1 Overview

Let's first look at standard preferences – additive expected utility and define some notation. Time is discrete, $t = 0, 1, ...T$. At each $t \geq 0$, an event z_t is drawn from a set Γ_t. A history, denoted z^t is a collection of events up to an including z_t, i.e., e.g., $\{z_0, z_1, ..., z_t\}$. The set of possible histories at t is Γ^t.

Let $c(z^t)$ denote the vector of consumption goods (possibly including service flows and leisure) consumed in period t if z^t is realized.Then, the standard assumption is that for any t, preferences over $\{c(z^t)\,|z^t \in \Gamma^t\}_{t=0}^T$ with associated probabilities $p(z^t)$ can be represented by the function

$$U\left(\left\{c\left(z^t\right)|z^t \in \Gamma^t\right\}_{t=0}^T\right) = \sum_{t=0}^T \beta^t \sum_{z^t \in \Gamma^t} p\left(z^t\right) u\left(c\left(z^t\right)\right) = E\sum_{t=0}^T \beta^t u\left(c\left(z^t\right)\right)$$

where u is a smooth increasing concave function, called the per-period utility or felicity function. Several important features of these preferences will be relaxed during the course.

1. Curvature of u determines elasticity of substitution both between states within a period (inverse of risk aversion) and between periods, intertemporal elasticity of substitution. Therefore, these intrinsically different concepts cannot be disentangled theoretically.

2. No preference for time revelation – it does not matter when we get to know our faith (unless, of course, if we can condition our actions on the information).

3. No history effect; the utility

$$E\sum_{t=s}^T \beta^t c\left(z^t\right)$$

   is independent of consumption taking place before s. (However, history of course matters by affecting the budget, i.e., the consumption possibility set, usually simply captured by a state variable like wealth.

4. Time consistency; if

$$U\left(c_1, \left\{c\left(z^t\right)\right\}_{t=2}^T\right) \geq U\left(c_1, \left\{c'\left(z^t\right)\right\}_{t=2}^T\right)$$

   then

$$U\left(\left\{c\left(z^t\right)\right\}_{t=2}^T\right) \geq U\left(\left\{c'\left(z^t\right)\right\}_{t=2}^T\right),$$

   i.e., if the sequence of consumption $\{c(z^t)\}_{t=2}^T$ is preferred over $\{c'(z^t)\}_{t=2}^T$ at $t = 1$, it is preferred also at $t = 2$.

5. Risk matters only through second order effects. E.g., suppose in period t, there is a lottery with payoff $\tilde{x}$ and $E\tilde{x} > 0$. Then all individuals will prefer to hold a strictly positive amount of this lottery over having a certain consumption level $\bar{c}$. I.e., there is a $\bar{k} > 0$ such that for all $k \in (0, \bar{k})$
$$E\left(u\left(\bar{c} + k\tilde{x}\right)\right) > u\left(\bar{c}\right).$$
To see this, let's take the derivative of the left hand side w.r.t. k and evaluate at $k = 0$.
$$\begin{aligned} \frac{\partial\left(E\left(u\left(\bar{c} + k\tilde{x}\right)\right)\right)}{\partial k}|_{k=0} &= E\left(u'\left(\bar{c} + k\tilde{x}\right)\tilde{x}\right)|_{k=0} \\ &= u'\left(\bar{c}\right)E\tilde{x} \end{aligned}$$
which is positive if $E\tilde{x} > 0$, regardless of riskaversion.

Empirical findings, introspection and lab experiments have all shown that these implications are often invalidated.

1.1 Two important macro-puzzles.

1. **Equity premium puzzle**
2. **Too little insurance puzzle.**

Most well-known example. Mehra-Prescott, "equity premium puzzle". Consider the following example (almost) in there original formulation and we will return to it later. Suppose consumption grows fast with $p = 1/2$ and slowly otherwise. Starting from a consumption level c_t, history at $t+1$ is either $z^{t+1} = z_h$ in which case $c_{t+1} = c_t\left(1 + g + \sigma\right)$ or $z^{t+1} = z_l$ and $c_{t+1} = c_t\left(1 + g - \sigma\right)$. We can now using the standard Euler condition for maximizing utility to price a bond that gives 1 unit of consumption in both cases and a share that gives $1+g+\sigma$ or $1+g-\sigma$. The share is thought to be a claim to the process that provides the consumption possibilities. Individuals eat only apples. Apples are grown on a tree (or a number of identical trees(which has exogenous but stochastic output that grows with a rate $g + \sigma$ or $g - \sigma$. A share is ownership of the tree.

The Euler equation is
$$u'\left(c_t\right) = \beta E u'\left(c_{t+1}\right)R_{t+1}.$$

Well known intuition, give up one unit of consumption "costs" $u'\left(c_t\right)$, invest it and increasing consumption by R_{t+1} next period gives an increase in utility given by $\beta E u'\left(c_{t+1}\right)R_{t+1}$.

We find asset prices by first defining state-contingent prices of Arrow-Debreu type. In period t there is one lottery that gives 1 unit of consumption in $t+1$ if growth is high and zero otherwise. There is also a lottery that gives one unit of consumption in the low growth state. The price of these lotteries and p_h and p_l and the returns $\frac{p(z_h)}{p_h}$ and $\frac{p(z_l)}{p_l}$ respectively. Therefore,

$$\begin{aligned} p_h &= \beta \frac{u'(c(z_h))\, p(z_h)}{u'(c_t)} \\ p_l &= \beta \frac{u'(c(z_l))\, p(z_l)}{u'(c_t)} \end{aligned}$$

Using the standard CRRA felicity function

$$\begin{aligned} u(c) &= \frac{1}{\alpha} c^{\alpha} \\ \alpha &\leq 1, \end{aligned}$$

recalling that the coefficient of RRA is

$$-c\frac{u''(c)}{u'(c)} = -c\frac{(\alpha-1)\, c^{\alpha-2}}{c^{\alpha-1}} = 1-\alpha.$$

Setting $p(z_h) = p(z_l) = \frac{1}{2}$ we have

$$\begin{aligned} p_h &= \beta \frac{(1+g+\sigma)^{\alpha-1}}{2}, \\ p_l &= \beta \frac{(1+g-\sigma)^{\alpha-1}}{2}. \end{aligned}$$

A portfolio consisting of one each of the lotteries mimics perfectly the safe one-period bond. The price of a bond is thus

$$p_b = p_h + p_l$$

with a return,

$$r_b = \frac{1}{p_h + p_l}.$$

Let us now compute the return on a claim to next periods dividends – a one-period share. A portfolio consisting of $(1+g+\sigma)$ of the h lottery and $(1+g-\sigma)$ of the l lottery exactly replicates such a one period share.

The price of this risky portfolio is

$$p_r = p_h\left((1+g+\sigma)\right) + p_l\,(1+g-\sigma)$$

and its expected return is therefore

$$
\begin{aligned}
r_r &= \frac{1+g}{p_h\left((1+g+\sigma)\right)+p_l\left(1+g-\sigma\right)} \\
&= \frac{1+g}{\beta\frac{(1+g+\sigma)^{\alpha-1}}{2}\left(1+g+\sigma\right)+\beta\frac{(1+g-\sigma)^{\alpha-1}}{2}\left(1+g-\sigma\right)} \\
&= \frac{2\left(1+g\right)}{\beta\left(\left(1+g+\sigma\right)^{\alpha}+\left(1+g-\sigma\right)^{\alpha}\right)}
\end{aligned}
$$

In this simple economy, the return on a more standard type of share, i.e., a one that gives rights to all future dividends is the same. Why? To see this, we recall that the price of the share with CARA utility will be proportional to current income/consumption. The price of the share will therefore be $P_r c_t$. and the return

$$
\frac{P_r c_{t+1}+c_{t+1}}{P_r c_t}=\frac{\left(1+P_r\right)c_{t+1}}{P_r c_t}.
$$

From the Euler equation,

$$
\begin{aligned}
P_r &= \beta E\frac{u'\left(c_{t+1}\right)\left(1+P_r\right)c_{t+1}}{u'\left(c_t\right)c_t} \\
&= \left(1+P_r\right)\beta\frac{\frac{(c_t(1+g+\sigma))^{\alpha-1}}{2}\left(c_t\left(1+g+\sigma\right)\right)+\frac{(c_t(1+g-\sigma))^{\alpha-1}}{2}\left(c_t\left(1+g-\sigma\right)\right)}{c_t^{\alpha-1}c_t} \\
&= \left(1+P_r\right)\beta\frac{\left(1+g+\sigma\right)^{\alpha}+\left(1+g-\sigma\right)^{\alpha}}{2}. \\
\frac{1+P_r}{P_r} &= \frac{2}{\beta\left(\left(1+g+\sigma\right)^{\alpha}+\left(1+g-\sigma\right)^{\alpha}\right)}
\end{aligned}
$$

Finally, calculate the expected return on the share:

