Rince Preferences

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The Quarterly Journal of Economics, Vol. 105, No. 1 (Feb., 1990), 43-60.

Stable URL:
http://links.jstor.org/sici?sici=0033-5533%28199002%29105%3A1%3C43%3ARP%3E2.0.CO%3B2-M

RINCE PREFERENCES*

Roger E. A. Farmer

This paper presents a class of preferences that yield closed-form solutions to dynamic stochastic choice problems. These preferences are based on a set of axioms that were proposed by Kreps and Porteus. The Kreps-Porteus axioms allow one to separate an agent's attitudes to risk from his or her intertemporal elasticity of substitution. RINCE preferences have the properties of RIsk Neutrality and Constant Elasticity of substitution.

I. Introduction

There are many instances of stochastic intertemporal choice problems that one would like to be able to solve in closed form. But it is generally recognized that, if one maintains the axioms of von Neumann and Morgenstern (VNM), such problems quickly become intractable. In this paper I show that a slight weakening of the VNM axioms that were originally explored by Kreps and Porteus [1978, 1979a, 1979b] (henceforth KP) allows one to find a convenient parameterization of utility that may be explicitly solved to yield closed-form decision rules. These decision rules determine optimal actions as functions of current state variables and of the expected values of certain functions of future state variables.

The parametric structure that I propose exploits the fact that KP preferences are able to separate an agent's attitudes to risk from his or her intertemporal elasticity of substitution. This separation allows one to make the simplifying assumption that agents are indifferent to income risk, while maintaining a nontrivial preference for the time at which consumption occurs. A decision maker with the preferences that I describe is risk neutral, in the above sense, but he or she displays a constant elasticity of intertemporal substitution in environments where there is no uncertainty. For this reason, I refer to these preferences as Risk Neutral Constant Elasticity, abbreviated as RINCE.

*An earlier version of this paper was circulated under the title "Closed Form Solutions to Dynamic Stochastic Choice Problems," in May 1987. An earlier version with the current title was also circulated as University of Cambridge discussion paper #121. The current version is considerably modified from both of these previous incarnations, and it includes a substantial amount of new material as well as some corrections in earlier errors. I wish to thank, without implicating, Larry Epstein, Philippe Weil, and two anonymous referees—all of whom have provided valuable feedback. I also wish to acknowledge the kind support of the Risk Project at the University of Cambridge and the National Science Foundation under grant number SES 87 2243.

The KP axioms take the basic space over which preferences are defined to be a space of temporal lotteries. This space is more complex than the space of lotteries over consumption sequences since elements of the space are distinguished not only by probability distributions over possible payoff sequences but also by the time at which uncertainty resolves. This more complicated structure implies that individuals may express a preference or aversion for the resolution of uncertainty even if knowledge of the future does not yield a planning advantage. This structure may be contrasted to the VNM approach under which the time when uncertainty resolves does not directly influence one's choices.

II. The Standard Approach to Intertemporal Stochastic Choice

Consider the problem faced by a mortal consumer who must make a finite sequence of savings decisions when the future is uncertain. In the standard representation of this problem, one assumes that rational choice is characterized by the solution to a dynamic programming problem of the following type:

$$\max_{\{\tilde{c}_t\}_{t=0}^{T}} u(c_0) + E\sum_{t=1}^{T} \beta^t u(c_t) \tag{1}$$

such that

$$a_1 = R_0 a_0 + \omega_0 - c_0; \tag{2}$$

$$a_{t+1} = \tilde{R}_t a_t + \tilde{\omega}_t - c_t; \qquad t = 1,2, \ldots, T; \tag{3}$$

$$R_0 a_0 = \overline{R}_0 \overline{a}_0; \tag{4}$$

$$a_{T+1} \geq 0. \tag{5}$$

The function $U = \Sigma_{t=0}^{T} \beta^t u(c_t)$ may be interpreted as VNM utility function defined over the space of consumption sequences $\{c_t\}_{t=0}^{T}$, where the consumption set is taken to be R_+^{T+1}. The tildes over the variables $\tilde{R}_t$ and $\tilde{\omega}_t$ are used to denote the assumption that they are random variables, and the interpretation of the sequence of constraints (3) is that the individual receives endowments $\{\tilde{\omega}\}_{t=1}^{T}$ which may be invested in a single risky asset. The asset a_t is assumed to pay a gross return $\tilde{R}_t$, and in general I shall allow for the possibility that the sequences $\{\tilde{\omega}\}_{t=1}^{T}$ and $\{\tilde{R}_t\}_{t=1}^{T}$ are jointly distributed random variables that may take values in R_+^{2T}. The expectation operator that appears in equation (1) has the interpretation of an expectation taken over the joint probability distribution of $\{\tilde{\omega}_s,\tilde{R}_s\}_{s=t+1}^{T}$ conditional on the realizations of $(\tilde{\omega}_s,\tilde{R}_s)$ for all $s \le t$.

