \cdot s_t^{c,2} &= 2(\kappa_1)^2 \left(\frac{c_t^2 - c_t^1}{Y_t}\right) + 2\kappa_1(s_1^Y - s_2^Y) \\ &= 2(\mu_t^{c,1} - \mu_t^{c,2}). \end{aligned}$$

Though both means and variances are stochastic, the last equality implies that at all times and states, the country with higher mean growth rate also has the larger variance of consumption growth. The close connection between the difference in variances and the difference in mean growth rates implies also that factors underlying both are similar. For example, (i) $s_1^Y > s_2^Y$ contributes to a larger variance for consumption growth in country 1 relative to that in country 2 and (ii) a redistribution of initial endowments in favor of country 2 (that is, an increase in $\lambda$) increases the variance of consumption growth for country 1 relative to that for country 2. Consolidating with the previous discussion of mean growth rates and information about levels provided by (31), it follows that an initial redistribution towards country 2 results for that country in a higher initial level of consumption, and (in relative terms) a *lower* mean and variance for the rate of growth of consumption.

In terms of absolute (rather than relative) variance, consumption growth has a higher variance than aggregate output growth for at least one country, and for both countries if $s_1^Y = s_2^Y$.

### 4.3. Riskless rate, market price of uncertainty and excess returns

Eq. (35) shows that like risk, ambiguity drives down the riskless rate; their effects are captured respectively by $s^Y \cdot s^Y$ (the variance of total output growth) and the last bracketed expression on the right. The riskless rate is stochastic and varies over time between the extremes $\beta + \mu^Y - s^Y \cdot s^Y - s_2^Y \kappa_1$ and $\beta + \mu^Y - s^Y \cdot s^Y - s_1^Y \kappa_2$, depending on the distribution of aggregate consumption. To interpret the latter dependence, assume (42). Then $r_t$ is increasing in 1's share of total consumption. The reason for this dependence is that by (38), the noted consumption share serves as a proxy for 1's share of total wealth. Moreover, by (42) country 1 faces less ambiguity than does 2. Thus as the distribution of wealth shifts in favor of 1, the 'aggregate' ambiguity in the economy falls. Because ambiguity depresses the riskless rate, the latter is induced to rise.

Under (42), it is also the case that $r_t$ is increasing as a function of 1's initial endowment (decreasing in $\lambda$). In the special case that 1 and 2 face the identical ambiguity ($\kappa_2 s_1^Y = \kappa_1 s_2^Y$), then $r_t$ is constant and independent of the initial distribution.

Ambiguity acts to increase the market price of uncertainty, with the qualitative features of its effect being similar to those discussed for the riskless rate. The time variation of $\eta_t$ is of particular interest. Refer to the component $\eta_t^i$ as the domestic market price of uncertainty for country $i$. The significance of $\eta_t^1$, for example, is that it determines equilibrium excess returns for 'domestic securities' in country 1. That is, for a security whose return process $(R_t^*)$ satisfies $\mathrm{d}R_t^* = b_t^* \mathrm{d}t + s_t^* \mathrm{d}W_t^1$, its mean excess return equals

$$b_t^* - r_t = s_t^* \eta_t^1 \equiv s_t^*(s_1^Y + \kappa_2 - \kappa_2 c_t^1 / Y_t).$$

It is noteworthy that each domestic market price $\eta_t^i$ is a decreasing function of $c_t^i/Y_t$. Campbell (1999) argues that asset market data in a number of countries suggest that the (domestic) market price of uncertainty is negatively correlated with the level of domestic consumption. Our model delivers negative correlation, though with the share of aggregate consumption that occurs domestically. An immediate further implication is that the market price of uncertainty in country 1 is increasing in country 2's share of aggregate consumption. Finally, an increase in $\lambda$ increases the domestic market price of uncertainty in country 1 and reduces that in country 2.

Turn to the excess returns (37). Rewrite them in vector form

$$b_t - r_t 1 = s_t s^Y + s_t \sum_{i=1}^{2} \frac{c_t^i}{Y_t} \theta_t^{*i},$$

where $\theta_t^{*i}$ satisfies (26) for each $i$. Chen and Epstein derive a corresponding decomposition of excess returns in a representative agent model and they interpret the two components as premia for risk and ambiguity respectively. A similar interpretation applies here. The first risk premium term is the familiar instantaneous covariance of asset returns with the growth rate of aggregate consumption. The second component (which vanishes if each $\kappa_i = 0$) is a consumption-share weighted sum of individual ambiguity premia. If returns to the country $i$ security are positively correlated with shocks in both countries ($s_t^{im} > 0$, for $m = 1, 2$), then the ambiguity premium for the security is positive. This is true in particular for each country given the specification of dividend processes described in the next section (see Corollary 2).

### 4.4. Country-specific securities and home bias

The properties of equilibrium discussed to this point depend on the hypothesis that aggregate output is geometric as in (16), but not on how that output is distributed between the dividend streams $(Y_t^i)$ of the two traded securities and the non-traded endowment $\Phi$. We turn now to properties that depend on the specification of $(Y_t^i)$.

Henceforth assume that

$$Y_t^i/Y_t = v(W_t^i), \quad i = 1,2, \tag{47}$$

where the 'share' function $v : \mathbb{R}^1 \to (0, 1/2)$ is twice continuously differentiable with $v' > 0$. An immediate consequence is that

$$0 < Y_t^i, \quad i = 1,2, \quad \text{and} \quad Y_t^1 + Y_t^2 < Y_t.$$

A second consequence is that, by Ito's Lemma,

$$\begin{aligned} \mathrm{d}Y_t^1/Y_t^1 &= a_t^1\,\mathrm{d}t + \left[s_1^Y + \frac{v'(W_t^1)}{v(W_t^1)}\right]\mathrm{d}W_t^1 + s_2^Y\,\mathrm{d}W_t^2 \\ \mathrm{d}Y_t^2/Y_t^2 &= a_t^2\,\mathrm{d}t + s_1^Y\,\mathrm{d}W_t^1 + \left[s_2^Y + \frac{v'(W_t^2)}{v(W_t^2)}\right]\mathrm{d}W_t^2 \end{aligned} \tag{48}$$

for suitable drifts $a_t^1$ and $a_t^2$. Consequently, $\mathrm{d}Y_t^1/Y_t^1 - \mathrm{d}Y_t/Y_t$ is positively correlated with $W_t^1$ and uncorrelated with $W_t^2$. This justifies interpretation of $Y_t^1$ as the domestic security in country 1—the idiosyncratic part of its growth rate is driven by domestic shocks. Moreover, because a similar statement applies to $Y_t^2$ and because the representative investor in country 1 views $W_t^1$ as unambiguous and $W_t^2$ as ambiguous, the foreign security is 'more ambiguous' for her. Thus the above specification of dividend streams is consistent with our guiding intuition, namely that foreign securities are more ambiguous than domestic securities.[20]

Given the specification for $Y_t^i$, we can elaborate on or reformulate the consumption home bias that is delivered by our model. From (46) and (48), it follows that

$$cov_t(\mathrm{d}c_t^1/c_t^1 - \mathrm{d}Y_t/Y_t, \mathrm{d}Y_t^1/Y_t^1 - \mathrm{d}Y_t/Y_t) = \kappa_2 \frac{c_t^2}{Y_t}\frac{v'(W_t^1)}{v(W_t^1)} > 0.$$

In other words, there is positive correlation between country-specific consumption growth and country-specific output growth (see Lewis, p. 574).