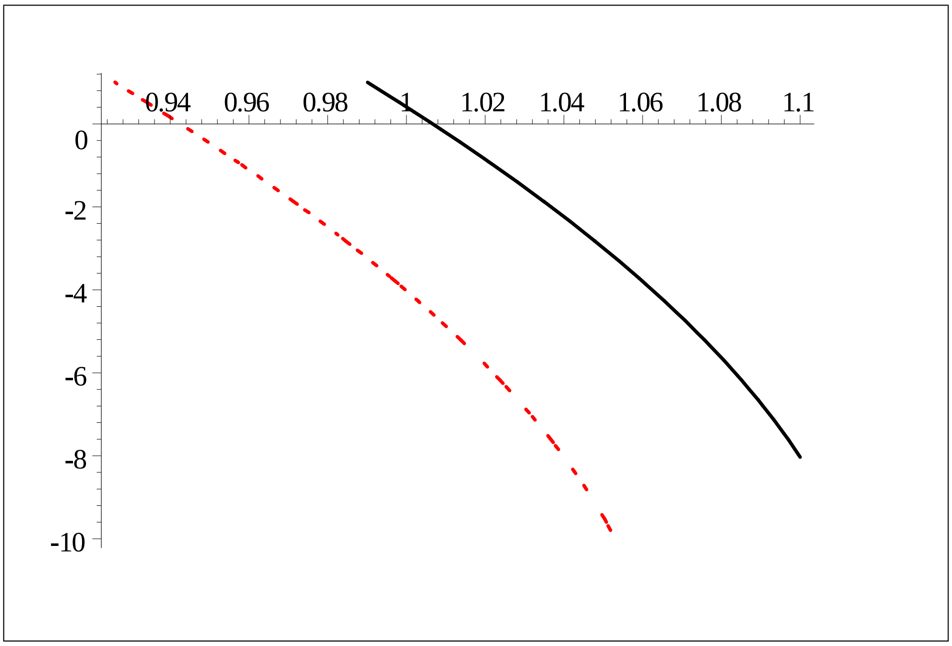
$$
\begin{aligned}
&\frac{1+P_r}{P_r}E\frac{c_{t+1}}{c_t} \\
&= \frac{1+P_r}{P_r}\left(1+g\right) \\
&= \frac{2\left(1+g\right)}{\beta\left(\left(1+g+\sigma\right)^{\alpha}+\left(1+g-\sigma\right)^{\alpha}\right)},
\end{aligned}
$$

as with the one-period share.

Using (US) data g is around 1.8% per year and σ is around 3.6%. Stock market returns have averaged around 8% per year and the risk-free rate around 1% over the last 100 years or so. Therefore

$$\begin{aligned}
p_h &= \beta\frac{(1+0.018+0.036)^{\alpha-1}}{2} \\
p_l &= \beta\frac{(1+0.018-0.036)^{\alpha-1}}{2} \\
r_b &= \frac{1}{p_h+p_l} \\
r_r &= \frac{1+0.018}{p_h(1+0.018+0.036)+p_l(1+0.018-0.036)}
\end{aligned}$$

Can we find α and β to generate the observed values of $r_r = 1.08$ and $r_b = 1.01$?



Combinations of α and β such that $r_a = 1.08$ (red,dashed) and $r_b = 1.01$ (black,solid)

If, for example, we set $\beta = 0.98$, riskaversion. $1-\alpha$ should be 3.5, to motivate 8% stock return. But then the bond return should be 7.6%, leaving a mere 0.4% risk premium. In fact, it is very difficult to get the right risk premium.

Let's look closer at the risk premium. Let's express it as the ratio of the price of the bond to the ratio of the price of the risky asset

$$\begin{aligned}\frac{p_b}{p_r} &= \frac{p_h + p_l}{p_h\,(1+g+\sigma) + p_l\,(1+g-\sigma)} \\ &= \frac{(1+g+\sigma)^{\alpha-1} + (1+g-\sigma)^{\alpha-1}}{(1+g+\sigma)^{\alpha} + (1+g-\sigma)^{\alpha}}\end{aligned}$$

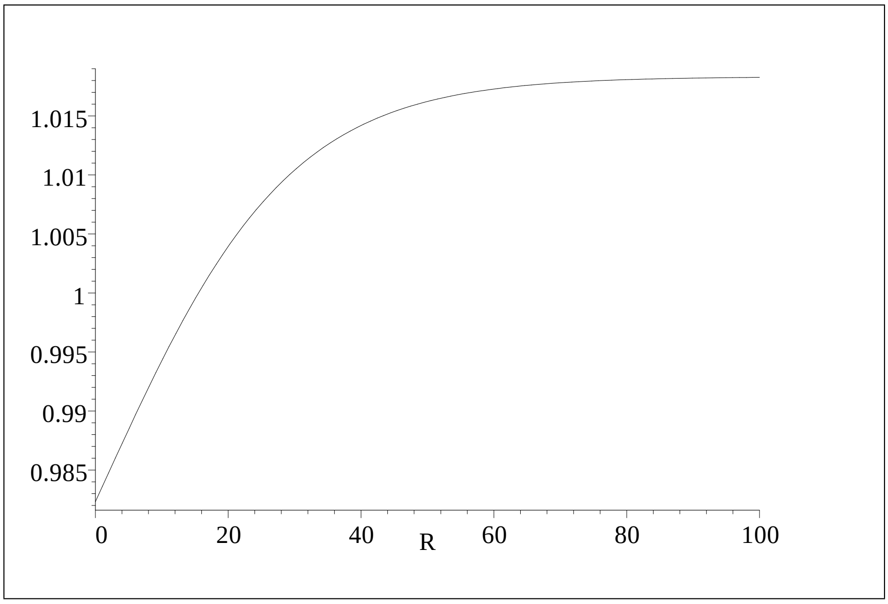
In reality, this ratio is

$$\frac{\frac{1}{1.01}}{\frac{1+g}{1.08}} \approx 1.05$$

However, by plotting

$$\left[\frac{(1+g+\sigma)^{\alpha-1} + (1+g-\sigma)^{\alpha-1}}{(1+g+\sigma)^{\alpha} + (1+g-\sigma)^{\alpha}}\right]_{g=0.018,\sigma=0.036}$$

against $RRA = 1-\alpha$,



we see that it is very difficult to get the right risk premium. In fact, in the realistic case where $\sigma > g$, it is easy to bound the risk premium,

$$\begin{aligned}\lim_{\alpha\to-\infty}\left[\frac{(1+g+\sigma)^{\alpha-1} + (1+g-\sigma)^{\alpha-1}}{(1+g+\sigma)^{\alpha} + (1+g-\sigma)^{\alpha}}\right]_{g<\sigma} &= \lim_{\alpha\to-\infty}\left[\frac{(1+g-\sigma)^{\alpha-1}}{(1+g-\sigma)^{\alpha}}\right] \\ &= \frac{1}{1+g-\sigma} \approx 1.0183.\end{aligned}$$

The fist line comes from the fact that $(1+g-\sigma)^{\alpha}$ goes to infinity as α approach $-\infty$, while $(1+g+\sigma)^{\alpha}$ approach zero. Of course, with lower growth and higher risk, the risk premium can get larger, but we are stuck with data.

2. **Too little risk-sharing**

In a complete markets equilibrium where individuals have homothetic preferences, e.g., CRRA, there should be full risk sharing. Consumption growth should be perfectly correlated between individuals and everyone should hold a share in a global portfolio of assets. This is not the case, obviously frictions and asymmetric information may be one explanation. But sometimes these explanations don't seem to suffice. An example is the home bias puzzle. All around the world local investors hold unbalanced portfolios with to much domestic assets. It is shown in the literature that expected returns could increase a lot, without increasing risk by having more balanced portfolios, containing more foreign assets. The explanation cannot be that information is superior. Then, domestic holders should sometimes have more negative information than foreign investors, in which case they should sell, moving to foreign assets, this we don't see. Conversely, they should sometimes go short abroad, having an investment share above unity at home, which we don't see either.

1.2 Lab puzzles

Ambiguity aversion – Ellsberg Paradox.

Consider following lottery. There are two urns, each with 100 balls. In urn 1, there are 50 red and 50 black. In urn 2, there are only red and black balls but the proportions are unknown. The subject is given a color and can pick one ball. If a ball with the given color comes up, the gain is 50$, if not the gain is zero. The subject is asked to rank lotteries. Typically the following response comes up.

1. Red from urn 1 $\sim$ Black from urn 1.
2. Red from urn 2 $\sim$ Black from urn 2.
3. Red from urn 1 $\succ$ Red from urn 2.
4. Black from urn 1 $\succ$ Black from urn 2.

This contradicts expected utility since from 2, we expect subjective probability to imply that they believe $p(red) = 1/2$. Then, 3 and 4 should be with indifference.

Time inconsistency.

In lab experiments, preference reversal occurs. Example.

Suppose you can choose between 10CD's 1 year from now or 11 CD's 1 year and a week. Often the latter is preferred. However, if the same individual is asked after a year, 10 CD's today might be preferred over 11 CD's in a week. This is inconsistent with standard time-additive utility with geometric discounting. A week's discounting depends on how close in time it is.

