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THE STRUCTURE OF INTERTEMPORAL PREFERENCES UNDER UNCERTAINTY AND TIME CONSISTENT PLANS

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1. INTRODUCTION

THIS PAPER EXPLORES the structure of intertemporal preferences necessary for consistent planning. Our setting is that of an individual decision maker endowed with a finite sequence of preference orderings over continuations of contingent consumption plans. That is, for any time $t \in \{0, 1, \dots, T\}$, we have defined a preference ordering R_t over consumption plans for remaining periods $\{t, t+1, \dots, T\}$, where R_t will typically depend upon the past consumption history. Our question may then be phrased: If, at each date, our decision maker could anticipate and plan against any future contingency, what is the required relationship among the family of orderings $\{R_t\}_{t=0,1,\dots,T}$ in order that plans optimal with respect to R_0 remain optimal from the perspective of all succeeding preferences, given intermediate consumption and the realization of uncertainty.

In general, this will simply require that the successive R_t's be exactly and formally identical to R_0 over their common domain—continuations of consumption plans for periods $t, t+1, \dots, T$ with a common consumption history. Since the domain of R_0 includes plans for all contingent events, its restriction to this common domain must thus in general depend upon plans for events not included in the common domain. This means, in effect, that preferences R_t must also depend upon unrealized plans; viz., plans for contingencies which have not occurred and which cannot in the future occur.

In many applications, this latter dependence may not be appropriate. *Accordingly, this paper seeks to characterize those preference orderings which not only admit time consistent planning, but also enjoy the feature that the successive orderings are independent of unrealized alternatives.* Using our notion of consistency, and relying on standard separability results of Debreu [2], Gorman [4], and Koopmans [**8**], we derive a recursive "tree structure" of utility-functions representing preferences having these properties. Such a structure has the analogous certainty results of Strotz [**11**] and Blackborby et al. [**1**] as a special case.

As an immediate consequence of our recursive tree structure, it follows that the customary expected utility representation is not needed for time consistent planning. Thus our discussion runs counter in spirit to the work of Weller [**12**] (and Hammond [**5**]) in which he seeks conditions under which one may conclude that "consistency (in the face of uncertainty) is equivalent to maximizing expected utility on the set of feasible plans..." (Weller [**12**, p. 263]). A study of Weller's [**12**] analysis quickly reveals, however, that his conclusion obtains only if each conditional ordering is assumed to be expected utility representable with respect to the remaining uncertainty.

Lastly, our work clarifies the nonmarket reopening property typically assumed in the standard Arrow-Debreu uncertainty equilibrium paradigm. Our results suggest that the nonmarket reopening property is appropriate only if, once again, agent's preferences also satisfy our state separability condition.

The paper is divided into five sections. In the second we derive our basic preference separability results within a two period setting. In Section 3 we expand our notions of separability to encompass the more specialized sequential structures which have appeared elsewhere in the literature. In Section 4 we compare the works of Kreps and Porteus [**9**], and Selden [**10**] with our own results. Section 5 considers the implications of our work for general equilibrium analysis; concluding comments can be found there as well.

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2. TWO PERIOD PLANNING HORIZON

Let us first consider the case of an individual decision maker (*dm*) drawing up a plan for the next two periods. There is no uncertainty in his decision environment in the current period, while any one of S possible states of nature may obtain in the second period (we will also let S represent the set of states). Let $x \in X$ represent his current action, and $y_s \in Y_s$ his future action given that state $s \in S$ obtains. Both X and Y_s are connected sets. A plan is a $(x, y) \in X \times Y$, where $Y = \prod Y_s$.[2]

Dm's *current* preferences over plans (x, y) are given by the continuous and strictly increasing ordering R of $X \times Y$. His *future* preferences over actions y_s in state s, given past consumption x and planned consumption $y_{\sim s} \in Y_{\sim s} \equiv \prod_{s' \neq s} Y_s$ for states other than s, are described by the ordering $Q^s[x, y_{\sim s}]$; $\{R, Q\} \equiv \{R, (Q^s) s \in S\}$ is said to represent *dm*'s dynamic preferences.