A solution to equation (1) is represented by a number $\hat{c}_0$ and a sequence of functions $\hat{c}_t$: $R_+^{2t} \mapsto R_+, t = 1, \ldots, T$, where $\hat{c}_t$ is interpreted as a contingent plan. It represents a list of actions, one for every possible realization of past values of ω and R, that the consumer proposes to undertake in period t.

Stated in this way, this problem is a direct application of expected utility theory which has a distinguished history dating back at least to Bernoulli. But the application of expected utility theory to the choice of intertemporal consumption sequences makes no reference to the temporal nature of the consumer's problem. The axioms of atemporal expected utility theory are typically justified by an appeal to simple thought experiments in which it is suggested that a violation of one or other of the VNM axioms would be irrational; the discussion of the Allais paradox in Raiffa [1970, pp. 80 ff] is a good example of this approach. But temporal versions of such arguments are not as compelling as their atemporal counterparts. The KP framework provides a rationalization of a violation of the VNM axioms that can be directly traced to the sequential nature of decisions.

III. The Relationship of the Kreps-Porteus Axioms to those of Von Neumann and Morgenstern

Kreps and Porteus provide two alternative axiomatizations of their approach. One set of axioms views choice as a sequence of decisions. At each stage in the sequence, the agent ranks alternative pairs of payoffs; each such pair consists of a current consumption bundle and a ticket to a lottery that will take place in the following stage. The prizes in the lottery represent the maximum possible utilities that the agent could hope to achieve in different states of nature. In this formulation of the problem, preferences for one-step-ahead lotteries obey the complete set of VNM axioms. The sequence of one-step decision problems is knitted together with a time consistency axiom. KP also provide a second formulation of the agent's preference ordering in which axioms are formulated directly over a space of temporal lotteries. For the sake of completeness, a description of this second approach is provided below.

To describe the KP axioms, it is necessary to introduce some notation. Let d_T be a probability distribution over c_T, and let D_T be the set of all such distributions. One may think of the individual, at the beginning of period T, expressing preferences over lotteries for period T consumption; these lotteries are the elements of D_T. Now imagine the individual who stands at the beginning of period $T - 1$. This person must express preferences over uncertain gambles which may resolve partly in $T - 1$ and partly in period T. In the VNM approach these preferences are defined by formulating axioms over the set of lotteries that yield a compound prize of consumption commodities, part of which is paid in period $T - 1$ and part of which is paid in period T. In the KP approach preferences are defined over a more complicated object; that is, the set of lotteries that yield an uncertain consumption payoff in period $T - 1$ and a ticket to another lottery that takes place in period T. In the absence of the reduction of compound lotteries axiom, these approaches are not identical.

To formalize this idea, one defines recursively the sets, $\{D_t\}_{t=T-1}^{0}$, of probability distributions over $R_+ \times D_{t+1}$. For example, an element of D_{T-1} is a probability distribution, d_{T-1}, which represents the probability of receiving consumption c_{T-1} in conjunction with the lottery ticket d_T. The payoff to the lottery d_{T-2} is the pair (c_{T-2}, d_{T-1}) which is an element of $R_+ \times D_{T-1}$. Carrying this recursion backwards, one arrives at the set of *temporal lotteries* D_0, which is the basic space over which the KP axioms are defined.

An additional piece of notation is required in order to characterize those subsets of D_0 that describe the possible positions at which a decision maker may find him or herself at a given point in time. Let $h_t = \{c_0, c_1, \ldots, c_t\}$ be a consumption *history*. Now define the set $P_t(h_t)$ to consist of those lotteries in D_0 for which the decision maker will receive the history h_t with probability one. An element of $P_t(h_t)$, denoted $p_t(h_t)$, will give the decision maker a nonstochastic consumption sequence, h_t, and a ticket to a lottery $d_{t+1} \in D_{t+1}$. Notice that if one denotes the first k elements of h_t by $h_k(h_t)$, then $P_k(h_k(h_t)) \supseteq P_t(h_t)$. This follows since one of the possible sequences of lotteries that leads from k to t is the sequence in which the decision maker receives the realizations $\{c_{k+1}, \ldots, c_t\}$ with probability one. It follows that $P_{t-1}(h_{t-1}(h_t)) \supseteq P_t(h_t)$ and, by induction, that the sets $\{P_t(h_t)\}_{t=1}^{T}$ are all contained in D_0.

The key difference between the KP and VNM representations of choice hinges on an agent's attitude toward the timing of the resolution of uncertainty. Imagine standing at the beginning of period 0 and choosing between two elements of $P_t(h_t)$ for some $t > 0$. Each of these lotteries contains the same nonstochastic consumption sequence up to date t but possibly different distributions over uncertain events that resolve beyond date t. Now think of mixing two of these lotteries by flipping a coin that comes up heads with probability α and tails with probability $1 - \alpha$ *but flip the coin at date* $k < t$. This new mixture is an element of $P_k(h_k)$, where $h_k \equiv h_k(h_t)$. Let the mixed distribution be represented by the quadruple $(k,\alpha;p,p')$ where p and p' are elements of $P_t(h_t)$. A decision maker whose preferences admit an expected utility representation over consumption sequences must be indifferent to the timing of the coin flip in the experiment described above. A KP individual may, on the other hand, prefer either early or late resolution of uncertainty. The following three axioms characterize KP choice.