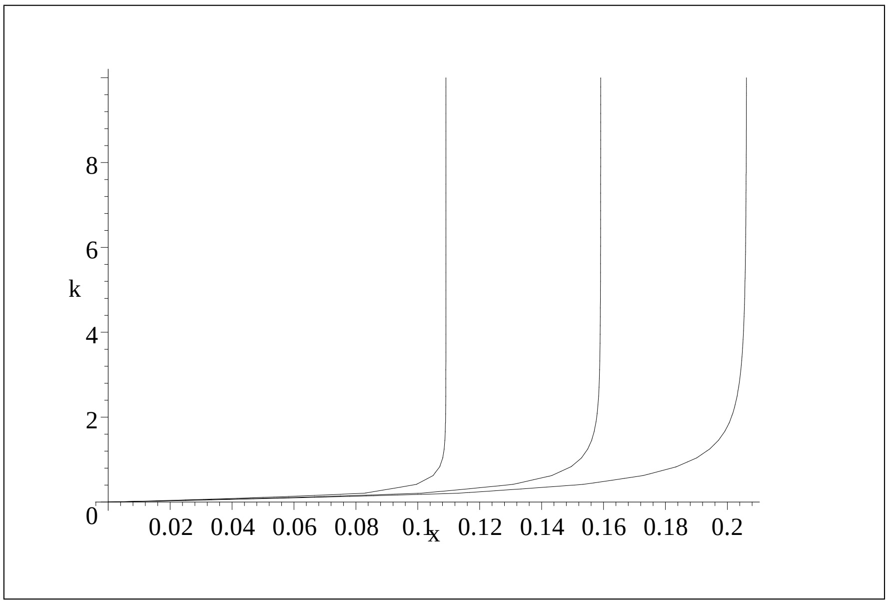
Other examples, people sometimes seems to pay to commit. They tend to over-consume during the year, and, for example, ask their employer to keep money for tax-payments at the end of the year or, say for big holidays.

First-order risk-aversion

With smooth preferences, people should as we have seen not care much about small gambles, ilke holding a balanced stockmarket protfolio. Big risks, on the other hand, are detrimental. In fact, with CRRA coefficient bigger than unity, sufficiently big losses can never be compensated since $U(c_t)$ is bounded from above but not from below. For example, consider a lottery that gives a relative loss of x, forcing a consumption loss of xc with $p = \frac{1}{2}$ and otherwise gives consumption $(1+k)c$. For different values of x, how large must k be to compensate for so that

$$U(c) = \frac{1}{2}U((1-x)c) + \frac{1}{2}U((1+k)c)$$

Here, I plot this k as a function of x for $\alpha = -3, -4$ and -6.



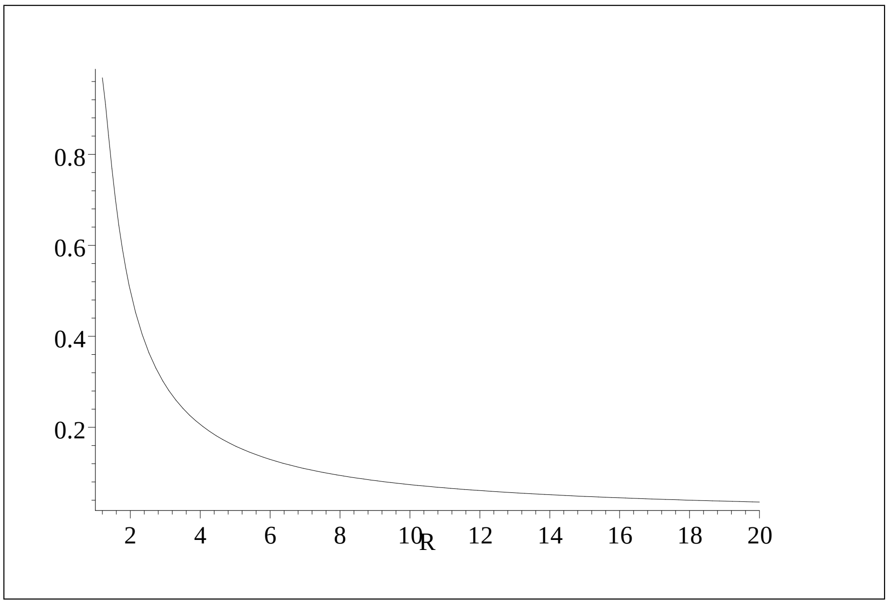
:To get people to behave like they do for small gambles, σ has to be so large as to give unreasonable predictions for large gambles. In fact, sometimes no upside can compensate for a sufficiently large but finite downside. This fact is due to that when CRRA coefficient larger than 1, i.e., when $\alpha < 0$, utility is bounded, since

$$\frac{c^{\alpha}}{\alpha} < 0, \forall c, \alpha < 0.$$

This means that solving

$$\begin{aligned} \frac{c^{\alpha}}{\alpha} &= \frac{1}{2}\frac{((1-x)\,c)^{\alpha}}{\alpha} + \frac{1}{2}0 \\ 1 &= \frac{1}{2}(1-x)^{\alpha} \\ x &= 1 - 2^{\frac{1}{\alpha}} \end{aligned}$$

we find an upper bound to the downside risk that could be compensated by any upside. In the graph, we see the maximum loss occuring with 50% chance that could be compensated by any gain as a function of the level of riskaversion.



For $\alpha = -11$, x is as low as 6.1%. Would you refuse a 50/50 bet of loosing 25% of your lifetime income vs. getting the fortune of Bill Gates? If, not, you cannot have absolute riskaversion above 3.4, which is quite low.

2 Non-additive recursive preferences

2.1 Aggregation over time

Now, disregard risk. In general, preferences can be described as a function that associates a particular level of overall utility to any sequence of consumption levels

$$U(c_1, c_2.,,, c_T) \equiv U\left(\{c_t\}_0^T\right)$$

MRS is defined

$$MRS_{t,t+1} \equiv \frac{\frac{\partial U(c_1,c_2.,,,c_T)}{\partial c_{t+1}}}{\frac{\partial U(c_1,c_2.,,,c_T)}{\partial c_t}}$$

We define time preference as MRS along a path of constant consumption c

$$\beta(c)_{t,t+1} = \frac{\frac{\partial U(c_1,c_2.,,,c_T)}{\partial c_{t+1}}}{\frac{\partial U(c_1,c_2.,,,c_T)}{\partial c_t}}|_{c_t=c\forall t}$$

noting that this may depend onf c. For the time additive utility with constant discounting, however, we have

$$U = \sum_{t=s}^{T} \beta^t u(c_t)$$

with

$$\beta(c)_{t,t+1} = \beta \forall c.$$

Koopman's time aggregator

Assume preferences *at all dates* are represented by a time zero utility function, so preferences are time consistent.

First notation,

$${}_t c \equiv \{c_t, c_{t+1}, c_{t+2}, ...c_{t+\infty}\}$$

Utility at time zero is

$$U({}_0 c) = U(c_{0\cdot 1} c)$$

Assume *history independence,* here marginal rate of substitution $MRS_{t,t+1}$ *does not* depend on consumption prior to t (is this innocuous?) and if c_t is a vector, also the intra-temporal MRS between goods in t, is independent of prior consumption. Then, but not otherwise, we can write

$$U({}_0 c) = \tilde{V}[c_0, U_1({}_1 c)]$$

for an aggregator function V and a function that gives the continuation utility $U_1({}_1c)$ Choices over ${}_1c_1$, in particular, what maximizes U_1 in some choice set, does not depend on c_0. But the choice set can, of course, be affected.

Now also assume *future independence* preferences over c_t does not depend on ${}_{t+1}c$. (Is this innocuous? Yes, clearly if c_0 is a scalar, then more is just better, but if c_0 is a vector this is a restriction. One could prefer chicken over fish if one plans to eat a lot of fish in the future. However, future independence seems like a less strong assumption than history independence).

Now, we can write utility as

$$U({}_0c) = V[u(c_0), U_1({}_1c)]$$

V aggregates utility coming from current consumption, and future consumption. It is *not* restricted to simply add them like standard preferences.

Finally, assume *stationarity*, then for all t,

$$U({}_tc) = V[u(c_t), U({}_{t+1}c)],$$

and recursivity is implied

$$U({}_tc) = V[u(c_t), V[u(c_{t+1}), U({}_{t+2}c)]]$$

$\text{MRS}_{t,t+1}$ is

$$\frac{\frac{\partial U(c_1,c_2.,,,c_T)}{\partial c_{t+1}}}{\frac{\partial U(c_1,c_2.,,,c_T)}{\partial c_t}} = \frac{V_2[u(c_t), U({}_{t+1}c)]\, V_1[u(c_{t+1}), U({}_{t+2}c)]\, u'(c_{t+1})}{V_1[u(c_t), U({}_{t+1}c)]\, u'(c_t)}$$

As we know, time preference is MRS evaluated at a constant consumption path, where by stationarity, also $u(c_t)$ and $U({}_tc)$ is constant at $u(c)$ and $U(c)$ (excuse the notation, I am here letting c denote a path of constant levels of consumption). Then,

$$\beta(u(c)) = V_2[u(c), U(c)]$$

which can depend on c unless V is a linear aggregator (standard).

The Uzawa simplification is a particular example of the Koopmans aggregator.

$$U({}_tc) = u(c_t) + \beta(u(c_t))\, U({}_{t+1}c)$$

First order condition:

$$\frac{\partial u(c_t)}{\partial c_{t,i}}(1 + \beta'(u(c_t))\, U({}_{t+1}c)) + \beta(u(c_t))\frac{\partial U({}_{t+1}c)}{\partial c_{t,i}} = 0$$

The expression $\frac{\partial U({}_{t+1}c)}{\partial c_{t,i}}$ is a a (sloppy) way of denoting the effect (via the budget constraint) of consumption today on future utility.