Turn next to home bias in equities. Trading strategies for the two risky securities are given by (39). Suppose that our model is correct, including, in particular, regarding security prices and returns volatilities. Then, if one mistakenly adopts the standard model with no ambiguity, the first two expressions on the right side of (39) would be used to predict the components of the (value) portfolio of risky assets. The error that results is captured in the third term on the right which represents the effect of

[20] This discussion is admittedly informal. We do not yet have a well-founded formal definition of 'more ambiguous than'.

Conformity with the guiding intuition is the reason that we cannot specify dividends so that they exhaust total output and thus obviate the need for a non-traded endowment. For example, if we adopt (47) for country 1 and then define $Y_t^2$ as $Y_t - Y_t^1$, then $Y_t^2/Y_t$ is driven by the shock in country 1.

ambiguity. If volatilities satisfy

$$s_t^{ij} > 0, \quad i,j = 1,2, \quad \text{and} \quad \det(s_t) > 0, \tag{49}$$

then ambiguity induces country $i$ to invest more in the domestic asset and less in the foreign asset. Thus from the perspective of a model that ignores ambiguity and focuses exclusively on the risk characteristics of securities, there is a seemingly irrational bias towards domestic securities. In this sense, if (49) is satisfied, our model can resolve the equity home bias puzzle, at least in qualitative terms.

Finally, (49) is valid, as shown in the following corollary of Theorem 1.

**Corollary 2.** *Let dividend processes be given by* (47) *and refer to the Arrow–Debreu equilibrium in Theorem* 1. *Then the returns volatility matrix* $s_t$ *satisfies* (49). *In particular*, *the Radner equilibrium described in the Theorem exists.*

The positivity of returns volatilities has other noteworthy implications. In particular, it follows immediately that security returns in the two countries are positively correlated ($cov_t(\mathrm{d}R_t^1, \mathrm{d}R_t^2) > 0$) and, from (46), that returns are positively correlated with consumption growth in each country ($cov_t(\mathrm{d}R_t^i, \mathrm{d}c_t^i/c_t^i) > 0$).

Finally, with regard to home bias in equities, in the introduction we pointed to evidence that investors are more optimistic about domestic securities. Such a bias in expectations about mean returns can be identified in our model as follows: While the returns process for security 1 is given by (23), investors in the two countries view the driving processes $W^1$ and $W^2$ differently. In particular, in terms of the ambiguity adjusted probability measures (Section 4.2), 1 views $(W_t^1, W_t^2 + \kappa_1 t)$ as a Brownian motion while 2 views $(W_t^1 + \kappa_2 t, W_t^2)$ as a Brownian motion. Rewriting the returns process in terms of the Brownian driving process that is appropriate for each investor, leads to

$$\begin{aligned} \mathrm{d}R_t^1 &= (b_t^1 - \kappa_1 s_t^{12})\,\mathrm{d}t + s_t^{11}\,\mathrm{d}W_t^1 + s_t^{12}\,\mathrm{d}(W_t^2 + \kappa_1 t), \\ \mathrm{d}R_t^1 &= (b_t^1 - \kappa_2 s_t^{11})\,\mathrm{d}t + s_t^{11}\,\mathrm{d}(W_t^1 + \kappa_2 t) + s_t^{12}\,\mathrm{d}W_t^2. \end{aligned} \tag{50}$$

Consequently, after adjusting for ambiguity, country 1 attaches a higher mean return to security 1 than does country 2 if and only if

$$\kappa_2 s_t^{11} > \kappa_1 s_t^{12}. \tag{51}$$

From the explicit expressions for the returns volatilities that are derived in Appendix B, conclude that (51) and hence the noted relative optimism are confirmed for our model in the symmetric case[21]

$$\kappa_2 s_1^Y = \kappa_1 s_2^Y.$$

We can interpret this prediction as being confirmed by the survey evidence regarding relative optimism about domestic securities cited in the introduction; to do so interpret elicited probability measures as including an adjustment for ambiguity.

[21] See Section 4.1 for interpretation of this equality.

It is noteworthy that differences in ambiguity-adjusted expectations are restricted to means. Agreement regarding volatilities is consistent with the well-known relative ease of estimating the variance–covariance matrix of returns.

### 4.5. A further parameterization and trading strategies

To study further the nature of trading strategies, we specialize (47) and assume: [22]

$$v(x) = \begin{cases} \frac{1}{4}\mathrm{e}^{x} & \text{if } x \leqslant 0 \\ \frac{1}{2}(1 - \frac{1}{2}\mathrm{e}^{-x}) & \text{if } x \geqslant 0. \end{cases}$$

It is readily computed that (with respect to the reference measure $P$),

$$\mathrm{E}(Y_t^i/Y_t) = \tfrac{1}{4} \text{ and } \lim_{t\to\infty} var(Y_t^i/Y_t) = \tfrac{1}{16}.$$

As explained following Theorem 1, security prices (33) are the key to the complete description of equilibrium. Since (32) provides an explicit characterization of state prices, closed-form solutions for all endogenous variables can be obtained if the dividend streams $(Y_t^i)$ are specified so that the integration in (33) can be carried out analytically. The preceding specification for $v$ permits such integration. For example, the formulae in Theorem 1 and (47) imply that $S_t^1$ can be written in the form

$$S_t^1 = c_t^1 \int_t^T \mathrm{e}^{-\beta(\tau-t)} \mathrm{E}_t[v(W_\tau^1)]\,\mathrm{d}\tau + c_t^2 \int_t^T \mathrm{e}^{-\beta(\tau-t)} \mathrm{E}_t \left[ \frac{z_\tau^{\theta^{*2}}}{z_t^{\theta^{*2}}} v(W_\tau^1) \right] \mathrm{d}\tau$$

and the conditional expectations can be computed explicitly in terms of the standard univariate normal cdf. Therefore, $S_t^1 = h(t, W_t, Y_t)$ and $h(\cdot)$ is in closed-form up to the presence of some Riemann integrals.

However, the resulting expressions are lengthy and not easily interpreted and thus we have simulated our model numerically. For parameter values, we take [23]

$$\mu^Y = 0.0179,\ s_1^Y = s_2^Y = 0.0406,\ \beta = 0.02 \quad \text{and} \quad T = 42.5.$$

To treat the two countries symmetrically, we assume that initial endowments are such that the relative utility weight $\lambda$ equals 1 and that $\kappa_1 = \kappa_2$. Finally, the common value of the ambiguity parameter is specified to be 0.02.

To clarify the meaning and plausibility of the value 0.02, note that just as we derived (50), we can derive the corresponding ‘ambiguity-adjusted’ laws of motion for the aggregate endowment process. For country 1, for example, it is

$$\mathrm{d}Y_t/Y_t = (\mu^Y - s_2^Y \kappa_1)\,\mathrm{d}t + s_1^Y \mathrm{d}W_t^1 + s_2^Y(\mathrm{d}W_t^2 + \kappa_1 t),$$

[22] Contrary to previous assumptions, $v$ fails to be twice continuously differentiable though only at the origin. This does not affect preceding arguments, including (48), for example.

[23] The values for $\mu^Y$ and $s^Y$ are based on the discretized version of (16) and IFS quarterly consumption data (transformed into per capita terms) for the period 1957:1 to 1999:3.