A1. There exists a complete transitive ordering, $\succcurlyeq$, over the elements of D_0.

A2. The relation, $\succcurlyeq$, is continuous on D_0.

A3. If $p,p' \in P_t(h_t)$ satisfy $p \succeq p'$, then $(t,\alpha;p,p'') \succeq (t,\alpha;p',p'')$ for all $\alpha \in [0, 1)$ and $p'' \in P_t(h_t)$.

The key axiom A3 is a temporal version of the independence of irrelevant alternatives. Kreps and Porteus [1978, p. 195] present a representation theorem based on axioms A1, A2, and A3. This theorem asserts that one may represent choice by a sequence of utility functions, $\{w_t\}_{t=T-1}^{0}$, each of which maps $R^+ \times R \mapsto R$. The first argument of each function is consumption in period t, and the second argument, denoted v_{t+1}, represents the solution to a programming problem that takes place in period $t + 1$. In general, KP utility functions may be time dependent, and in addition, the individual's utility at date t may depend on his entire consumption history.[1] In the special case in which preferences are *history independent* and in which the utility function $\{w_t\}$ is independent of t, KP preferences admit the following representation:

$$v_{t+1} = \max_{c_{t+1}} w(c_{t+1}, E_{t+1}v_{t+2}).$$

The value of consumption that may be chosen in each of the date t programming problems is constrained by the sequence of budget sets:

$$a_{t+1} = R_t a_t + \omega_t - c_t; \qquad t = T - 1, \ldots, 0.$$

In period T the consumer maximizes a function w_T which is defined

1. Throughout this paper I shall maintain the assumption of *history independence*. Time independence is discussed in more detail in Section VII.

over terminal consumption alone. Given the maximal value of utility in period T, v_T, one can construct the sequence of value functions, $\{v_t\}_{t=T-1}^{0}$, which represents the sequence of maximal utilities attainable in each period. In contrast to the VNM approach these value functions will not generally be linear in probabilities. The relationship with VNM choice is given by the following axiom, which, in conjunction with the other three axioms, implies the existence of a single VNM utility function over the space of intertemporal consumption sequences.

A4. For all t, h_t; $\alpha \in [0,1]$, and $p,p' \in P_t(h_t)$, $(t,\alpha;p,p') \sim (t-1, \alpha;p,p')$.

If axiom A4 holds, then agents are indifferent to the timing of the resolution of uncertainty. In this case their preferences may be reduced to lotteries over intertemporal consumption sequences and KP preferences are identical to VNM. On the other hand—if axiom A4 does not hold—KP preferences define a much broader class of intertemporal stochastic decision rules. In this sense, axiom A4 implies that the difference between KP and VNM choice hinges solely on the issue of preference for, or aversion to, the timing of the resolution of uncertainty.

IV. The Value Function

Stochastic intertemporal choice problems are usually solved recursively by constructing a sequence of value functions. Beginning with the last period of the problem, one finds the optimal decision rule of a planner who enters period T with a given level of wealth. Given this decision rule, one can proceed to find the optimal allocative decision in period $T-1$ and, working backwards, one constructs a sequence of decision rules and an associated sequence of value functions. In the case of the expected utility example, equation (1), the sequence of value functions $\{v_t(a_t)\}_{t=0}^{T}$ is defined by the formulae:[2]

$$v_T \equiv u(R_T a_T + \omega_T); \tag{6}$$

$$v_t(a_t) \equiv \max_{c_t} \{u(c_t) + \beta E_t[\tilde{v}_{t+1}(a_{t+1})]\}; \tag{7}$$

2. It is worth pointing out that the value functions depend, not only on a_t, but also on the entire history of wages and interest rates. This dependence follows since (i) this history conditions the expectation E_t, and (ii) the relevant wealth variable depends on the current wage and the current interest rate.

such that

(8) $$a_{t+1} \equiv R_t a_t + \omega_t - c_t; \qquad t = 0,1, \ldots, T-1.$$

A great deal is known about the properties of the functions $\{v_t\}$, and for special cases one may obtain closed-form solutions for the optimal decision rules. By restricting attention to the case of multiplicative uncertainty (random interest but deterministic endowments), one may obtain closed-form solutions to the class of preferences $u(c_t) = c_t^\rho/\rho$. On the other hand, with only additive uncertainty (random endowment but deterministic interest rates) one can solve the quadratic case. But the general case of random interest *and* random endowments does not admit a closed-form solution except in the trivial situation when w is an affine function. In this case the agent's preferences are linear, not only across states of nature, but also through time.