A bit more general, by not imposing future independence.

$$U({}_t c) = u(c_t) + \beta(c_t) U({}_{t+1}c)$$

Note that here, preference over elements in c_t may depend on $U({}_{t+1}c)$, which matters if c_t is a vector.

First order condition:

$$\frac{\partial u(c_t)}{\partial c_{t,i}} + \frac{\partial \beta(c_t)}{\partial c_{t,i}} U({}_{t+1}c) + \beta(c_t)\frac{\partial U({}_{t+1}c)}{\partial c_{t,i}} = 0$$

Examples:

Growth and fiscal policy (Dolmas and Wynne (1998)). Using Usawa

$$\begin{aligned} \max U({}_t c) &= u(c_t) + \beta(u(c_t)) U({}_{t+1}c) \\ s.t. c_t &= f(k_t) - k_{t+1} - g_t \end{aligned}$$

We can derive a Bellman equation:

$$J(k) = \max_{k_{t+1}} u(f(k_t) - k_{t+1} - g_t) + \beta(u(f(k_t) - k_{t+1} - g_t)) J(k_{t+1}).$$

Only non-standard is endogenous discounting.

FOC:

$$u'(c_t)(1 + \beta'(u(c_t)) J(k_{t+1})) = \beta(u(c_t)) J'(k_{t+1})$$

Envelope:

$$\begin{aligned} J'(k_t) &= u'(c_t) f'(k_t) + \beta'(u(c_t)) u'(c_t) f'(k_t) J(k_{t+1}) \\ &= u'(c_t) f'(k_t)(1 + \beta'(u(c_t)) J(k_{t+1})). \end{aligned}$$

Giving

$$J'(k_t) = \beta(u(c_t)) J'(k_{t+1}) f'(k_t)$$

In a steady state

$$1 = \beta(u(c_t)) f'(k_t)$$

Compare this to the standard case

$$1 = \beta f'(k_t)$$

being independent of fiscal policy. In particular, an increase in g, must reduce c one-for-one, since $c_{ss} = f(k_{ss}) - k_{ss} - g_t$.

Now changes in g can affect the steady state. To see this, consider

$$1 = \beta (u (f (k_{ss}) - k_{ss} - g)) f' (k_{ss})$$

An increase in g reduces u, suppose this makes people more patient, i.e., $\beta' < 0$. Then, the increase in g makes $\beta(u) f' > 1$. This will lead to *more saving* and a growing capital stock. Crowding out of consumption more than one-for-one. In this case, since both β' and f'' are negative, there is a unique steady state. If, instead $\beta^{'} > 0$, there may be multiple solutions to

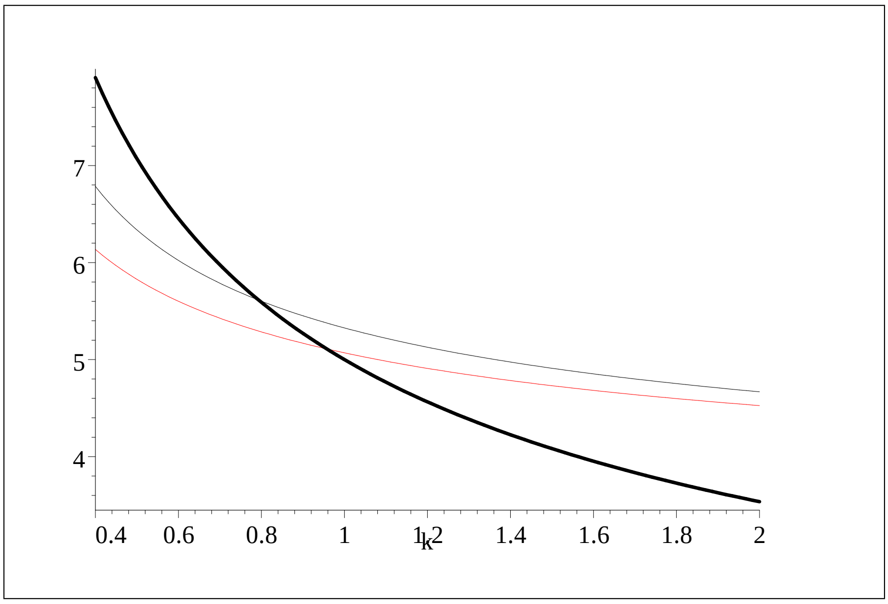
$$1 = \beta (u (f (k_{ss}) - k_{ss} - g)) f' (k_{ss}) .$$

In this case, some steady states are unstable - a higher level of k increases u, and β more than the fall in f'. Therefore, $\beta > f\prime$ and individuals accumulates capital.

In the graph, I have specified

$$\begin{aligned} \beta(u) &= [1 - e^{\gamma u}]_{\gamma=-0.1} \\ u(c) &= \ln c \\ f(k) &= [\theta k^{\alpha}]_{\alpha=\frac{1}{2},\theta=10.} \end{aligned}$$

I plot $f'(k)$ (the fat black line) and $(\beta (u (f(k) - k - g)))^{-1}$ for $g = 0$ and $g = 2$. A decrease in g shifts the latter curve down and steady state k increases.



Other examples is small open economies with a fixed net interest rate, r. The steady state is

$$1 = \beta \left(u \left(f \left(k_{ss} \right) - k_{ss} - g \right) \right) \left(1 + r \right)$$

With standard preferences no steady state exists generically. With $\beta' < 0$ a unique stable one exists. Intuition, with low capital and low consumption, β is higher than r and people save a lot. As capital and consumption increases, the discount factor falls.

2.2 Aggregation over states

As in the intro, consider states (within a period) to be $z \in \Gamma$, with an associated probability measure $p(z)$. Utility is now

$$U \left(\{ c(z) \}_{z \in \Gamma} \right)$$

We can solve for the certainty equivalent consumption level μ of $\{ c(z) \}_{z \in \Gamma}$ from

$$U \left(\{ c(z) \}_{z \in \Gamma} \right) = U \left(\{ \mu \} \right)$$

Standard theory says

$$U \left(\{ c(z) \} \right) = \sum_{z \in \Gamma} p(z) \, u \left((z) \right)$$

and since

$$u(\mu) = \sum_{z \in \Gamma} p(z) \, u \left(c(z) \right)$$

we have

$$\mu \left(\{ c(z) \} \right) = u^{-1} \left(\sum_{z \in \Gamma} p(z) \, u \left(c(z) \right) \right).$$

For example: Suppose preferences are CRRA, then

$$u(c) = \frac{1}{\alpha} c^{\alpha}$$

So

$$\begin{aligned} \frac{1}{\alpha} \mu^{\alpha} &= \sum_{z \in \Gamma} p(z) \frac{1}{\alpha} c(z)^{\alpha} \\ \mu &= \left(\alpha \sum_{z \in \Gamma} p(z) \frac{1}{\alpha} c(z)^{\alpha} \right)^{\frac{1}{\alpha}} \end{aligned}$$

Note the linear homogeneity in this case. For a constant $k > 0$.

$$\mu\left(k\left\{c\left(z\right)\right\}\right) = k\mu\left(\left\{c\left(z\right)\right\}\right).$$

Chew and Dekel generalizes this by allowing the certainty equivalent of $\{c(z)\}$ to be a more general function *while maintaining* first order conditions that are linear in probabilities by implicitly defining

$$\mu\left(\left\{c\left(z\right)\right\}\right) = \sum_{z\in\Gamma} p\left(z\right) M\left[c\left(z\right),\mu\right].$$

As we see, this generalizes standard utility by implying that the marginal value of consumption in state z depends on consumption in other states through their effect on μ. An example of this is that people might care more (or less) about consumption in states that provide less consumption than the certainty equivalence (disappointment aversion). Notice the relative comparison here. With concave utility, marginal utility in states with low consumption is high, but *independent of consumption in other states.* This is not necessarily the case here since consumption in state z, relative to μ depends on consumption in all other states since they affect μ.

We assume that $M(\mu,\mu) = \mu$ (Why?), $M_1 > 0$, $M_{11} < 0$ (first order stochastic dominance and riskaversion). Often we want to maintain the linear homogeneity of preferences like in CRRA.

$$M\left(kc, k\mu\right) = kM\left(c,\mu\right)$$

Examples:

To show that the Chew-Dekel generalizes and includes e.g., CRRA preferences: Note that if we set

$$M\left(c,\mu\right) = \frac{c^{\alpha}\mu^{1-\alpha}}{\alpha} - \frac{\mu}{\alpha} + \mu \tag{1}$$

we get

$$\begin{aligned}
\mu &= \sum_{z\in\Gamma} p(z)\left(\frac{c(z)^{\alpha}\mu^{1-\alpha}}{\alpha} - \frac{\mu}{\alpha} + \mu\right) \\
0 &= \sum_{z\in\Gamma} p(z)\left(c(z)^{\alpha}\mu^{1-\alpha} - \mu\right) \\
\mu &= \mu^{1-\alpha}\sum_{z\in\Gamma} p(z)\, c(z)^{\alpha} \\
\mu^{\alpha} &= \sum_{z\in\Gamma} p(z)\left(c(z)^{\alpha}\right) \\
\mu &= \left(\sum_{z\in\Gamma} p(z)\, c(z)^{\alpha}\right)^{\frac{1}{\alpha}}
\end{aligned}$$

which is the CRRA certainty equivalence.