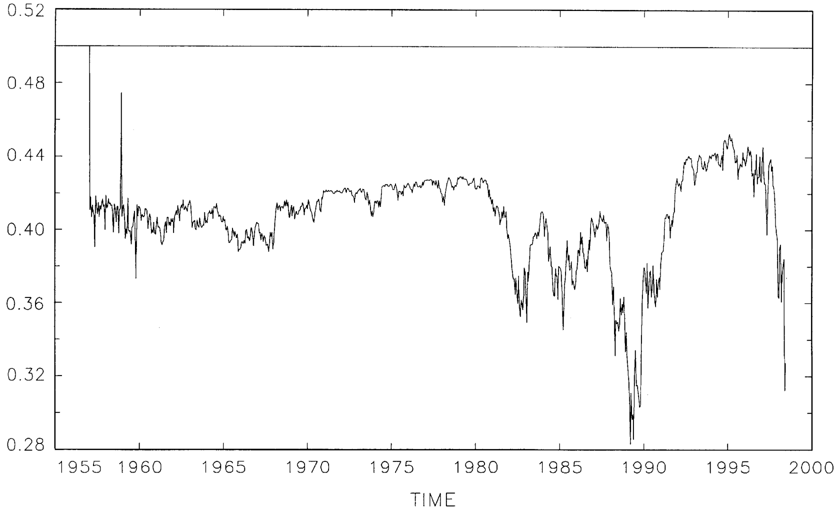


Fig. 1. Country 1's trading strategy for foreign assets.

implying that the adjustment for ambiguity calls for lowering the mean to 0.0170, a reduction of only about 6%. From this perspective, a value of 0.02 for the $\kappa_i$'s does not seem excessive.

As a benchmark, note that if $\kappa = 0$, then the equilibrium trading strategies are to buy and hold $\frac{1}{2}$ share of each of the domestic and foreign securities. In contrast, Fig. 1 describes the optimal holding $\gamma^1_{2,t}$ of the foreign security in one realization of the Brownian motion. There is a downward bias ($\gamma^1_{2,t} < 1/2$) and continual retrading. To illustrate the latter, Fig. 2 plots the corresponding turnover process $|d\gamma^1_{2,t}|$.

## 5. Concluding comments

We have extended the standard, log-utility, two-country general equilibrium model by incorporating a feature that seems to us to be intuitive, namely (greater) ambiguity about foreign securities. This extension moves predictions in the right direction in terms of helping to resolve the puzzles concerning home bias in consumption and equity. A more thorough (and quantitative) assessment of the model's usefulness for this purpose is left for future work. A multi-country extension would permit a fairer comparison with data. In this concluding section, we comment briefly on two questions about the modeling approach that might have occurred to some readers. See Chen and Epstein for expanded discussions.

One natural question concerns observational equivalence. The supergradient (9) is identical to that for an expected additive utility maximizer who uses the single prior $Q^*$ (see also (11)). It follows immediately, that our model's predictions can be generated alternatively by a model without ambiguity and in which beliefs are probabilistic,

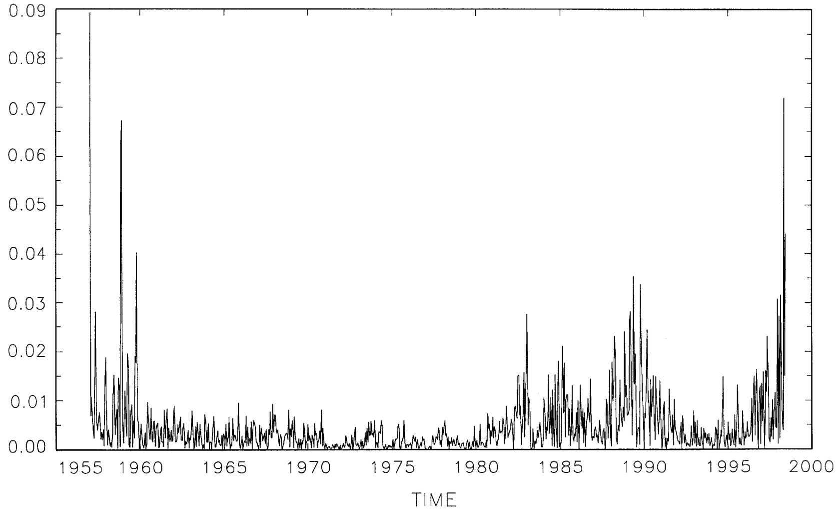


Fig. 2. Turnover of country 1's demand for foreign assets.

heterogeneous and (if $P$ is the true measure) wrong.[24] Nevertheless, there are good reasons for being interested in our approach based on ambiguity. For example, it seems to us to be: (i) less ad hoc than basing an explanation of behavior on a particular specification of heterogeneous and erroneous beliefs; and (ii) more coherent in that it models individuals as being aware of the possibility that any single probability measure that they consider could be wrong and seeking, therefore, to adopt robust decisions. Finally, the observational equivalence fails once one connects the dynamic equilibrium to behavior in other settings. Because in the Bayesian approach agents view all prospects as purely risky, risk aversion parameters may be tied to magnitudes that are deemed plausible given risk attitudes revealed by choices in other settings; this is a large part of the equity premium puzzle. Such a transfer of parameters across settings is inappropriate, however, if prospects faced in asset market are ambiguous and thus are qualitatively different than lotteries. Thus our reinterpretation in terms of ambiguity rather than risk has potential empirical significance and is not merely a change in vocabulary.

Finally, consider the question of learning. As noted earlier, we interpret our model as describing the steady state of an unmodeled learning process during which the individual has learned all she can about the environment. It seems to us plausible (indeed intuitive) that in many settings an individual may *not* be completely confident that she knows *precisely* the probability law describing her environment, where one exists, and that ambiguity may persist for her even in the long run. In the specific context of foreign security markets, many have claimed that it is not at all clear that

[24] On a technical note, observational equivalence does *not* render our equilibrium analysis a corollary of analyses where agents are assumed to be Bayesian but with different priors. That is because the heterogeneous singleton priors that replicate our equilibrium are endogenous; see the discussion surrounding (26).

investors could learn the true statistical model driving security returns even where one exists. For example, French and Poterba (p. 225) write that 'the statistical uncertainties associated with estimating expected returns in equity markets makes it difficult for investors to learn that expected returns in domestic markets are not systematically higher than those abroad'. In such an environment, where there may be less than complete confidence in estimated moments of expected returns, the investor may not treat these estimates as true in making portfolio decisions.[25] Rather she may be aware of the possibility that the estimates are wrong and thus seek to make robust decisions. Our model can be interpreted in these terms.

In fact, the assumption that there exists a true probability law describing the data generating mechanism is made for analytical convenience rather than because of compelling evidence. Bewley (1998) argues that economic time series may be generated by stochastic processes that could never be discovered from the data that they generate.

Even granting these points, the question remains whether one could extend the model to include learning. As pointed out by Chen and Epstein, the general recursive multiple-priors model is rich enough to accommodate learning. In a discrete-time setting, a theory of learning under ambiguity is being developed in Epstein and Schneider (2002), which includes also applications to asset market settings. The extension of this paper's model to include learning is a natural next step.

## 6. For further reading

The following references may also be of interest to the reader: Duffie and Skiadas, 1994; Karoui et al., 1997; Karatzas and Shreve, 1987.

## Acknowledgements

Epstein gratefully acknowledges the financial support of the NSF (grant SES-9972442). We have benefitted from comments by Gur Huberman, Martin Schneider and especially a referee. First version: March 2000.

## Appendix A. Proof of Theorem 1

The parametric restrictions

$$0 \leqslant \kappa_1 < s_2^Y \quad \text{and} \quad 0 \leqslant \kappa_2 < s_1^Y \tag{A.1}$$

are adopted throughout.