If one is willing to drop the assumption of timing indifference, then the weaker axiom set A1–A3 implies that the choice of intertemporal consumption sequences admits a value function representation where the value functions are defined as follows:

(9) $$v_t(a_t) \equiv w_T(R_T a_T + \omega_T);$$

(10) $$v_t(a_t) \equiv \max_{c_t} w(c_t, E_t[\tilde{v}_{t+1}(a_{t+1})]);$$

such that

(11) $$a_{t+1} = R_t a_t + \omega_t - c_t; \qquad t = 0,1, \ldots, T-1.$$

Equation (10) differs from the VNM approach (equation (7)) in that v_t is nonlinear in the expectation operator E_t. This generalization would appear to complicate the problem and make things more, rather than less, difficult. However, by choosing the function w correctly, one can find a class of decision problems that yield closed-form solutions in a wide variety of situations.

I shall return to the value function approach in Section VII in which I define a class of preferences that admit closed-form representations for the sequence of functions $\{v_t\}$. Before taking up this issue, however, I shall explore an alternative representation of choice that permits a more direct comparison of the KP approach with the expected utility framework. This representation is the KP analogue of the expected utility index.

V. The Utility Index

In this section I introduce the appropriate notion of the utility index for KP choice. In the case of VNM preferences the utility index is a function that takes, as its domain, the cartesian product of the real line with the space of probability distributions over R_+^T. Current consumption is an element of R_+, and lotteries over future consumption sequences are elements of the set of probability distributions over R_+^T. Decision making under uncertainty is frequently represented as the choice of a set of contingent plans that maximizes such an index subject to a sequence of constraints, that is, in the form of equation (1).

From the perspective of a decision maker at date 0, the utility index for this problem is given by the function,

$$U_0 = u(c_0) + E_0 \sum_{t=1}^{T} \beta^t u(c_t). \tag{12}$$

Because this function is both separable through time and linear in probabilities, one can ignore past choices if plans are reformulated at a later date. That is to say, a decision maker at date τ who uses the index,

$$U_\tau = u(c_\tau) + E_\tau \sum_{t=\tau+1}^{T} \beta^t u(c_t), \tag{13}$$

will make decisions that are consistent with the plans that were formed at date 0 to maximize the index U_0. Linearity in probabilities and separability through time are sufficient but not necessary conditions to guarantee consistent planning. KP preferences are also time consistent, but the KP utility index is not linear in probabilities; it is constructed recursively.

Recall the definition of the sequence of sets $\{D_t\}_{t=0}^{T}$ that was introduced in Section IV. One may define a utility index $U_T{:}R_+ \mapsto R$ and a sequence of indices $\{U_t\}_{t=T-1}^{0}$, using the following recursion:

$$U_T = w_T(c_T); \tag{14}$$

$$U_t = w(c_t, E_t\tilde{U}_{t+1}), \qquad t = 0,1, \ldots, T-1. \tag{15}$$

The index U_t maps from the space $R_+ \times D_{t+1}$ to the real line, and it is the KP analogue of the VNM index defined in equation (12). The structure of this index is closely related to a class of preferences over nonstochastic sequences that Lucas and Stokey [1984] refer to as *recursive*. Koopmans [1960] was the first to study preferences in this class, and in view of the similarity of equation (15) to the Koopmans class I shall refer to $w{:}R_+ \times R \mapsto R$ as an *aggregator function*. Recursive preferences are easy to study because they allow one to construct a solution to a programming problem in steps using Bellman's principle of optimality.

One is entitled to ask why the theory of choice under uncertainty should be complicated by introducing the concept of temporal lotteries. Why not stick to the more basic choice objects, that is, to lotteries over consumption sequences? The answer is that the concept of temporal lotteries allows one to separate risk from intertemporal preferences while retaining the very useful property of recursivity.

What would happen if the assumption of recursivity were to be relaxed? A natural way of generating a change in an agent's attitude to risk without affecting his or her ordinal ranking of nonstochastic sequences would be to apply the multicommodity analysis discussed in Kihlstrom and Mirman [1974] to the space of distributions over intertemporal consumption sequences. The Kihlstrom-Mirman approach is to define a family of utility functions U_F by taking increasing, concave transformations H_F of a basic utility function U. In the intertemporal context the decision maker would solve the problem,

$$\max_{\{c_t\}t=0} E_0 H_F[U(c_0,c_1,\ldots,c_T)]. \tag{16}$$

By varying the curvature of H_F, one could make the individual more or less risk averse without changing his or her preferences over nonrandom sequences. But the cost of this approach is that an agent's relative ranking of choices at date t necessarily depends on the entire history of past consumptions *and* on all of the possible choices that might be made in the future. KP preferences allow one to break the link between risk and intertemporal substitution without giving up on recursivity.

Recursive preferences are defined in the nonstochastic environment by the assumption that the decision maker's ranking over future consumption sequences is independent of his or her ranking over current consumption bundles. The natural extension of this property to choice over temporal lotteries leads to the sequence of recursive indices defined by equations (14) and (15). It is the property of independence of future decisions from events that have occurred in the past that allows one to apply the maximum principle of dynamic programming to choice problems with a recursive structure.[3]

3. Note that the application of dynamic programming requires the assumption of history independence, in addition to the basic KP axioms.