Examples:

"Weighted expected utility"

Let

$$M = \left(\left(\frac{c}{\mu}\right)^{\gamma}\left(\frac{c^{\alpha}\mu^{1-\alpha}}{\alpha} - \frac{\mu}{\alpha}\right) + \mu\right)$$

Compare to (1). Now, we have

$$\begin{aligned}
\mu &= \sum_{z\in\Gamma} p(z)\left(\frac{c(z)^{\gamma}}{\mu^{\gamma}}\left(\frac{c(z)^{\alpha}\mu^{1-\alpha}}{\alpha}\right) - \frac{c(z)^{\gamma}}{\mu^{\gamma}}\frac{\mu}{\alpha} + \mu\right) \\
0 &= \sum_{z\in\Gamma} p(z)\left(\frac{c(z)^{\gamma}}{\mu^{\gamma}}\left(\frac{c(z)^{\alpha}\mu^{1-\alpha}}{\alpha}\right) - \frac{c(z)^{\gamma}}{\mu^{\gamma}}\frac{\mu}{\alpha}\right) \\
\mu\sum_{z\in\Gamma} p(z)\, c(z)^{\gamma} &= \mu^{1-\alpha}\sum_{z\in\Gamma} p(z)\, c(z)^{\gamma} c(z)^{\alpha} \\
\mu^{\alpha} &= \frac{\sum_{z\in\Gamma} p(z)\, c(z)^{\gamma} c(z)^{\alpha}}{\sum_{z\in\Gamma} p(z)\, c(z)^{\gamma}}
\end{aligned}$$

Here, we can interpret

$$\frac{p(z)\, c(z)^{\gamma}}{\sum_{z\in\Gamma} p(z)\, c(z)^{\gamma}} \equiv \hat{p}(z)$$

as a weighted probability. If $\gamma < 0$, these weights decrease in c, i.e., bad outcomes are weighted higher than otherwise. Suppose we have two outcomes

c_1 and c_2 with equal probabilities $\frac{1}{2}$each. The weighted probabilities are then

$$\hat{p}(z_1) = \frac{\frac{1}{2}c_1^\gamma}{\frac{1}{2}c_1^\gamma + \frac{1}{2}c_1^\gamma} = \frac{c_1^\gamma}{c_1^\gamma + c_1^\gamma}$$
$$\hat{p}(z_2) = \frac{\frac{1}{2}c_2^\gamma}{\frac{1}{2}c_1^\gamma + \frac{1}{2}c_1^\gamma} = \frac{c_2^\gamma}{c_1^\gamma + c_1^\gamma}$$

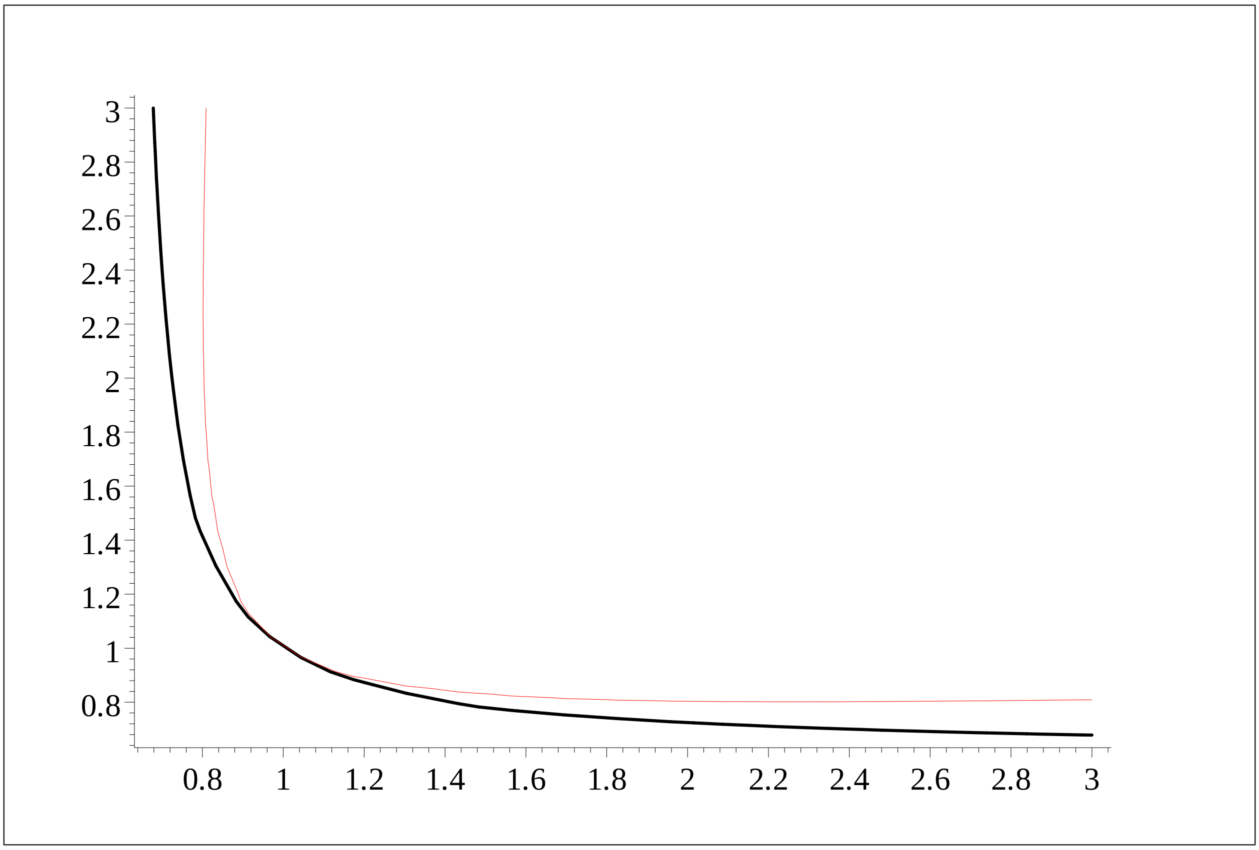
Consequently,

$$\mu^\alpha = \frac{c_1^\gamma}{c_1^\gamma + c_1^\gamma}c_1^\alpha + \frac{c_2^\gamma}{c_1^\gamma + c_1^\gamma}c_2^\alpha$$

For any constant k, indifference curves then satisfy

$$k = \frac{c_1^{\alpha+\gamma}}{c_1^\gamma + c_2^\gamma} + \frac{c_2^{\alpha+\gamma}}{c_1^\gamma + c_2^\gamma}$$

Plotting one together with standard utility we find these preferences produce a higher level of riskaversion.



We can also have Faruk Gul's disappointment aversion. Then

$$M(c,\mu) = \begin{cases} (1+\delta)\left(\frac{c^\alpha \mu^{1-\alpha}}{\alpha} - \frac{\mu}{\alpha}\right) + \mu \text{ if } c(z) < \mu \\ \frac{c^\alpha \mu^{1-\alpha}}{\alpha} - \frac{\mu}{\alpha} + \mu \text{ else} \end{cases}$$

This means that all outcomes worse than the certainty equivalence get scaled up.