[25] See Lewis (1999) for a discussion of some of the literature on portfolio choice under estimation risk where it *is* assumed that estimates are treated as true.

Given consumption processes $c^i, i=1,2, (\sigma_t^i)$ is the volatility associated with $i$'s utility process $(V_t^i(c^i))$. Write

$$\sigma_t^1 = (\sigma_{1,t}^1, \sigma_{2,t}^1)^\top \quad \text{and} \quad \sigma_t^2 = (\sigma_{1,t}^2, \sigma_{2,t}^2)^\top.$$

Thus $\sigma_{2,t}^1$ is that part of the volatility of 1's utility process that corresponds to $W_t^2$, the component that is ambiguous for 1.

**Lemma 3.** *For the specific consumption processes $c^1$ and $c^2$ defined by* (31), *the volatilities of utility satisfy*

$$\sigma_{2,t}^1 > 0 \quad \text{and} \quad \sigma_{1,t}^2 > 0. \tag{A.2}$$

**Proof.** Consider individual 1's utility process and show

$$\sigma_{2,t}^1 > 0 \tag{A.3}$$

the other inequality can be proven analogously.

Let $Q$ be the measure defined as in (2)–(4) by taking

$$\theta_t = (0, \kappa_1)^\top \tag{A.4}$$

for all $t$. Define

$$V_t = \mathrm{E}_Q\left[\int_t^T \mathrm{e}^{-\beta(\tau-t)} \log c_\tau^1 \, \mathrm{d}\tau | \mathscr{F}_t\right]. \tag{A.5}$$

Then $(V_t)$ is an Ito process and we can write

$$\mathrm{d}V_t = \mu_t^V \mathrm{d}t + \sigma_{1,t}^V \mathrm{d}W_t^1 + \sigma_{2,t}^V \mathrm{d}W_t^2.$$

**Claim.** $\sigma_{2,t}^V > 0$. If true, then we can conclude that $V_t = V_t^1(c^1)$ and hence that $\sigma_{2,t}^1 = \sigma_{2,t}^V > 0$. The point, roughly, is that the positivity of the volatility $\sigma_{2,t}^V$ validates the specification (A.4) as the one that is consistent with the minimization over all priors in $\mathscr{P}$, as described in (10). In more formal terms, Girsanov's Theorem implies that $(V_t)$ solves the BSDE:

$$\mathrm{d}V_t = [-\log c_t^1 + \beta V_t + \kappa_1 |\sigma_{2,t}^V|] \, \mathrm{d}t + \sigma_t^V \cdot \mathrm{d}W_t, \quad V_T = 0.$$

But given that $\kappa = (0, \kappa_1)^\top$, this is the BSDE that defines, via an appropriate form of (7), the utility process $(V_t^1(c^1))$. By uniqueness of the solution, conclude that $V_t^1(c^1) = V_t$.

Turn to the proof of the claim. Given the explicit expression for $V_t$, a direct approach is possible. Substitute into (A.5) for $c_\tau^1$ using (31) to obtain $V_t = K_t - L_t$, where $K_t = \mathrm{E}_Q[. \int_t^{\mathscr{T}} \mathrm{e}^{-\beta(\tau-t)} \log Y_\tau \, \mathrm{d}\tau | \mathscr{F}_t]$ and $L_t = \mathrm{E}_Q[. \int_t^T \mathrm{e}^{-\beta(\tau-t)} \log(1 + \lambda\varsigma_\tau) \, \mathrm{d}\tau \,|\, \mathscr{F}_t]$. By Girsanov's Theorem $(W_t^1, W_t^2 + \kappa_1 t)$, is a Brownian motion under $Q$. Thus we can compute $K_t$ and, to a lesser degree, $L_t$. Because $Y$ is a geometric process, compute that

$$K_t = a_t + d_t(s_1^Y W_t^1 + s_2^Y W_t^2),$$

where $a_t$ is deterministic and $d_t = \int_t^T e^{-\beta(\tau-t)}\,d\tau$. Write $L_t = H_t(W_t^1, W_t^2)$, where

$$H_t(w^1,w^2) = \int_t^T e^{-\beta(\tau-t)} E^{\tau-t} \log\left(1 + \lambda e^{\left(\frac{1}{2}((\kappa_1)^2-(\kappa_2)^2)\tau+\kappa_1 x^2-\kappa_2 x^1\right)}\right) d\tau$$

and the expectation $E^{\tau-t}$ refers to integration on the plane of points $(x^1,x^2)$ with respect to the bivariate normal distribution $N(m_{\tau-t},\Sigma_{\tau-t})$ with

$$m_{\tau-t} = (w^1, w^2 - \kappa_1(\tau-t)) \quad \text{and} \quad \Sigma_{\tau-t} = (\tau-t)I_{2\times 2}.$$

By Ito's Lemma and the preceding, it suffices to prove that (for all $(t,w^1,w^2) \in [0,T]\times\mathbb{R}^2$)

$$d_t s_2^Y - \partial H_t(w^1,w^2)/\partial w^2 > 0. \tag{A.6}$$

Direct computation and reversing the order of differentiation and integration by Billingsley (1986, p. 215) yields

$$\frac{\partial}{\partial w^2} E^{\tau-t} \log\left(1 + \lambda e^{\left(\frac{1}{2}((\kappa_1)^2-(\kappa_2)^2)\tau+\kappa_1 x^2-\kappa_2 x^1\right)}\right)$$
$$= \frac{\partial}{\partial w^2}\left[\int_{\mathbb{R}^2} \log\left(1+\lambda e^{\left(\frac{1}{2}((\kappa_1)^2-(\kappa_2)^2)\tau+\kappa_1(x^2+w^2-\kappa_1(\tau-t))-\kappa_2(x^1+w^1)\right)}\right) dN(0,\Sigma_{\tau-t})\right]$$

$< \kappa_1$, which leads to (A.6). □

**Lemma 4.** *For any given* $\lambda > 0$, *the consumption processes defined in* (31) *solve* (*uniquely*) [26]

$$\max\{V^1(e^1) + \lambda V^2(e^2) : e^1, e^2 \in \mathscr{C}, e^1 + e^2 \leqslant Y\}. \tag{A.7}$$

**Proof.** Clearly, $c^1$ and $c^2$ are feasible. Therefore, it suffices to verify that there exists a $\mathbb{R}^1_{++}$-valued shadow price process $\pi = (\pi_t)$ satisfying

$$\pi \in \partial V^1(c^1) \cap \lambda \partial V^2(c^2), \tag{A.8}$$

where $\partial V^i(c^i)$ denotes the set of supergradients for $V^i$ at $c^i$. Given such a $\pi$, then

$$V^1(e^1) + \lambda V^2(e^2) - V^1(c^1) - \lambda V^2(c^2)$$
$$\leqslant E_P\left[\int_0^T \pi_t(e_t^1 - c_t^1)\,dt\right] + \lambda E_P\left[\int_0^T (\pi_t/\lambda)(e_t^2 - c_t^2)\,dt\right]$$
$$= E_P\left[\int_0^T \pi_t(\Sigma_i(e_t^i - c_t^i))\,dt\right] = E_P\left[\int_0^T \pi_t(\Sigma_i e_t^i - Y_t)\,dt\right] \leqslant 0.$$

To establish (A.8), recall (9) and thus that $\pi^i \in \partial V^i(c^i)$, where

$$\pi_t^i(c^i) = e^{-\beta t} z_t^{\theta * i}/c_t^i$$

[26] Because $c^i$ denotes the equilibrium consumption process, we use $e^i$ below to denote the generic process in $\mathscr{C}$.