VI. A Homogeneous Class of Preferences

In this section I introduce a class of preferences for which the utility index U_t is homogeneous of degree γ in current consumption and in the value of future state dependent consumption. This class, which has been proposed by Epstein and Zin [1989], has the convenient property of allowing the intertemporal elasticity of substitution and the coefficient of relative risk aversion to be represented by two separate parameters.[4] It is capable of capturing the behavior either of an individual who prefers early resolution of uncertainty or of one who prefers late resolution. These preferences are defined by choosing the functions w_T and w in equations (14) and (15) to be given by

$$w_T = z^\gamma \tag{17}$$

$$w(x,y) = (x^\rho + \beta y^{\rho/\gamma})^{\gamma/\rho}. \tag{18}$$

The case in which $\gamma = 1$ is the case that defines RINCE preferences, and it is the only member of this class[5] for which one can obtain closed-form solutions to intertemporal stochastic choice problems when there is both rate of return and endowment uncertainty. For the special case in which there is no uncertainty, the KP utility index that is induced by equations (17) and (18) takes the degenerate form,

$$U_0 = \left(\sum_{t=0}^{T} \beta^t c_t^\rho\right)^{\gamma/\rho}. \tag{19}$$

The parameter γ defines a family of utility functions, each member of which has the same ordinal properties. The parameter ρ, on the other hand, captures the intertemporal curvature of these functions; ρ is related to the intertemporal elasticity of substitution η by the relationship,

$$\eta \equiv \frac{\partial \log (c_t/c_{t+1})}{\partial \log R_{t+1}} = \frac{1}{\rho - 1}. \tag{20}$$

In situations for which uncertainty is nontrivial, the parameter γ

4. Philippe Weil [1990] has proposed a related class of preferences which is generated by a nonhomogeneous aggregator. My initial work on RINCE preferences was developed prior to the Epstein-Zin and Weil papers. However, since the Epstein-Zin and Weil preferences contain RINCE as a subclass, they provide a useful framework for explaining what is special about RINCE if one is interested in closed-form solutions.

5. This is a conjecture. I do not have a proof of the nonexistence of some other class that can be easily solved, but I have been unable to find a counterexample.

captures the decision maker's attitude toward risk. Its effect follows from the presence of terms of the form,

$$(E_t x^{\gamma/\rho})^{\rho/\gamma},$$

in the recursive equations that are used to construct the utility index U_0. If there is no uncertainty, then these terms collapse. The special case of $\gamma = 1$ corresponds to a type of risk neutrality, and it is this property that enables one to generate closed-form solutions.

Kreps and Porteus show that the decision maker will prefer early (late) resolution of uncertainty if the aggregator function $w(x,y)$ is convex (concave) in its second argument. For the class of homogeneous preferences described above, it follows that a preference for early (late) resolution occurs if $\rho > \gamma (\rho < \gamma)$. Since ρ is bounded above by 1, it follows that the risk-neutral decision maker must prefer late resolution.

What is going on here? A priori—it seems plausible that a risk-neutral agent could also prefer early resolution of uncertainty. But our concept of risk neutrality, in which the value function is linear in the appropriate measure of expected human wealth, excludes this possibility. What seems to be happening is that the curvature of the period utility function, when preferences are von Neumann-Morgenstern, provides a natural planning advantage to early resolution. That is, VNM preferences generate value functions in which an agent has a natural preference for early resolution of income lotteries. In order to counteract this natural tendency for prefering early resolution, the agent's preferences over consumption lotteries must incorporate a basic desire for late resolution.[6]

VII. The Parameterization of RINCE Preferences

In this section I describe the class of preferences that I call RINCE, and in Section VIII I derive an exact solution for the sequence of consumption decisions that would be taken by a decision maker whose preferences were of this type. RINCE preferences are members of the homogeneous class described in Section

6. Although a preference for late resolution of uncertainty is not necessarily unreasonable, it may have some counterintuitive implications. For example, a referee has pointed out to me that, in the case in which $T = 1$ (2 periods), all wealth is nonhuman, $\rho > 1/2$, and $\gamma = 1$, it can be shown that an agent would strictly prefer not to be told the realization of the rate of interest before making his or her consumption decision for the first period. This preference occurs even though the information would be used to alter the agent's decision if it were available. One must be careful, when using RINCE preferences, to make sure that the context in which they are applied makes economic sense.