Let us consider *dm*'s dynamic choice problem, as time passes and the state of the world unfolds. Having carried out the current action of his chosen plan and knowing that state s obtains, he is then free to choose any action in the set Y_s. Having ruled out any surprise as to what his remaining options are, if his choice deviates from the original plan, this may be taken as *prima facia* evidence of "changing taste." If, on the other hand, the original plan is carried through whatever state obtains, we may say that *dm*'s taste remains constant. His dynamic preferences $\{R, Q\}$ will then be said to admit *time consistent planning* (*TCP*). We are interested in formalizing the restrictions *TCP* imposes on $\{R, Q\}$. In particular, what are the restrictions on the initial ordering R such that *TCP* follows for a given future ordering $Q = \{(Q^s): s \in S\}$?

Clearly, *TCP* requires that the current preference ordering R, when restricted to Y_s, must agree with the future orderings Q^s for each s. By fixing a reference plan for states other than s, say $(\bar{x}; \bar{y}_{\sim s}) \in X \times Y_{\sim s}$, we get the restricted current ordering $R^s[\bar{x}, \bar{y}_{\sim s}]$ of Y_s, given $(\bar{x}, \bar{y}_{\sim s})$:

DEFINITION: $y_s R^s[\bar{x}, \bar{y}_{\sim s}] y'_s$ if and only if $(\bar{x}, y_s, \bar{y}_{\sim s}) R (\bar{x}, y'_s, \bar{y}_{\sim s})$.[3]

The following two special cases will be important:

History Independence: For each s, $R^s[\bar{x}, \bar{y}_{\sim s}]$ is independent of $\bar{x}$.

Conditional Weak Independence: For each s, $R^s[\bar{x}, \bar{y}_{\sim s}]$ is independent of $\bar{y}_{\sim s}$.

The meaning of each condition will be clear from the following discussion. We may now formalize the notion of "constant taste" or *TCP*:

DEFINITION: $\{R, Q\}$ allow *TCP* if $R^s[\bar{x}, \bar{y}_{\sim s}] = Q^s[\bar{x}, \bar{y}_{\sim s}]$ for every s and every $(\bar{x}, \bar{y}_{\sim s}) \in X \times Y_{\sim s}$.

Thus, $\{R, Q\}$ will satisfy *TCP* only if each future ordering Q^s depends on *everything* which the current induced preferences R^s depend on outside Y_s. In particular, if future preferences are allowed to depend also on unrealized actions (i.e. actions which would have been realized or could be realized had the world evolved differently), then any initial order R will yield *TCP*: For each state s, take as future preferences the induced ordering R^s.

If, in fact, preferences are allowed to depend on unrealizable actions, then we may find four different feasible plans for the next period, $(y_s, y_{\sim s})$, $(y'_s, y_{\sim s})$, $(y_s, y'_{\sim s})$, and $(y'_s, y'_{\sim s})$

[2] Here, and later in our multi-period analysis, we will neglect the obvious economic intertemporal restrictions on *dm*'s choices, therefore using the Cartesian product set representation $X \times Y$. Our consistency conditions therefore are stronger than necessary.

[3] $(\bar{x}, y'_s, \bar{y}_{\sim s})$ is shorthand notation for the plan $(\bar{x}, y)$, with $y_s = y'_s$ and $y_s = \bar{y}_s, (s' \neq s)$.

such that

(2) $(y_s, y_{\sim s})P(y'_s, y_{\sim s})$,

(3) $(y'_s, y'_{\sim s})P(y_s, y'_{\sim s})$,

where P is the strict current preference ordering. (For simplicity, we have suppressed the common current action x.) By definition, we get the following conditional "future" orderings, given state s:

(2′) $y_s P^s[y_{\sim s}]y'_s$,

(3′) $y'_s P^s[y'_{\sim s}]y_s$;

i.e., depending on which action was planned for the states which did not obtain, the ranking of the two alternative actions y_s and y'_s will differ, but be *consistent* with the original ranking. This example should also demonstrate that if preferences are allowed to depend on unrealized alternatives then the typical person in the Allais and related paradoxes would feel no "regret" (i.e. exhibit no inconsistency over time).

Usually, however, one would not want future preferences to depend on unrealized or "irrelevant" alternatives. If so, we must impose the stated conditional weak independence condition on current preferences R for *TCP* to hold:

PROPOSITION 1: *For TCP to hold when future preferences do not depend on unrealizable alternatives, current preferences must satisfy conditional weak independence (CWI).*

In this case, clearly, there are no two pair of feasible plans which will satisfy (2) and (3) simultaneously. Unless otherwise stated, this condition on future preferences will be assumed to hold in the following analysis.