and, following (10),

$$\theta^{*1} \in \Theta_c^1 \equiv \{(\theta_t)\colon\ \theta_t = (0,\kappa_1)^\top \otimes \mathrm{sgn}(\sigma_t^1) \text{ all } t\}, \tag{A.9}$$

$$\theta^{*2} \in \Theta_c^2 \equiv \{(\theta_t)\colon\ \theta_t = (\kappa_2,0)^\top \otimes \mathrm{sgn}(\sigma_t^2) \text{ all } t\}.$$

By the positivity of volatilities in (A.2) and (A.9) is equivalent to

$$\theta_t^{*1} = (0,\kappa_1) \quad \text{and} \quad \theta_t^{*2} = (\kappa_2, 0) \quad \text{for all } t.$$

Thus (A.8) is satisfied by $\pi$, where

$$\pi_t = \mathrm{e}^{-\beta t} z_t^{\theta^{*1}}/c_t^1 = \lambda \mathrm{e}^{-\beta t} z_t^{\theta^{*2}}/c_t^2. \qquad \square \tag{A.10}$$

Turn to description of an equilibrium of the form $((c^i,\gamma^i)_{i=1,2},S)$, where the $c^i$'s are defined by (31) and where $\lambda$ is defined by (28). Because these consumption processes are efficient, they can be implemented as part of an Arrow–Debreu equilibrium and subsequently also as part of a Radner equilibrium. To see this, let $p=\pi/\pi_0$, where $\pi$ is defined in (A.10). Use $p$ as a state price process; in particular, define security prices $S^n$, $n=0,1,2$, by (34) and (33) and associated returns as in (23). Define person $i$'s *financial wealth* by $X_t^i \equiv \gamma_t^i \cdot S_t$. Let

$$\psi_{n,t}^i \equiv S_t^n \gamma_{n,t}^i / X_t^i, \quad n = 1,2 \tag{A.11}$$

and $\psi_t^i = (\psi_{1,t}^i, \psi_{2,t}^i)^\top$ denote portfolio shares. (If $X_t^i = 0$, let $\psi_{n,t}^i = 0$.) The proportion invested in the riskless asset is $1 - \psi_t^i \cdot \mathbf{1}$. Then $i$'s initial total wealth is (letting $i,j=1,2, j \neq i$)

$$\begin{aligned}\bar{X}_0^i &= X_0^i + \frac{1}{2}\bar{S}_0 = \frac{1}{2}\gamma_0^i \cdot S_0 + \frac{1}{2}\bar{S}_0 \\ &= \gamma_{i,0}^i \mathrm{E}\left[\int_0^T p_s Y_s^i \,\mathrm{d}s\right] + \gamma_{j,0}^i \mathrm{E}\left[\int_0^T p_s Y_s^j \,\mathrm{d}s\right] + \frac{1}{2}\mathrm{E}\left[\int_0^T p_s \Phi_s \,\mathrm{d}s\right].\end{aligned}$$

Therefore the static budget constraint (20) is equivalent to

$$\mathrm{E}\left[\int_0^T p_s e_s^i \,\mathrm{d}s\right] \leqslant \bar{X}_0^i,\ e^i \in \mathscr{C}. \tag{A.12}$$

By the definition of $\lambda$, these constraints hold with equality if $e^i = c^i$. Finally, at $c^i$, each individual $i$ satisfies the first-order conditions

$$\mathrm{e}^{-\beta t} z_t^{\theta^{*i}}/c_t^i = \delta^i p_t \tag{A.13}$$

for suitable multipliers $\delta^1 = 1$ and $\delta^2 = \lambda^{-1}$. Conclude that $c^1$ and $c^2$ are utility maximizing. Because they clear output markets, they constitute an Arrow–Debreu equilibrium allocation.

The following two lemmas show that the static Arrow–Debreu equilibrium can be implemented by some security trading strategies $\gamma^i$, $i=1,2$, to form the Radner equilibrium described in Theorem 1.

First, notice that the budget constraint (18) is equivalent to the following familiar dynamic budget constraint:

$$\mathrm{d}X_t^i = \{[r_t + (\psi_t^i)^\top (b_t - r_t\mathbf{1})]X_t^i - (e_t^i - \tfrac{1}{2}\Phi_t)\}\,\mathrm{d}t + X_t^i(\psi_t^i)^\top s_t\,\mathrm{d}W_t. \tag{A.14}$$

In order to rule out arbitrage opportunities (Dybvig and Huang, 1988), impose also the credit constraint

$$X_t^i \geqslant -\tfrac{1}{2}\bar{S}_t, \quad t \in [0,T]. \tag{A.15}$$

**Lemma 5.** *Let* $S = (S^0, S^1, S^2)^\top$ *be given by* (33) *and* (34) *and suppose that* $s_t$ *is invertible. Then*:

(i) *The state price process* $p$ *satisfies* (24).
(ii) *If* $(e^i, \psi^i, X^i)$ *satisfies the dynamic budget constraint* (A.14) *and the credit constraint* (A.15), *then* $e^i$ *satisfies the static budget constraint* (A.12).
(iii) *Conversely, if* $e^i$ *satisfies* (A.12), *there exist a portfolio share process* $\psi^i$ *and financial wealth process* $X^i$ *such that* $(e^i, \psi^i, X^i)$ *satisfies* (A.14) *and* (A.15). *Moreover, if* $e^i = c^i$, *then* $\psi^i$ *is unique up to equivalence and person* $i$*'s financial wealth* $X_t^i$ *is given by*

$$X_t^i = \frac{1}{p_t}\mathrm{E}\left[\int_t^T p_s(c_s^i - \tfrac{1}{2}\Phi_s)\,\mathrm{d}s|\mathscr{F}_t\right]. \tag{A.16}$$

**Proof.** (i) Since $S$ is given by (33) and (34), the deflated gains process $(\int_0^t p_s\,\mathrm{d}D_s + p_tS_t)$ is a three-dimensional $P$-martingale and hence it has a zero drift. Then (24) follows from Ito's Lemma and the definitions of $D$ and returns.

(ii) Adapt arguments from Karatzas (1989). By (i), (A.15) and Ito's Lemma,

$$p_tX_t^i + \int_0^t p_\tau(e_\tau^i - \tfrac{1}{2}\Phi_\tau)\,\mathrm{d}\tau = X_0^i + \int_0^t p_\tau X_\tau^i(s_\tau^\top\psi_\tau^i - \eta_\tau)^\top\,\mathrm{d}W_\tau. \tag{A.17}$$

The left side of this equation is a local martingale. Because of the credit constraint (A.15), it is bounded below by a martingale and hence is a supermartingale. By the optional sampling theorem, therefore,

$$\mathrm{E}\left[p_TX_T^i + \int_0^T p_t(e_t^i - \tfrac{1}{2}\Phi_t)\,\mathrm{d}t\right] \leqslant X_0^i.$$

From $X_T^i \geqslant 0$, derive the static budget constraint (A.12).