VI for which the homogeneity parameter γ is equal to one. The decision rule that describes optimal behavior in any period is constructed by solving the sequence of value functions described in equations (9), (10), and (11) when the functions w_T and w are given by

$$w_T(z) = z \tag{21}$$

$$w(x, y) = (x^\rho + \beta y^\rho)^{1/\rho}; \tag{22}$$

where $\beta \in [0,\infty)$ and $\rho \in [-\infty,0) \cup (0,1]$.[7]

When the joint distribution of $\{\omega,R\}_{t=1}^T$ is degenerate (the case of no uncertainty), this aggregator generates the following utility function:

$$U = \left(\sum_{t=0}^{T} \beta^t c_t^\rho\right)^{1/\rho}. \tag{23}$$

As ρ tends to 0, this function becomes infinite. To find the correct parameterization for the case of $\rho = 0$, one must normalize the coefficients of each element of $\{c_t\}_{t=0}^T$ by $\Sigma_{t=0}^T \beta^t$ and apply L'Hospital's rule. The normalized utility function is given by

$$U = \left(\frac{\sum_{t=0}^{T} \beta^t c_t^\rho}{\sum_{t=0}^{T} \beta^t}\right)^{1/\rho}, \tag{24}$$

which, for the case of $\rho = 0$, becomes

$$U = \prod_{s=0}^{T} c_s^{(\beta^s/\Sigma_{v=0}^T \beta^v)}. \tag{25}$$

The aggregator that delivers this representation is the following one:

$$w_t(x,y) \equiv [(1 - \delta_t)x^\rho + \delta_t y^\rho]^{1/\rho}; \tag{26}$$

where the coefficient δ_t is defined by

$$\delta_t \equiv \left(1 - \frac{1}{\sum_{s=0}^{T-t} \beta^s}\right). \tag{27}$$

7. The restriction to discount rates less than one is not necessary for the finite horizon case.

The time-dependent aggregator given by equation (26) generates a sequence of utility indices, each of which is equivalent (up to a linear transformation) to the sequence of indices that is generated by the time-independent aggregator given in equation (22). When there is no uncertainty, the date 0 values of these utility indices are given by equations (24) and (23), respectively. Since the time-independent parameterization is notationally more compact, it seems preferable to work with the time-independent aggregator, equation (22), when describing preferences for which $\rho \neq 0$.

The Cobb-Douglas case is more complicated, however, since one must use the time-dependent paramaterization in order to apply L'Hospital's rule.[8] For the case in which $\rho = 0$, the aggregator function given in equation (26) reduces to

$$w_t(x,y) = x^{(1-\delta_t)} y^{\delta_t}; \tag{28}$$

where δ_t is defined above. When there is no uncertainty, this aggregator generates the utility function in equation (25) which is ordinally equivalent to the familiar example of logarithmic preferences with exponentially declining weights; however, there does not exist an aggregator function that is both linearly homogeneous and time independent which generates an equivalent ordinal representation of utility. To preserve ordinal properties, one is restricted to aggregators that generate increasing monotonic transformations of (25). But to maintain linear homogeneity of the aggregator, one is further restricted to linear transformations of (28). The unique (up to a linear transformation) time-independent Cobb-Douglas aggregator is of the form,

$$w(x,y) = x^{1-\beta} y^{\beta}, \tag{29}$$

which generates the following utility function:

$$U = \left(\prod_{s=0}^{T-1} c_s^{(1-\beta)\beta^s} \right) c_T^{\beta T}. \tag{30}$$

This is *not* ordinally equivalent, for any finite horizon, to the familiar logarithmic representation; which takes the form,

$$U = \sum_{t=0}^{T} \beta^t \log c_t. \tag{31}$$

8. Recall that L'Hospital's rule applies to the limit of the ratio of two functions, each of which converges to zero (or infinity). The coefficients on x and y must sum to unity in order for the numerator of the CES aggregator to converge to zero as ρ converges to 0.

The representation in equation (30) achieves time independence of the aggregator by placing more weight (than the logarithmic representation) on the final period of life.[9]

VIII. The Solution to the Stochastic Choice Problem When the Decision Maker Has RINCE Preferences

In this section I present a set of functional equations that describes the solution to the general problem of choice under uncertainty described in Section IV. These equations may be used to compute recursively a sequence of value functions $\{v_t\}_{t=T}^{0}$ and a sequence of decision rules for consumption $\{c_t\}_{t=T}^{0}$ as functions of the assets of the decision maker and of the moments of the joint distribution of the sequence of future endowments and future interest rates.

The solution presented below is parameterized separately for the CES and Cobb-Douglas cases since, in the CES case, I have exploited the existence of a time-invariant aggregator. Specifically, I assume that the decision maker's preferences are generated recursively by the sequence of functions:

$$w(x,y) \equiv (x + \beta y^{\rho})^{1/\rho}, \qquad t = T-1, \ldots, 0, \tag{32}$$

where $\rho \in [-\infty,0) \cup (0,1]$. For the case $\rho = 0$ the aggregator is given by

$$w_t(x,y) \equiv x^{1-\delta_t} y^{\delta_t}, \qquad t = T-1, \ldots, 0, \tag{33}$$

where

$$\delta_t \equiv \left(1 - \frac{1}{\sum_{s=0}^{T-t} \beta^s}\right). \tag{34}$$

For all values of ρ, the terminal condition sets $w_T(z) \equiv z$.