We will use an alternative characterization of *TCP* in terms of "current" and "future" utility functions. By our assumption, R may be represented by a monotone increasing utility function.

$$u: X \times Y \to R, \qquad R \text{ reals.}$$

A trivial modification of a standard result on weakly independent preferences, by e.g., Koopmans [**8**] yields:

PROPOSITION 2: *Initial preferences R admit TCP iff there exists continuous and monotone increasing functions*

$$f: X \times R^S \to R,$$

$$u_s: X \times Y_s \to R, \quad s \in S, \quad \textit{such that}$$

(4) $$u(x, y) = f(x, (u_s(x, y_s)/s \in S)).$$

PROOF: The "if" portion is obvious. For the "only if", use Result A in Koopmans [**8**], given any $x \in X$.

$u_s(x, \cdot)$ represents the induced future preferences $R^s[x]$, given current action x. Equation (4) simply says that "constant taste" requires (the utility of) future actions for each state to enter separately into the current utility function.

Representation (4) admits a wide class of functions which, e.g., are not expected utility representable. To illustrate, consider the following utility function defined over two period consumption streams when three second period states are possible ($|S| = 3$):

(5) $$u(x, y) = u(x, y_1, y_2, y_3) = \{x + u_1(x, y_1) + [u_2(x, y_2)]^{1/3} u_3(x, y_3)\}^{1/2}$$

where $u_1(x, y_1) = \log(x + y_1)$, $u_2(x, y_2) = x^{1/2}y^{1/2}$ and $u_3(x, y_3) = xy_3$. Clearly (5) is not expected utility, yet (by Proposition 2) such an ordering will admit *TCP*.

This should help clarify a result by Weller [**12**]. In particular, he seems to conclude that intertemporal "consistency [therefore] is equivalent to maximizing expected utility" (Weller [**12**, p. 263]). This claim is supported by Hammond [**5**]: "To the extent that an agent's choices depart from multi-period expected utility maximization, that agent is concerned with more than just the consequences of his actions. In particular, consequences which might have occurred but did not may be influencing choice" (Hammond [**5**, p. 56]). Proposition 2 shows both conclusions to be unwarranted.

3. EXTENDED NOTIONS OF PREFERENCE SEPARABILITY

In this section we consider various strengthenings of the preference separability notion underlying Propositions 1 and 2. Firstly, the effect of not allowing future preferences to depend on *dm*'s action history is accounted for. We then introduce two versions of strong preference independence. We should note, however, that while yielding nice and very useful utility representations, these latter properties lack the simple behavioral appeal of weak preference independence.

3.1. *History Independence*

So far, current realized actions have been allowed to affect future preferences. There are a multitude of reasons for such a dependency to arise, among them learning and habit formation (e.g. Hammond [**5**]). If this dependency is properly encoded in current preferences, no problem arises for time consistent planning. However, if preferences at the two dates belong to different generations, for example, it may be appropriate to exclude this dependence while retaining time consistent planning. This is done by combining the above history independence condition with *CWI*, making each restricted ordering R^s independent of any other actions but the remaining ones in Y_s. If so, we may also drop the current action argument in the future utility function $u^s(\cdot)$, to get a stronger version of (4):

$$u(x, y)=f(x, (u_s(y_s);\ s\in S)). \tag{4'}$$

3.2. *Strong Preference Independence*

Under *CWI*, induced preferences over future consumption in any state is independent of consumption in all other states. Conditional Strong Independence (*CSI*) allows the same to hold for any subset of states $e\subseteq S$; i.e. from R define the conditional orderings

$$R^e[\bar{x}, \bar{y}_{\sim e}] \text{ of } Y_e;\ e\subseteq S, \text{ given } (\bar{x}, \bar{y}_{\sim e})\in X\times Y_{\sim e}, \tag{6}$$

analogous to (1). If these orderings are independent of our choice of reference plan $\bar{y}_{\sim e}$ for every e, then preferences R are said to satisfy *Conditional Strong Independence.*

Modifying a standard result on strongly independent preferences, we get the following specialization of the utility representation (4),

$$u(x, y)=f\left(x, \sum_s u_s(x, y_s)\right), \tag{7}$$

where $f(\cdot)$ is unique up to an increasing transformation and increasing in each argument. Each $u_s(x, \cdot)$, on the other hand, is unique up to an affine transformation, and also increasing.[4]