(iii) Conversely, let $e_t^i$ satisfy the static budget constraint (A.12), or equivalently,

$$\mathrm{E}\left[\int_0^T p_t(e_t^i - \tfrac{1}{2}\Phi_t)\,\mathrm{d}t\right] \leqslant X_0^i.$$

Introduce the $P$-martingale

$$H_t^i \equiv \mathrm{E}\left[\int_0^T p_t(e_t^i - \tfrac{1}{2}\Phi_t)\,\mathrm{d}t|\mathscr{F}_t\right] - \mathrm{E}\left[\int_0^T p_t(e_t^i - \tfrac{1}{2}\Phi_t)\,\mathrm{d}t\right].$$

By the martingale representation theorem, $H_t^i = \int_0^t (\phi_s^i)^\top \, dW_s$, for some progressively measurable $\mathbb{R}^2$-valued process $\phi^i$ with $\int_0^T \|\phi_s^i\|^2 \, ds < \infty$, a.s. Let the financial wealth process $(X_t^i)$ and portfolio share process $(\psi_t^i)$ satisfy

$$X_t^i = \frac{1}{p_t}\left(X_0^i - \int_0^t p_s(e_s^i - \tfrac{1}{2}\Phi_s)\, ds + H_t^i\right) \tag{A.18}$$

and

$$\psi_t^i = (s_t^\top)^{-1}\left(\eta_t + \frac{\phi_t^i}{p_t X_t^i}\right). \tag{A.19}$$

Then $H_t^i = \int_0^t p_\tau X_\tau^i (s_\tau^\top \psi_\tau^i - \eta_\tau)^\top \, dW_\tau$, and, by (A.18),

$$\begin{aligned} p_t X_t^i &= X_0^i - \int_0^t p_\tau(e_\tau^i - \tfrac{1}{2}\Phi_\tau)\, d\tau + \int_0^t p_\tau X_\tau^i (s_\tau^\top \psi_\tau^i - \eta_\tau)^\top \, dW_\tau \\ &= X_0^i - \mathrm{E}\left[\int_0^T p_t(e_t^i - \tfrac{1}{2}\Phi_t)\, dt\right] + \mathrm{E}\left[\int_t^T p_s(e_s^i - \tfrac{1}{2}\Phi_s)\, ds | \mathscr{F}_t\right]. \end{aligned}$$

From this one can verify that $X_t^i$ satisfies the dynamic budget constraint (A.14) and the credit constraint (A.15).

If $e^i = c^i$, the static budget constraint (A.12) holds with equality. Consequently, (A.16) follows from the preceding equation.

Finally, consider the uniqueness of portfolio shares. By (A.16), $X_T^i = 0$ and $M^i$ is a $P$-martingale, where

$$M_t^i \equiv p_t X_t^i + \int_0^t p_s(c_s^i - \tfrac{1}{2}\Phi_s)\, ds, \quad t \in [0, T]. \tag{A.20}$$

Suppose there are two such portfolios $\psi^i$ and $\hat{\psi}^i$ satisfying the stated properties. Let $X^i$ and $\hat{X}^i$ represent the corresponding financial wealth processes and $(M_t^i)$ and $(\hat{M}_t^i)$ the corresponding $P$-martingales as in (A.20). By (A.17) and

$$M_T^i = \hat{M}_T^i = \int_0^T B_s(c_s^i - \tfrac{1}{2}\Phi_s)\, ds,$$

the martingale

$$M_t^i - \hat{M}_t^i = \int_0^t B_\tau X_\tau^i (\psi_\tau^i - \hat{\psi}_\tau^i)^\top s_\tau \, dW_\tau, \quad t \in [0, T]$$

is identically zero. Thus the quadratic variation

$$\langle M^i - \hat{M}^i \rangle_t = \int_0^t (B_\tau X_\tau^i)^2 \; \| \, (\psi_\tau^i - \hat{\psi}_\tau^i)^\top s_\tau \, \|^2 \, d\tau = 0, \quad t \in [0, T].$$

Since $s_t$ is invertible for all $t, \psi_t^i = \hat{\psi}_t^i$ a.s. $dt \otimes dP$. □

**Lemma 6.** *Let the returns volatility matrix $s_t$ be invertible. Then the Arrow–Debreu equilibrium $((c^i)_{i=1,2}, p)$ can be implemented to form a Radner equilibrium*

$((c^i, \gamma^i)_{i=1,2}, S)$, *for some trading strategies* $(\gamma^1, \gamma^2) \in \Gamma \times \Gamma$ *and for security prices* $S^n, n = 0, 1, 2$, *given by* (33) *and* (34).

**Proof.** By the equivalence of the static and dynamic budget constraints proven in the preceding lemma, the two associated optimization problems are equivalent. Hence, we need only find trading strategies to clear all markets. Note that the static budget constraint (A.12) holds with equality in equilibrium.

Let $\gamma^i_{n,t} = X^i_t \psi^i_{n,t}/S^n_t, n = 1, 2$, and $\gamma^i_{0,t} = X^i_t(1 - \psi^i_{1,t} - \psi^i_{2,t})/S^0_t$, where $\psi^i_t$ is given by (A.19) and $X^i_t$ is given by (A.16). Then $\gamma^i \in \Gamma$. By the preceding lemma, $(c^i, \psi^i, X^i)$ satisfies the dynamic budget constraint (A.14). Stock markets clear if and only if

$$X^1_t \psi^1_t + X^2_t \psi^2_t = (S^1_t, S^2_t)^\top. \tag{A.21}$$

Sum financial wealth (A.16) over $i$ and use pricing equation (33) and contingent consumption market clearing condition (21) to obtain

$$X^1_t + X^2_t = S^1_t + S^2_t. \tag{A.22}$$

This equation and (A.21) imply that the bond market also clears. Therefore, we need only verify (A.21).

By Ito's Lemma, (24) and (A.14) to obtain (A.17). Sum (A.17) over $i$ and apply Ito's Lemma and (A.22) to obtain

$$\begin{aligned} &\mathrm{d}[p_t(S^1_t + S^2_t)] \\ &\quad = -\, p_t(Y^1_t + Y^2_t)\,\mathrm{d}t + p_t[X^1_t(\psi^1_t)^\top + X^2_t(\psi^2_t)^\top]s_t\,\mathrm{d}W_t - p_t(S^1_t + S^2_t)\eta_t \cdot \mathrm{d}W_t. \end{aligned}$$

On the other hand, apply Ito's Lemma and use (22)–(24) to obtain

$$\mathrm{d}[p_t(S^1_t + S^2_t)] = a_t\,\mathrm{d}t + p_t(S^1_t, S^2_t)s_t\,\mathrm{d}W_t - p_t(S^1_t + S^2_t)\eta_t \cdot \mathrm{d}W_t$$

for some process $(a_t)$. Match the volatility terms in the above two expressions and apply invertibility of $s_t$ to derive (A.21). □

It remains to verify the security market conditions asserted in the theorem. Apply Ito's Lemma to the first-order conditions (A.13) and compare with (24) to derive

$$r_t = \beta + \mu^{c,i}_t - s^{c,i}_t \cdot s^{c,i}_t - s^{c,i}_t \cdot \theta^{*i}_t \tag{A.23}$$

and

$$\eta_t = s^{c,1}_t + \theta^{*1}_t = s^{c,2}_t + \theta^{*2}_t. \tag{A.24}$$

Substitute (46) and (26) into (A.24) to obtain (36).

Apply Ito's Lemma to the market clearing condition (21) and derive

$$\mu^Y = \sum_{i=1}^{2} \mu^{c,i}_t c^i_t / Y_t, \quad s^Y = \sum_{i=1}^{2} s^{c,i}_t c^i_t / Y_t.$$

Multiply $c_t^i$ on each side of (A.23) and sum over $i$ to obtain

$$r_t = \beta + \mu^Y - s^Y \eta_t.$$

Substitute expression (36) for $\eta_t$ into the preceding to obtain (35).

By the definition of the market price of uncertainty process, $b_t - r_t\mathbf{1} = s_t\eta_t$. Substitute (36) for $\eta_t$ into the preceding to obtain (37).