Before providing explicit functional forms for the decision rules that determine $\{c_t\}_{t=0}^{T}$, it helps to define two new functions, F and $G{:}R_+ \mapsto R_+$:

$$F(x) = \begin{cases} (1 + \beta^{1/(1-\rho)} x^{\rho/(1-\rho)})^{(1-\rho)/\rho}; & \text{if } \rho \neq 0; \\ (1-\delta_t)^{1-\delta_t} \delta_t^{\delta_t} x^{\delta_t}; & \text{if } \rho = 0; \end{cases} \tag{35}$$

9. The same comment applies to the following time-independent CES aggregator $[(1-\beta)x^{\rho} + \beta y^{\rho}]^{1/\rho}$ which generates a CES utility function with constantly declining weights in all but the final period of life. It is this version of the CES aggregator that converges to equation (29) as ρ converges to zero.

$$(36) \qquad G(x) = \begin{cases} (1 + \beta^{1/(1-\rho)} x^{\rho/(1-\rho)})^{-1}; & \text{if } \rho \neq 0; \\ 1 - \delta_t; & \text{if } \rho = 0; \end{cases}$$

where δ_t is the function of β defined in equation (34).

These functions appear repeatedly in the solution to the agent's decision problem. For the case of $\rho \neq 0$ they have been derived from the time-independent CES aggregator, equation (32), by solving the problem explicitly, beginning with period T and working backwards. For the Cobb-Douglas case they are worked out for the time-dependent formulation, equation (33). There is a formulation of the CES case, using the time-dependent aggregator (26), which involves functions that converge to the Cobb-Douglas forms of F and G as ρ converges to zero. However, the time-dependent CES formulation of the problem may be written, by transforming variables, in a form that involves time-invariant coefficients.[10]

Using the above definitions, one may write the decision rule for consumption in terms of two variables that resemble a compounded interest rate and a human wealth term. However, this analogy is not exact since these variables involve the parameters of the utility index. More precisely, the sequences of interest terms $\{Q_t\}_{t=0}^{T}$ and human wealth terms $\{h_t\}_{t=0}^{T}$ are defined recursively as follows:

$$(37) \qquad F(Q_T) = 1;$$

$$(38) \qquad Q_t = E_t[\tilde{R}_{t+1} F(\tilde{Q}_{t+1})], \qquad t = 0, \ldots, T-1;$$

$$(39) \qquad h_T = \omega_T;$$

$$(40) \qquad h_t = \omega_t + E_t[\tilde{h}_{t+1} F(\tilde{Q}_{t+1})/Q_t], \qquad t = 0, \ldots, T-1.$$

In the Cobb-Douglas case the functions F and G are time dependent since they depend on the parameter δ_t; this dependence has been suppressed in the above formulation which holds for all values of ρ in the interval $[-\infty, 1]$.

10. Specifically, this transformation takes the form,

$$Q_t^* = Q_t(1 - \delta_t)^{-1/\rho},$$

where the variable Q_t is defined by (38) and (37). The functions F and G, for the time-dependent case are given by

$$F_t(x) = [(1 - \delta_t)^{1/1-\rho} + \delta_t^{1-\rho} x^{\rho/1-\rho}]^{(1-\rho)/\rho};$$

$$G_t(x) = \frac{1 - \delta_t}{[(1 - \delta_t)^{1/1-\rho} + \delta_t^{1/1-\rho} x^{\rho/1-\rho}]}.$$

The variable Q_t depends on the moments of the distribution of all future interest rates and on the preference parameters β and ρ. Notice from equation (40) that the terms $F(\hat{Q}_{t+1})/Q_t$ act as stochastic discount rates on the future endowment sequence $\{\omega_t\}$. The term h_t may be thought of as "perceived human wealth" in view of the analogous role that it plays to human wealth in the nonstochastic case. One may then define total wealth W_t as

$$(41) \qquad W_t \equiv Ra_t + h_t, \qquad t = 0,1, \ldots, T.$$

W_t consists of the market value of physical assets, plus the subjectively discounted value of future endowments. Given these definitions, the sequences of decision rules $\{c_t\}_{t=0}^T$ and value functions $\{\upsilon_t\}_{t=0}^T$ are given by the following equations;

$$(42) \qquad c_t = G(Q_t)W_t, \qquad t = 0,1, \ldots, T;$$

$$(43) \qquad \upsilon_t = F(Q_t)W_t, \qquad t = 0,1, \ldots, T.$$

The system of equations (37)–(41) gives explicit rules for determining the values of the variables Q_t and W_t in terms of the conditional moments of the joint endowment-return process $\{\tilde{\omega}_s, \tilde{R}_s\}_{s=t+1}^T$. One may therefore summarize the behavior of an individual with preferences of this type by keeping track of two rather simple functional equations.

Some special cases of this model may prove helpful in establishing the relationship of these preferences to more familiar examples of nonstochastic utility functions. Over nonstochastic choice problems the RINCE decision maker will behave exactly like a VNM individual whose preferences are described by the additively separable function,

$$(44) \qquad U = \sum_{t=0}^{T} \frac{\beta^t c_t^\rho}{\rho}.$$

For problems of this kind the utility index of RINCE preferences reduces to the function given in equation (19) when the parameter γ is set equal to one. In situations of risky choice, however, it follows from equations (42), (41), and (40) that the decision rule of the RINCE agent is linear in probabilities and that it is only the first moment of the one-step-ahead endowment that affects his or her consumption choice. It is in this sense that RINCE preferences display risk neutrality.