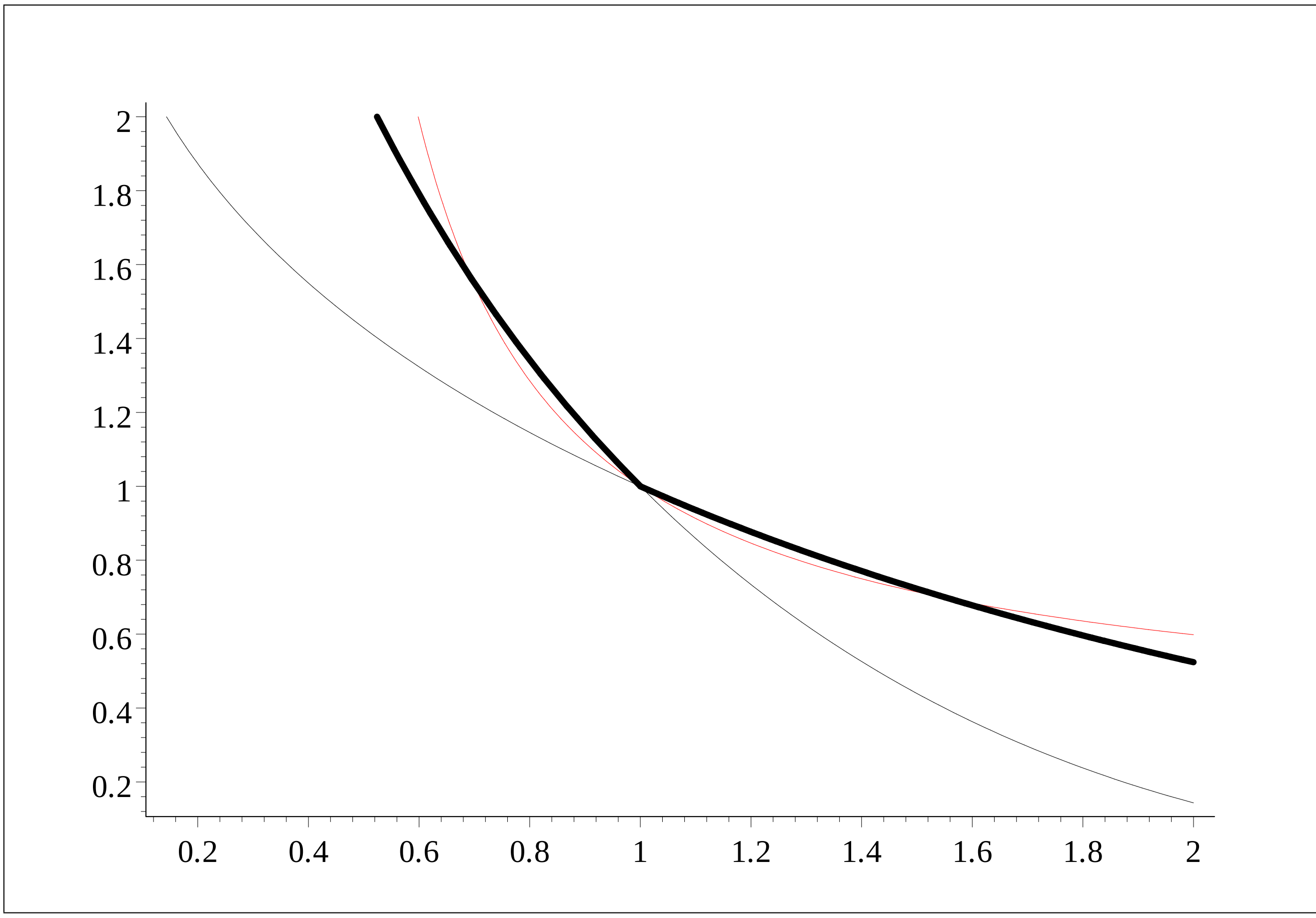
$$\begin{aligned} \mu &= \sum_{z\in\Gamma} p(z)\left((1+\delta I[c(z)<\mu])\left(\frac{c(z)^\alpha \mu^{1-\alpha}}{\alpha} - \frac{\mu}{\alpha}\right) + \mu\right) \\ \mu \sum_{z\in\Gamma} p(z)(1+\delta I[c(z)<\mu]) &= \mu^{1-\alpha} \sum_{z\in\Gamma} p(z)(1+\delta I[c(z)<\mu])\, c(z)^\alpha \\ \mu^\alpha &= \frac{\sum_{z\in\Gamma} p(z)(1+\delta I[c(z)<\mu])\, c(z)^\alpha}{\sum_{z\in\Gamma} p(z)(1+\delta I[c(z)<\mu])} \end{aligned}$$

The weighted probability here is

$$\frac{p(z)(1+\delta I[c(z)<\mu])}{\sum_{z\in\Gamma} p(z)((1+\delta I[c(z)<\mu]))}$$

For any constant k. indifference curves can be written as

$$k = \begin{cases} \frac{(1+\delta)\frac{1}{2}c_1^\alpha}{(1+\delta)\frac{1}{2}+\frac{1}{2}} + \frac{\frac{1}{2}c_2^\alpha}{(1+\delta)\frac{1}{2}+\frac{1}{2}} \text{if } c_1 < c_2 \\ \frac{\frac{1}{2}c_1^\alpha}{(1+\delta)\frac{1}{2}+\frac{1}{2}} + \frac{(1+\delta)\frac{1}{2}c_2^\alpha}{(1+\delta)\frac{1}{2}+\frac{1}{2}} \text{ if } c_1 > c_2 \end{cases}$$



Here, it is important to notice the kink around unity. This is first order risk-aversion. Individuals will be more averse to small risks than under standard preferences.

Risk premia under Chew-Dekel Preferences.

Define the risk-premium R from

$$R = Ec(z) - \mu(\{c(z)\})$$

Let us consider two state case above. In the two cases of CRRA E utility and weighted utility. Suppose $c = c_1 = 1 - \sigma$ with prob $\frac{1}{2}$ and $c = c_2 = 1 + \sigma$ with prob. $\frac{1}{2}$. The shock is thus σ and σ is the standard deviation of c.

$$
\begin{aligned}
R_{s\tan d} &= 1-\left(\frac{(1-\sigma)^{\alpha}+(1+\sigma)^{\alpha}}{2}\right)^{\frac{1}{\alpha}} \equiv R_s(\sigma) \\
&\approx R_s(0)+R_s'(0)\,\sigma+\frac{1}{2}R_s''(0)\,\sigma^2 \\
&= \left[\frac{\partial\left(1-\left(\frac{(1-\sigma)^{\alpha}+(1+\sigma)^{\alpha}}{2}\right)^{\frac{1}{\alpha}}\right)}{\partial\sigma}\right]_{\sigma=0}\sigma \\
&\quad +\frac{1}{2}\left[\frac{\partial^2\left(1-\left(\frac{(1-\sigma)^{\alpha}+(1+\sigma)^{\alpha}}{2}\right)^{\frac{1}{\alpha}}\right)}{\partial\sigma^2}\right]_{\sigma=0}\sigma^2 \\
&= \frac{1}{2}(1-\alpha)\,\sigma^2
\end{aligned}
$$

In the weighted expected utility, we have

$$
\begin{aligned}
R_{wheigt} &= \left(1-\left(\frac{(1-\sigma)^{\alpha+\gamma}+(1+\sigma)^{\alpha+\gamma}}{(1-\sigma)^{\gamma}+(1+\sigma)^{\gamma}}\right)^{\frac{1}{\alpha}}\right) \equiv R_w(\sigma) \\
&\approx R_w(0)+R_w'(0)\sigma+\frac{1}{2}R_w'(0)\sigma^2 \\
&= \left[\frac{\partial\left(1-\left(\frac{(1-\sigma)^{\alpha+\gamma}+(1+\sigma)^{\alpha+\gamma}}{(1-\sigma)^{\gamma}+(1+\sigma)^{\gamma}}\right)^{\frac{1}{\alpha}}\right)}{\partial\sigma}\right]_{\sigma=0}\sigma \\
&\quad +\frac{1}{2}\left[\frac{\partial^2\left(1-\left(\frac{(1-\sigma)^{\alpha+\gamma}+(1+\sigma)^{\alpha+\gamma}}{(1-\sigma)^{\gamma}+(1+\sigma)^{\gamma}}\right)^{\frac{1}{\alpha}}\right)}{\partial\sigma^2}\right]_{\sigma=0}\sigma^2 \\
&= \frac{1}{2}(1-\alpha-2\gamma)\,\sigma^2
\end{aligned}
$$

In the disappointment aversion case, the first derivative is not going to be zero, people are not locally riskneutral. Therefore,

$$
\begin{aligned}
R_{disapp} &= \left(1-\left(\frac{(1+\delta)(1-\sigma)^{\alpha}+(1+\sigma)^{\alpha}}{2+\delta}\right)^{\frac{1}{\alpha}}\right) \equiv R_d(\sigma) \\
&\approx R_d(0)+R_d'(0)\,\sigma+\frac{1}{2}R_d''(0)\,\sigma^2 \\
&= \left[\frac{\partial\left(1-\left(\frac{(1+\delta)(1-\sigma)^{\alpha}+(1+\sigma)^{\alpha}}{2+\delta}\right)^{\frac{1}{\alpha}}\right)}{\partial\sigma}\right]_{\sigma=0}\sigma \\
&\quad +\frac{1}{2}\left[\frac{\partial^2\left(1-\left(\frac{(1+\delta)(1-\sigma)^{\alpha}+(1+\sigma)^{\alpha}}{2+\delta}\right)^{\frac{1}{\alpha}}\right)}{\partial\sigma^2}\right]_{\sigma=0}\sigma^2 \\
&= \frac{\delta}{2+\delta}\sigma+2(1-\alpha)\frac{1+\delta}{(2+\delta)^2}\sigma^2
\end{aligned}
$$

The key here is the linear term.

2.3 Time and Risk

Let us now combine the aggregation over time and over states so we can define preferences over both states and time, allowing for rates of substitution to differ depending on whether we are talking about states or time periods.

Generalize the notation so that

$$ {}_tc $$

denotes a stochastic consumption stream starting from period t, i.e., $c_t, c(z_{t+1}), \ldots$, and denote

$$ U({}_tc) \equiv U_t $$

The aggregator can be written

$$ U_t = V(c_t, \mu(U_{t+1})) $$

or by imposing *future independence*

$$ U_t = V(u(c_t), \mu(U_{t+1})) \tag{2} $$

Note that we are now aggregating the direct utility of c_t and the certainty equivalent of the *stochastic continuation payoff* from $t+1$ and onwards.

Thus, since at t, U_{t+1} is stochastic (depending on the realization of z_{t+1}), we construct the period $t+1$ certainty equivalent of this stochastic utility, i.e., $\mu(U_{t+1})$.

If we use a standard expected utility certainty equivalent μ in (2), we get what is typically called *Kreps-Porteus* utility (some times also Epstein-Zin, although they often are associated with more general specification of state aggregation, *a la* Chew-Dekel). The key here is that we can now separate risk-aversion, which is determined by the properties of μ, from intertemporal substitution, determined by V.