Finally, turn to parts (c) and (d). Eq. (38) follows directly from the following Lemma.

**Lemma 7.** *In equilibrium, consumption and total wealth are related by*

$$c_t^i = \frac{\beta}{1 - \mathrm{e}^{-\beta(T-t)}} \bar{X}_t^i. \tag{A.25}$$

**Proof.** Use (A.16) and the definition of total wealth (30) to obtain

$$\bar{X}_t^i = \frac{1}{p_t} \mathrm{E}\left[\int_t^T p_s c_s^i \, \mathrm{d}s | \mathscr{F}_t\right]. \tag{A.26}$$

Thus

$$\begin{aligned} \delta^i p_t \bar{X}_t^i &= \mathrm{E}\left[\int_t^T \delta^i p_s c_s^i \, \mathrm{d}s | \mathscr{F}_t\right] \\ &= \mathrm{E}\left[\int_t^T \mathrm{e}^{-\beta s} z_s^{\theta^{*i}} \, \mathrm{d}s | \mathscr{F}_t\right] = \beta^{-1}(\mathrm{e}^{-\beta t} - \mathrm{e}^{-\beta T}) z_t^{\theta^{*i}}, \end{aligned}$$

where the second equality follows from the first-order conditions (A.13) and the third equality follows from the fact that $z_t^{\theta^{*i}}$ is a $P$-martingale. Apply (A.13) once more to derive (A.25). □

Write

$$\mathrm{d}\bar{X}_t^i / \bar{X}_t^i = \mu_t^{\bar{X},i} \mathrm{d}t + s_t^{\bar{X},i} \cdot \mathrm{d}W_t. \tag{A.27}$$

Thus $s_t^{\bar{X},i}$ is the volatility of $i$'s total wealth process. From (A.25) and Ito's Lemma, deduce that

$$s_t^{\bar{X},i} = s_t^{c,i}. \tag{A.28}$$

From (A.19), the key to solve for portfolio shares and trading strategies is to solve for $\phi_t^i$, the integrand in the martingale representation of $H_t^i$. Use (A.26) and the definition of $\bar{S}_t$ to rewrite $H_t^i$ as

$$\begin{aligned} H_t^i &= \mathrm{E}\left[\int_0^T p_t c_t^i \, \mathrm{d}t | \mathscr{F}_t\right] - \frac{1}{2} \mathrm{E}\left[\int_0^T p_t \Phi_t \, \mathrm{d}t | \mathscr{F}_t\right] - \mathrm{E}\left[\int_0^T p_t (c_t^i - \tfrac{1}{2}\Phi_t) \, \mathrm{d}t\right] \\ &= p_t \bar{X}_t^i - \tfrac{1}{2} p_t \bar{S}_t + \int_0^t p_s c_s^i \, \mathrm{d}s - \frac{1}{2}\int_0^t p_s \Phi_s \, \mathrm{d}s - E\left[\int_0^T p_t (c_t^i - \tfrac{1}{2}\Phi_t) \, \mathrm{d}t\right]. \end{aligned}$$

Apply Ito's Lemma to the above equation and use (24), (29) and (A.27) to obtain

$$p_t\bar{X}^i_t s^{\bar{X},i}_t - p_t\bar{X}^i_t\eta_t - \tfrac{1}{2}p_t\bar{s}_t + \tfrac{1}{2}p_t\bar{S}_t\eta_t = \phi^i_t.$$

Substitute this into (A.19) and use (A.28) and (30) to derive

$$\psi^i_t = \frac{1}{X^i_t}(s_t^\top)^{-1}(\bar{X}^i_t s^{c,i}_t - \tfrac{1}{2}\bar{s}_t).$$

Use (A.11), (A.24), (26) and $\eta_t = (s_t)^{-1}(b_t - r_t\mathbf{1})$ to substitute for $s^{c,i}_t$ in the above equation to obtain trading strategies (39) for the two risky securities. The trading strategy for bond is given by

$$\gamma^i_0 = X^i_t(1 - \psi^i_t \cdot \mathbf{1})/S^0_t.$$

By (30), (A.22), (A.25) and the market clearing condition (21),

$$\bar{X}^1_t + \bar{X}^2_t = X^1_t + X^2_t + \bar{S}_t = S^1_t + S^2_t + \bar{S}_t = \beta^{-1}(1 - \mathrm{e}^{-\beta(T-t)})Y_t.$$

Apply Ito's Lemma to this equation to obtain (40). □

## Appendix B. Proof of Corollary 2

Denote $\mathrm{E}(\cdot|\mathscr{F}_t)$ by $\mathrm{E}_t(\cdot)$. Substitute dividends processes (47) and the state price process (32) into pricing equations (33) to obtain

$$S^1_t = c^1_t K_t(W^1_t) + c^2_t L_t(W^1_t) \tag{B.1}$$

and

$$S^2_t = c^1_t M_t(W^2_t) + c^2_t N_t(W^2_t), \tag{B.2}$$

where

$$K_t(W^1_t) = \int_t^T \mathrm{e}^{-\beta(\tau-t)}\mathrm{E}_t[v(W^1_\tau)]\,\mathrm{d}\tau,$$

$$L_t(W^1_t) = \int_t^T \mathrm{e}^{-\beta(\tau-t)}\mathrm{E}_t\left[\frac{z^{\theta^{*2}}_\tau}{z^{\theta^{*2}}_t}v(W^1_\tau)\right]\,\mathrm{d}\tau,$$

$$M_t(W^2_t) = \int_t^T \mathrm{e}^{-\beta(\tau-t)}\mathrm{E}_t\left[\frac{z^{\theta^{*1}}_\tau}{z^{\theta^{*1}}_t}v(W^2_\tau)\right]\,\mathrm{d}\tau,$$

$$N_t(W^2_t) = \int_t^T \mathrm{e}^{-\beta(\tau-t)}\mathrm{E}_t[v(W^2_\tau)]\,\mathrm{d}\tau.$$

Since $v(\cdot)$ is increasing and positive, it is easy to show that $K_t(\cdot)$, $L_t(\cdot)$, $M_t(\cdot)$ and $N_t(\cdot)$ are all increasing and positive. (For example, to show that $\mathrm{E}_t[(z^{\theta^{*2}}_\tau/z^{\theta^{*2}}_t)v(W^1_\tau)]$

is increasing, use the facts (i) $z_\tau^{\theta^{*2}}/z_t^{\theta^{*2}}$ depends only on the increment $(W_\tau^1 - W_t^1)$ and (ii) $v$ is increasing.)