A second case that is of interest is that in which the sequences $\{\tilde{R}_t\}_{t=1}^T$ and $\{\tilde{\omega}_t\}_{t=1}^T$ are independent of each other and in which each of these sequences is independently distributed through time. In this case the variable h_t is given by the expression,

$$(45)\quad h_t = \omega_t + \frac{E_t[\tilde{\omega}_{t+1}]}{E_t[\tilde{R}_{t+1}]} + \frac{E_t[\tilde{\omega}_{t+2}]}{E_t[\tilde{R}_{t+1}]E_t[\tilde{R}_{t+2}]} + \cdots \frac{E_t[\tilde{\omega}]}{\Pi_{s=1}^{T-t}E_t[\tilde{R}_{t+s}]}.$$

If ω_t and R_t are nonstochastic, then this expression reduces to the familiar definition of human wealth. The expression for consumption given by equation (42) is, in this case, identical to the expression that is given by the "constant relative risk aversion" preferences described in equation (44). In the case of $\rho = 0$ and the stochastic but independent endowment-return processes, the consumption function takes the form,

$$(46)\qquad c_t = (1 - \delta_t)W_t,$$

where W_t is the sum of human and nonhuman wealth terms and human wealth is obtained by discounting the first moments of the endowment process by the first moments of the return process—equation (45). In the case where interest rates are serially correlated, the discount factor will no longer be equal to the mean of the return process because interest rates contain information about the future. Serial correlation will carry a "resolution premium" which reflects the agent's basic preferences over the timing of the revelation of information.

As the agent's horizon becomes longer, the time dependence for the Cobb-Douglas case becomes less important and, as $t \rightarrow \infty$, the parameter δ_t converges to β. This limiting case is of special interest since it provides an exact representation of Friedman's permanent income hypothesis which is valid for partial-equilibrium problems. By exploiting the separation between risk aversion and intertemporal elasticity of substitution that is provided by the KP structure, RINCE preferences are able to incorporate the simplifying assumption that agents are risk neutral without trivializing intertemporal choice.

Appendix

This appendix provides a sketch of the proof that the closed-form solution to the value function described in the text is valid. The proposed solution for v_t is given by

$$(47)\qquad v_t = F(Q_t)W_t.$$

Taking expectations of v_t at $t - 1$ using the identity (41) and the asset accumulation rule, one obtains

$$(48) \quad E_{t-1}(\tilde{v}_t) = E_{t-1}[\tilde{R}_t F(\tilde{Q}_t)]\,[R_{t-1}a_{t-1} + \omega_{t-1} - c_{t-1}] + E_{t-1}[\tilde{h}_t F(\tilde{Q}_t)],$$

which simplifies, using definitions (38) and (40), to

$$(49) \qquad E_{t-1}(\tilde{v}_t) = Q_{t-1}(W_{t-1} - c_{t-1}).$$

By substituting (49) into equation (10) and using the functional form (22) for w, one obtains the first-order conditions,

$$(50) \qquad c_{t-1}^{\rho-1} - Q_{t-1}\beta[Q_{t-1}(W_{t-1} - c_{t-1})]^{\rho-1} = 0,$$

which may be rearranged to give the functional form (42), using the definition of G given in equation (36). By substituting the solution for c_{t-1} at a maximum (equation (42)) into the function w, one obtains the expression,

$$(51) \qquad v_{t-1} = F(Q_{t-1})W_{t-1}.$$

This establishes that if (47) is a correct representation of the value function at t, then it is also correct at $t - 1$. One completes the proof by establishing that v_{T-1} is described by (47) given the definition of v_T in equation (43).

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REFERENCES

Epstein, L. G., and S. Zin, "Substitution, Risk Aversion and the Temporal Behavior of Consumption and Asset Returns I: A Theoretical Framework," *Econometrica*, LVII (1989), 937–69.

Kihlstrom, R. E., and L. J. Mirman, "Risk Aversion with many Commodities," *Journal of Economic Theory*, VIII (1974), 361–88.

Koopmans, T., "Stationary Ordinal Utility and Impatience," *Econometrica*, XXVIII (1960), 287–309.

Kreps, D. M., and E. L. Porteus, "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica*, XLVI (1978), 185–200.

——, and ——, "Dynamic Choice Theory and Dynamic Programming," *Econometrica*, XLVII (1979a), 91–100.

——, and ——, "Temporal von Neumann-Morgenstern and Induced Preferences," *Journal of Economic Theory*, XX (1979b), 81–109.

Lucas, R. E., and N. Stokey, "Optimal Growth with Many Consumers," *Journal of Economic Theory*, XXXII (1984), 139–71.

Raiffa, H., *Decision Analysis* (Reading, MA: Addison Wesley, 1970).

Weil, P., "Nonexpected Utility and Macroeconomics," *Quarterly Journal of Economics*, CV (1990) 29–42.