Let us use the CRRA specification for μ, so

$$\mu(U) = [EU^{\alpha}]^{\frac{1}{\alpha}}$$

and a constant elasticity aggregator

$$V(u(c_t), \mu(U_{t+1})) = [(1-\beta)u^{\rho} + \beta\mu(U)^{\rho}]^{\frac{1}{\rho}}.$$

If we let u be linear, $u = c$, so we let all curvature be taken care of by μ and V, we can now interpret $1-\alpha$ a s the degree of risk-aversion and $\frac{1}{1-\rho}$ as the elasticity of intertemporal substitution. To see this simply, look at a two period example where consumption is stochastic in the second period. In the second period, stochastic utility is

$$U_2 = c(z_2)$$

Suppose first that second period consumption is non-stochastic at c_2. Then,

$$\mu(U_2) = [EU^{\alpha}]^{\frac{1}{\alpha}} = c_2$$

Then,

$$U_1 = [(1-\beta)c_1^{\rho} + \beta c_2^{\rho}]^{\frac{1}{\rho}}$$

MRS is now,

$$\begin{aligned} MRS_{c_1,c_2} &\equiv \frac{\partial U_1}{\partial c_2} / \frac{\partial U_1}{\partial c_1} \\ &= \frac{\frac{1}{\rho}((1-\beta)c_1^{\rho} + \beta c_2^{\rho})^{\frac{1}{\rho}-1}\rho\beta c_2^{\rho-1}}{\frac{1}{\rho}((1-\beta)c_1^{\rho} + \beta c_2^{\rho})^{\frac{1}{\rho}-1}\rho(1-\beta)c_1^{\rho-1}} \\ &= \frac{\beta}{1-\beta}\left(\frac{c_2}{c_1}\right)^{\rho-1} \end{aligned}$$

as usual and

$$\frac{-\partial MRS_{c_1,c_2}}{\partial\left(\frac{c_1}{c_2}\right)}\frac{\frac{c_1}{c_2}}{MRS\left(\frac{c_1}{c_2}\right)}=IES^{-1}=1-\rho.$$

Note that we have imposed that U is linearly homogeneous by having the power $\frac{1}{\rho}$, which is with no consequence. To see this, consider

$$\tilde{U}_1=(1-\beta)\,c_1^{\rho}+\beta c_2^{\rho}$$

and calculate

$$\frac{\partial\tilde{U}_1}{\partial c_2}/\frac{\partial\tilde{U}_1}{\partial c_1}=\frac{\rho\beta c_2^{\rho-1}}{\rho\,(1-\beta)\,c_1^{\rho-1}}=\frac{\beta}{1-\beta}\left(\frac{c_2}{c_1}\right)^{\rho-1}$$

It should be clear from the aggregator

$$\mu\,(c_2\,(z_2))=[Ec\,(z_2)^{\alpha}]^{\frac{1}{\alpha}}$$

that we can think of $1-\alpha$ as measuring risk aversion.

A key characteristic of Kreps-Porteus preferences is that they give rise to preference for when risk is revealed. A basic intuition is that early revelation implies that some risk is *converted* into intertemporal substitution. To see an example of this, consider the example in the introduction, where second period consumption is high or low ($z_2=z_h$ or z_l). Suppose that this is reveled in the first period. Denote $\tilde{U}_1(z)$ first period utility given a realized value of z. Then,

$$\begin{aligned}\tilde{U}_1\,(z_l)&=&[(1-\beta)\,c_1^{\rho}+\beta c_l^{\rho}]^{\frac{1}{\rho}}\\ \tilde{U}_1\,(z_h)&=&[(1-\beta)\,c_1^{\rho}+\beta c_h^{\rho}]^{\frac{1}{\rho}}\end{aligned}$$

and

$$\begin{aligned}U_1^{early}&=&\mu\,(U_1)\\ &=&\left(\frac{1}{2}\left(((1-\beta)\,c_1^{\rho}+\beta c_h^{\rho})^{\frac{1}{\rho}}\right)^{\alpha}+\frac{1}{2}\left(((1-\beta)\,c_1^{\rho}+\beta c_h^{\rho})^{\frac{1}{\rho}}\right)^{\alpha}\right)^{\frac{1}{\alpha}}\\ &=&\left(\frac{1}{2}\,((1-\beta)\,c_1^{\rho}+\beta c_h^{\rho})^{\frac{\alpha}{\rho}}+\frac{1}{2}\,((1-\beta)\,c_1^{\rho}+\beta c_h^{\rho})^{\frac{\alpha}{\rho}}\right)^{\frac{1}{\alpha}}\end{aligned}$$

Compare this to the late resolution case. Then,

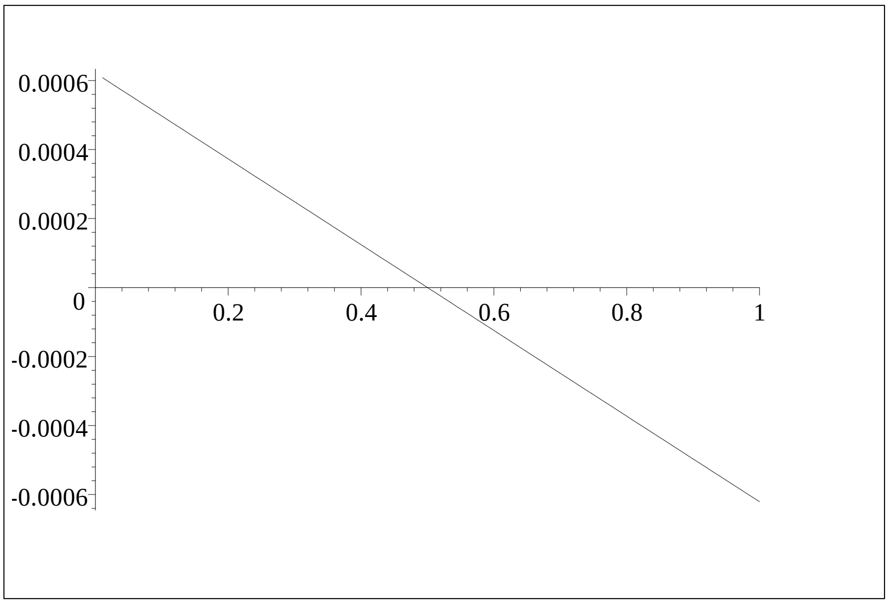
$$\mu\,(U_2)=\left(\frac{c_h^{\alpha}+c_l^{\alpha}}{2}\right)^{\frac{1}{\alpha}}$$

and

$$U_1^{late} = \left((1-\beta)\, c_1^{\rho} + \beta \left(\frac{c_h^{\alpha} + c_l^{\alpha}}{2} \right)^{\frac{\rho}{\alpha}} \right)^{\frac{1}{\rho}}$$

Comparing these, we see

$$\begin{aligned} & U_1^{early} - U_1^{late} \\ = \; & \left[\left(\frac{1}{2} \left((1-\beta)\, c_1^{\rho} + \beta c_h^{\rho} \right)^{\frac{\alpha}{\rho}} + \frac{1}{2} \left((1-\beta)\, c_1^{\rho} + \beta c_l^{\rho} \right)^{\frac{\alpha}{\rho}} \right)^{\frac{1}{\alpha}} - \left((1-\beta)\, c_1^{\rho} + \beta \left(\frac{c_h^{\alpha} + c_l^{\alpha}}{2} \right)^{\frac{\rho}{\alpha}} \right)^{\frac{1}{\rho}} \right]_{c_1=} \end{aligned}$$



Here I plotted the difference for $\rho = \frac{1}{2}$ and for $\alpha < \rho$, we see that early resolution is preferred. Note that when $\alpha < \rho$, risk aversion $(1-\alpha)$ is larger than the inverse of intertemporal elasticity of substitution, so in a sense, it is less costly to substitute between time periods than between states of nature. Is this important?

With Kreps-Porteus preferences we can clearly get low interest rates if individuals are facing little risk and high if they are facing large. But can we get what we want when we have a representative household that prices *both* risky and non-risky assets while having to bear reasonable amounts of risk himself.

An example. Look at the case in the introduction; given c_1

$$c_2 = \begin{cases} c_h = c_1 (1 + g + \sigma) \text{ with prob } \frac{1}{2} \\ c_l = c_1 (1 + g - \sigma) \text{ with prob } \frac{1}{2} \end{cases}$$

First, we note that Arrow-Debreu prices now have the property that the price of an asset that pays out in state $z = z_h$, depends on consumption also in the other state. This is clear from the expression of utility:

$$U_1 = \left((1-\beta) c_1^{\rho} + \beta \left(\frac{c_h^{\alpha} + c_l^{\alpha}}{2} \right)^{\frac{\rho}{\alpha}} \right)^{\frac{1}{\rho}}$$

where we see that the marginal contribution to utility U_1 of an asset that pays out in one of the states, say in $z = z_h$ depends on consumption also in the other state. The simple calculation of Arrow-Debreu securities is lost.