To highlight the distinction between *CWI* and *CSI* compare the earlier example (5) with the following example (8):

$$u(x, y)=\left\{x+\sum_{i=1}^{3} u_i(x, y_i)\right\}^{1/2} \tag{8}$$

[4] Note that the history independence condition above would not guaranteee that we may find "future" utility functions $u_s(\cdot)$ which are independent of current consumption x.

where the $u_i(\cdot)$, $i=1,2,3$, are as in (5). Notice that in example (8) the marginal rate of substitution e.g. between y_1 and y_3 is independent of $y_2(CSI)$, while this is not the case for example (5) earlier.

In general, R need not admit an f-transformation which is affine in its second argument, i.e. which would allow a state-additive version of the "current" utility function itself:

$$u(x, y) = \sum_s u_s(x, y_s). \tag{9}$$

In (6), $\sum_s u_s(x, y_s)$ represents conditional preferences $R[x]$ over future consumption plans $y=(y_s)$, given a current consumption plan x, while in (9) it represents (unconditional) preferences over joint current and future consumption plans (x, y).[5] We refer to representation (9) as the case in which preferences R satisfy (unconditional) *strong independence* (SI). Unfortunately, an axiomatic basis for (9) cannot be provided by extending our earlier standard separability arguments. Such arguments would, in fact require treating current consumption simultaneously as a fixed "given" and as a freely variable object of choice. In Johnsen and Thorlund-Petersen [7], however, an axiomatic basis is provided for (9) by utilizing separability techniques different from those employed here.

In an *expected utility* setting, CSI and SI, respectively, correspond to the utility representations

$$u(x, y) = f\left(x, \sum_s q(s) v(x, y_s)\right), \quad \text{and} \tag{10}$$

$$u(x, y) = \sum_s q(s) v(x, y_s). \tag{11}$$

Here $q(\cdot)$ is a probability measure on S while in (10) $v(x, y_s)$ is a conditional one-period von Neumann-Morgenstern (VNM) utility index given x, and in (11) a two-period VNM index.

The conditional representation (10) requires a state independent second period choice set space $Y_s \equiv Y$, for all $s \in S$, and a state independent conditional second period ordering (i.e. $R^s[x] \equiv \bar{R}[x]$, for all s and x). These conditions imply that the $u_s(x, y_s)$ functions in (8) must be affinely related by

$$u_s(x, y_s) = a_s(x) + b_s(x) v(x, y_s). \tag{12}$$

Thus, if

$$\frac{b_s(x)}{\sum_s b_s(x)} \tag{13}$$

can be made independent of x, we have the expected utility representation (10), with the ratio (13) being interpreted as the probability $q(s)$ of state s. This latter property can be guaranteed by the following condition: Fix current consumption $\bar{x}$. For any pair of states s and s', consider future consumption with a fixed level of consumption for all other states and with identical consumption in states s and s'. Then induced preferences over such plans are independent of $\bar{x}$.

The unconditional expected utility representation (11) can be obtained by extending preferences to the larger choice set consisting of probability mixtures of the original certain × uncertain consumption pairs. This procedure is utilized by, e.g., Fishburn [**3**], Selden [**10**], and Kreps and Porteus [**9**]. An alternative direct derivation utilizing only the given preference ordering is provided by Johnsen and Thorlund-Petersen [**7**].

[5] The $u_s(\cdot)$ functions in (7) and in (9) are of course distinctly different, the former being a (conditional) single-period function while the latter is a two-period utility function.

4. PREFERENCES FOR THE TIMING OF UNCERTAINTY RESOLUTION

An alternative interpretation of the postulated sequence of orderings is that each represents *ex ante* conditional preferences for that date induced by the initial ordering. Under this interpretation, two recent generalizations of the standard expected utility representation of intertemporal preferences—the works of Kreps and Porteus [9] and Selden [**10**]—can be seen to mirror the distinction between our representations (10) and (11). To articulate this distinction properly, however, we need to expand our setting to a T-period planning horizon.[6]