Apply Ito's Lemma to (B.1) and (B.2) to obtain all elements in the returns volatility matrix:

$$\begin{aligned}
s_t^{11} &= (s_1^Y + \kappa_2 c_t^2/Y_t)K_t c_t^1/S_t^1 + (s_1^Y - \kappa_2 c_t^1/Y_t)L_t c_t^2/S_t^1 \\
&\quad + (c_t^1 K_t'(W_t^1) + c_t^2 L_t'(W_t^1))/S_t^1, \\
s_t^{12} &= (s_2^Y - \kappa_1 c_t^2/Y_t)K_t c_t^1/S_t^1 + (s_2^Y + \kappa_1 c_t^1/Y_t)L_t c_t^2/S_t^1, \\
s_t^{21} &= (s_1^Y + \kappa_2 c_t^2/Y_t)M_t c_t^1/S_t^2 + (s_1^Y - \kappa_2 c_t^1/Y_t)N_t c_t^2/S_t^2, \\
s_t^{22} &= (s_2^Y - \kappa_1 c_t^2/Y_t)M_t c_t^1/S_t^2 + (s_2^Y + \kappa_1 c_t^1/Y_t)N_t c_t^2/S_t^2 \\
&\quad + (c_t^1 M_t'(W_t^2) + c_t^2 N_t'(W_t^2))/S_t^2,
\end{aligned}$$

where prime denotes derivative. Given the assumption (A.1) on parameters, each term in above equations is positive and hence

$$s_t^{ij} > 0, \quad \text{for all } i,j = 1,2.$$

For the determinant,

$$\begin{aligned}
\det(s_t) &= s_t^{11}s_t^{22} - s_t^{12}s_t^{21} \\
&= a_t + \frac{c_t^1 c_t^2}{S_t^2 S_t^1}(s_1^Y\kappa_1 + s_2^Y\kappa_2)(K_t N_t - M_t L_t),
\end{aligned}$$

where $a_t$ is a positive process. We claim that $K_t > L_t$ and $N_t > M_t$. In fact,

$$\begin{aligned}
K_t - L_t &= \int_t^T e^{-\beta(\tau-t)}E_t[v(W_\tau^1)]\,d\tau - \int_t^T e^{-\beta(\tau-t)}E_t\left[\frac{z_\tau^{\theta^{*2}}}{z_t^{\theta^{*2}}}v(W_\tau^1)\right]\,d\tau \\
&= \int_t^T e^{-\beta(\tau-t)}E_t[(1 - e^{-\frac{1}{2}(\kappa_2)^2(\tau-t)-\kappa_2(W_\tau^1-W_t^1)})v(W_\tau^1)]\,d\tau \\
&= \int_t^T e^{-\beta(\tau-t)}E_x[(1 - e^{-\frac{1}{2}(\kappa_2)^2(\tau-t)-\kappa_2 x})v(x+W_t^1)]\,d\tau > 0,
\end{aligned}$$

where $E_x$ denotes expectation with respect to $N(0, \tau-t)$. To obtain the last inequality, use the fact that both $v(x+W_t^1)$ and $1 - e^{-\frac{1}{2}(\kappa_2)^2(\tau-t)-\kappa_2 x}$ are increasing in $x$, which implies that

$$\begin{aligned}
&E_x[(1 - e^{-\frac{1}{2}(\kappa_2)^2(\tau-t)-\kappa_2 x})v(x+W_t^1)] > \\
&E_x[1 - e^{-\frac{1}{2}(\kappa_2)^2(\tau-t)-\kappa_2 x}] \cdot E_x[v(x+W_t^1)] = 0.
\end{aligned}$$

Similarly, $N_t > M_t$. Therefore, $\det(s_t) > 0$. □

## References


Anderson, E., Hansen, L.P., Sargent, T.J., 2000. Robustness, Detection and the Price of Risk. Stanford University.

Basak, S., 2000. A model of dynamic equilibrium asset pricing with heterogeneous beliefs and extraneous risk. Journal of Economic Dynamics and Control 24, 63–95.

Basak, S., Cuoco, D., 1998. An equilibrium model with restricted stock market participation. Review of Financial Studies 11, 309–341.

Bewley, T.F., 1998. Knightian uncertainty. In: Jacobs, D., Kalai, E., Kamien, M. (Eds.), Frontiers of Research in Economic Theory: the Nancy L. Schwartz Memorial Lectures. Cambridge University Press, New York, pp. 71–81.

Billingsley, P.B., 1986. Probability and Measure, 2nd Edition. Wiley, New York.

Brennan, M.J., Cao, H.H., 1997. International portfolio investment flows. Journal of Finance 52, 1851–1880.

Campbell, J.Y., 1999. Asset prices, consumption and the business cycle. In: Taylor, J., Woodford, M. (Eds.), Handbook of Macroeconomics. North-Holland, Amsterdam, pp. 1231–1303.

Chen, Z., Epstein, L.G., 2002. Ambiguity, risk and asset returns in continuous time, Econometrica, forthcoming.

Detemple, J., Murthy, S., 1994. Intertemporal asset pricing with heterogeneous beliefs. Journal of Economic Theory 62, 294–320.

Duffie, D., Epstein, L.G., 1992. Stochastic differential utility. Econometrica 60, 353–394.

Duffie, D., Skiadas, C., 1994. Continuous-time security pricing: a utility gradient approach Journal of Mathematical Economics 23, 107–131.

Dumas, B., 1989. Two-person dynamic equilibrium in the capital market. Review of Financial Studies 2, 157–188.

Dumas, B., Uppal, R., Wang, T., 2000. Efficient intertemporal allocations with recursive utility. Journal of Economic Theory 93, 240–259.

Dybvig, P., Huang, C., 1988. Nonnegative wealth, absence of arbitrage and feasible consumption plans. Review of Financial Studies 1, 377–402.

Epstein, L.G., Schneider, M., 2001. Recursive Multiple-Priors. University of Rochester, http://www.econ.rochester.edu/Faculty/Epstein.html.

Epstein, L.G., Schneider, M., 2002. Learning Under Ambiguity, University of Rochester.

Epstein, L.G., Wang, T., 1994. Intertemporal asset pricing under Knightian uncertainty. Econometrica 62, 283–322.

French, K.R., Poterba, J.M., 1991. Investor diversification and international equity markets. American Economic Review 81, 222–226.

Gilboa, I., Schmeidler, D., 1989. Maxmin expected utility with nonunique prior. Journal of Mathematical Economics 18, 141–153.

Grinblatt, M., Keloharju, M., 2001. How distance, language and culture influence stockholdings and trades. Journal of Finance 56, 1053–1073.

Hansen, L.P., Sargent, T.J., 2000. Acknowledging misspecification in macroeconomic theory. Sargent's Frisch lecture at the 2000 World Congress of the Econometric Society.

Hansen, L.P., Sargent, T.J., Talliarini, T., 1999. Robust permanent income and pricing. Review of Economic Studies 66, 873–907.

Heath, C., Tversky, A., 1991. Preferences and beliefs: ambiguity and competence in choice under uncertainty. Journal of Risk and Uncertainty 4, 5–28.

Huberman, G., 2001. Familiarity breeds investment. Review of Financial Studies 14, 659–680.

Karatzas, I., 1989. Optimization problems in the theory of continuous trading. SIAM Journal of Control and Optimization 27, 1557–1586.

Karatzas, I., Shreve, S.E., 1987. Brownian Motion and Stochastic Calculus. Springer, New York.

Karoui, El. N., Peng, S., Quenez, M.C., 1997. Backward stochastic differential equations in finance. Mathematical Finance 7, 1–71.

Kilka, M., Weber, M., 1997. Home-Bias in International Stock Return Expectations. University of Mannheim.

Lewis, K., 1999. Trying to explain home bias in equities and consumption. Journal of Economic Literature 37, 571–608.

Morris, S., 1995. The common prior assumption in economic theory. Economics and Philosophy 11, 227–253.
Shiller, R., Kon-Ya, F., Tsutsui, Y., 1996. Why did the Nikkei crash? Expanding the scope of expectations data collection. Review of Economics and Statistics 78, 156–164.
Strong, N., Xu, X., 1999. Understanding the Equity Home-Bias: Evidence from Survey Data. University of Manchester.
Wang, J., 1996. The term structure of interest rates in a pure exchange economy with heterogeneous investors. Journal of Financial Economics 41, 75–110.
Zapatero, F., 1998. Effects of financial innovation on market volatility when beliefs are heterogeneous. Journal of Economic Dynamics and Control 22, 597–626.