Let s now represent one particular history of the *dm*'s economic environment from date zero to T, where S is the set of all possible such states. Although all uncertainty is resolved by date T, at each earlier date the *dm* may distinguish only among subsets $e \in S$ called events. This defines a sequence $\{P_t\}$ of partitions of S, having increasing fineness, and with $P_0 = \{S\}$, $P_T = S$. At $t = 0$, the *dm* selects a plan $x = (x_0, x_1, \ldots, x_T): S \to R^{L(T+1)}$ specifying his desired consumption $x_t(s)$ for each of the L commodities for each future date t and state s. For each date t and event $e \in P_t$, let x_e be the restriction of any plan x to the states in e, $x_e^{t-1} = (x_{e,0}, \ldots, x_{e,t-1})$ its further restriction to the earlier dates—the consumption history at t—and ${}^t x_e = (x_{e,t}, \ldots, x_{e,T})$ its continuation from t—the yet to be realized part of x. Define $R_t^e[x_e^{t-1}]$ to be the ordering of plan continuations ${}^t x_e$, given consumption history x_e^{t-1}, induced by R_0. By *CWI* $R_t^e[x_e^{t-1}]$ is independent of consumption $x_{\sim e}$ outside event e. Finally, let P_{t+1}^e define the subset of the partition P_{t+1} which contains the events in e—the now feasible events at date $t+1$.

In an analogous setting Kreps and Porteus [9] develop a structure of intertemporal preferences for plans which may be distinguished according to when the uncertainty associated with them will resolve.[7] They define a sequence of utility representable time consistent preference ordering $\{u_t(\cdot)\}$. In order to preserve the necessary dating of when uncertainty resolves, however, they allow the time t expectation to be taken only with respect to the uncertainty resolving at the following date, rather than with respect to all remaining uncertainty as in the standard multiperiod expected utility formulation. This mirrors our distinction between Conditional Strong Independence—each R_t^e having an additive representation across all events in P_{t+1}^e, given past and current consumption x_e^t,

$$u_t^e(x_e) = f_t^e\left(x_e^t, \sum_{e' \in P_{t+1}^e} u_{t+1}^{e'}(x_e)\right) \tag{14}$$

analogous to (10) in Section 3, and Strong Independence—each R_t^e having an additive representation across events in P_{t+1}^e, given past consumption x_e^{t-1} only,

$$u_t^e(x_e) = \sum_{e' \in P_{t+1}^e} v_t^{e'}(x_{e'}). \tag{15}$$

Kreps and Porteus [9] then relate the preference for the timing of risk resolution to the curvature of the transform function $f_t(\cdot)$ with respect to its second argument. In particular, in direct correspondence to our definition of Strong Independence, they have that the decision maker is neutral to the timing of risk resolution, and, thus subscribes to multi-period expected utility if, and only if, $f_t(\cdot)$ is affine in its second argument. On the other hand, $f_t(\cdot)$ is convex (resp. concave) if, and only if, *dm* prefers early (respectively late) resolution of uncertainty. Earlier resolution of uncertainty would imply a finer partition of the state space from some date, e.g. P_{t+1}^e finer. This result should therefore have an intuitive counterpart in our setting.

[6] For a detailed discussion, see Johnsen and Donaldson [6].

[7] Needless to say, our representation in no way captures the complexity involved in the work by Kreps and Porteus [9]. Our simpler structure, on the other hand, may yield some additional insights into their problem of time/risk interactions.

In a similar two period setting, Selden [10] develops a representation form essentially identical to (11) yet from which it is possible to distinguish between "risk" and "time" preferences. The strong separability properties which his representation requires, however, limit its feasibility to the two period setting. The reader is referred to Johnsen and Donaldson [6] for a detailed discussion.

5. CONCLUDING COMMENTS

It is our hope that this paper will place in perspective much of the recent work seeking to characterize consistent planning over time in an uncertain world. As such our contribution is but an exercise in various forms of preference separability, drawing on the fundamental earlier work by Debreu [2] and Gorman [4].

One important implication of our work is to clarify the relationship between consistent planning and the expected utility hypothesis. In particular, we show that the latter condition is much stronger than is necessary for consistent planning.

Our results also have significant implications for general equilibrium theory. We have shown that if preferences are assumed to be "independent of unrealized alternatives," then consistent planning requires that agents' preferences satisfy our recursive date/state separability condition. Consequently, in a multi-period exchange equilibrium setting, the equilibrium allocation across current and future contingent consumption goods will remain optimal in future time periods *only if* preferences satisfy our specialized condition. If this condition is not satisfied, there will be incentives for markets to reopen at future dates as time passes and uncertainty resolves, unlike what is typically assumed.

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