What is now the price of a safe bond p_b and a risky asset (the apple tree) p_r that pays dividends $(1+g+\sigma)$ or $(1+g-\sigma)$ per share depending on which state occurs? The prices have to be such that if a consumer buys one marginal unit of the assets, utility is unchanged. Take the safe bond, it has to satisfy

$$\begin{aligned}
0 &= -p_b (1-\beta) \frac{\partial U_1}{\partial c_1} + \beta E \left(\frac{\partial U_1}{\partial c_2} \right) \\
0 &= -p_b (1-\beta) \frac{\partial c_1^{\rho}}{\partial c_1} + \beta \left(\frac{\partial \left(\frac{c_h^{\alpha}+c_l^{\alpha}}{2} \right)^{\frac{\rho}{\alpha}}}{\partial c_h} + \frac{\partial \left(\frac{c_h^{\alpha}+c_l^{\alpha}}{2} \right)^{\frac{\rho}{\alpha}}}{\partial c_l} \right) \\
p_b &= \left[\frac{\beta \left(\frac{c_h^{\alpha}+c_l^{\alpha}}{2} \right)^{\frac{\rho-\alpha}{\alpha}} \left(c_h^{\alpha-1} + c_l^{\alpha-1} \right)}{2(1-\beta) c_1^{\rho-1}} \right]_{c_h = c_1(1+g+\sigma), c_l = c_1(1+g-\sigma)} \\
&= \frac{\beta}{2(1-\beta)} \left(\frac{(1+g+\sigma)^{\alpha} + (1+g-\sigma)^{\alpha}}{2} \right)^{\frac{\rho-\alpha}{\alpha}} \left((1+g+\sigma)^{\alpha-1} + (1+g-\sigma)^{\alpha-1} \right)
\end{aligned}$$

With the risky asset we have

$$0 = -p_r (1-\beta) \frac{\partial c_1^{\rho}}{\partial c_1} + \beta \left(\frac{\partial \left(\frac{c_h^{\alpha}+c_l^{\alpha}}{2} \right)^{\frac{\rho}{\alpha}} (1+g+\sigma)}{\partial c_h} + \frac{\partial \left(\frac{c_h^{\alpha}+c_l^{\alpha}}{2} \right)^{\frac{\rho}{\alpha}} (1+g-\sigma)}{\partial c_l} \right)$$

$$\Rightarrow \; p_r = \frac{\beta}{2(1-\beta)} \left(\frac{(1+g+\sigma)^{\alpha} + (1+g-\sigma)^{\alpha}}{2} \right)^{\frac{\rho-\alpha}{\alpha}} \left((1+g+\sigma)^{\alpha} + (1+g-\sigma)^{\alpha} \right)$$

Note that the ratio

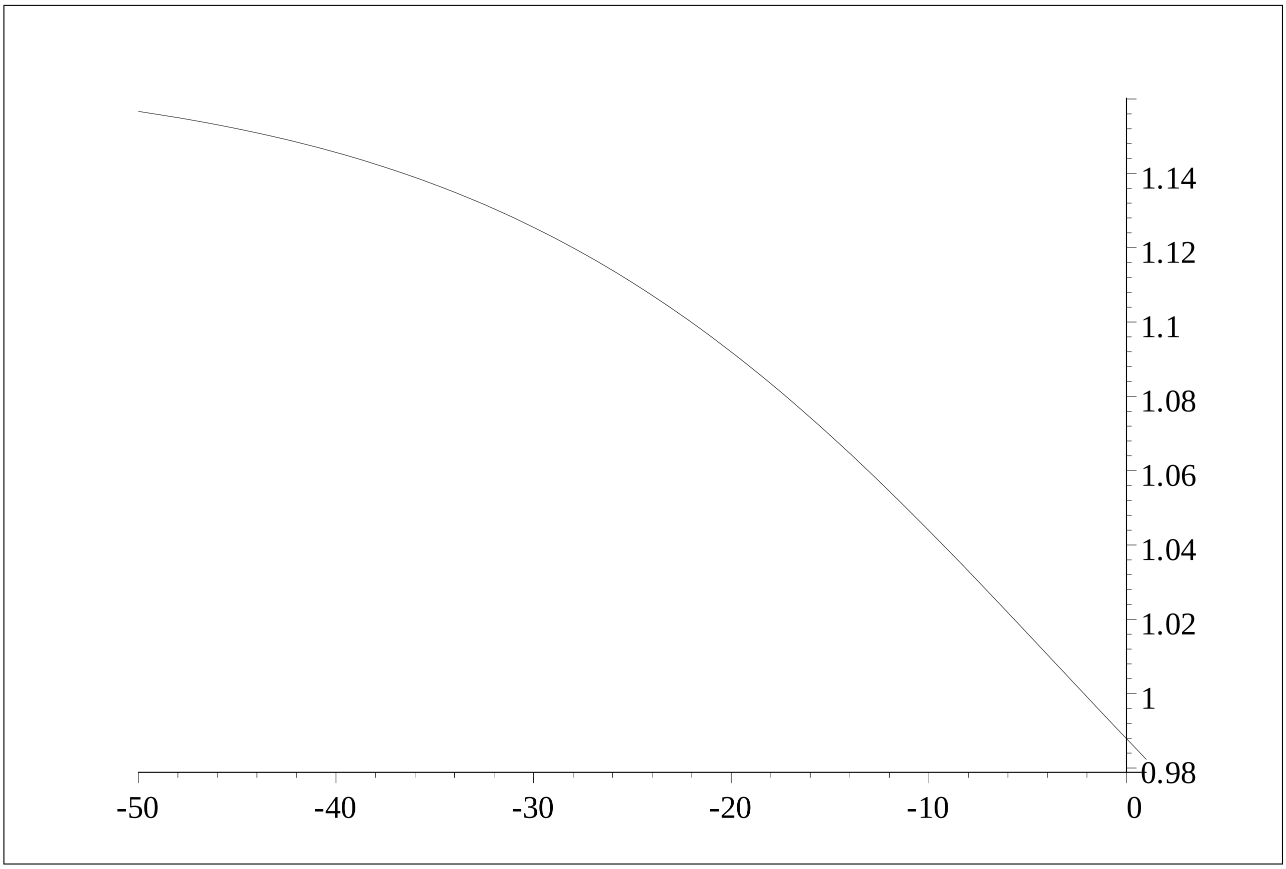
$$\frac{p_b}{p_r} = \frac{(1+g+\sigma)^{\alpha-1} + (1+g-\sigma)^{\alpha-1}}{(1+g+\sigma)^{\alpha} + (1+g-\sigma)^{\alpha}}$$

is determined by risk aversion and *exactly the same* as in the standard case. Therefore it seems as we have got little help from Kreps and Porteus. Not entirely right. We have learnt that it is difficult to get the risk premia by assuming that stock market returns are fully determined by consumption growth. Maybe this is not the way to go.

Suppose there is some other more stable income so that the share dividend is more volatile than consumption. Say that the standard deviation of consumption remains at σ while the standard deviation of dividends is $x\sigma$. Then

$$\frac{p_b}{p_r} = \frac{(1+g+\sigma)^{\alpha-1} + (1+g-\sigma)^{\alpha-1}}{(1+g+\sigma)^{\alpha-1}(1+g+x\sigma) + (1+g-\sigma)^{\alpha-1}(1+g-\sigma x)}$$

Setting $g = 0.018, \sigma = 0.036$ and $x = 4.5$, we get the following graph.



Targeting $\left[\frac{(1+g+\sigma)^{\alpha-1}+(1+g-\sigma)^{\alpha-1}}{(1+g+\sigma)^{\alpha-1}(1+g+x\sigma)+(1+g-\sigma)^{\alpha-1}(1+g-\sigma x)}\right]_{g=0.018,\sigma=0.036,x=4.5} = 1.05$ yields $\alpha = -11.1$, i.e., a CRRA coefficient of 10.

To also get the riskfree rate right, we need

$$p_b = \frac{1}{1.01}$$

with a reasonable value of β. Setting

$$\frac{\beta}{1-\beta} = .99 \Rightarrow \beta = 0.497\,49$$

and using $\alpha = -11.1$, we solve

$$\left[\frac{\beta}{2(1-\beta)}\left(\frac{(1+g+\sigma)^{\alpha}+(1+g-\sigma)^{\alpha}}{2}\right)^{\frac{\rho-\alpha}{\alpha}}\left((1+g+\sigma)^{\alpha-1}+(1+g-\sigma)^{\alpha-1}\right)\right]_{g=0.018,\sigma=0.036,\beta=}$$

we find $\rho = 0.34$, so the intertemporal elasticity of substitution is $\frac{1}{1-0.34} = 1.515$. This is quite far away from the EU case where we would have IES$=\frac{1}{1-(-11.1)} = 0.08$. Producing an interest rate of

$$\left(\left[\frac{\beta}{2(1-\beta)}\left(\frac{(1+g+\sigma)^{\alpha}+(1+g-\sigma)^{\alpha}}{2}\right)^{\frac{\rho-\alpha}{\alpha}}\left((1+g+\sigma)^{\alpha-1}+(1+g-\sigma)^{\alpha-1}\right)\right]_{g=0.018,\sigma=0.036,}\right.$$

which equals 13.8%.

Recent paper by Bansal and Yaron use Kreps-Porteus specification with CRRA=10 and IES 1.5, i.e., a large deviation from standard EU. They use data to show that dividends are more volatile than consumption, 4.5 times higher. As you can see, these assumptions seems able to account for the risk-premia.

2.4 Closed form value function in partial equilibrium

Consider an individual with no labor income who has access to a market for investments with an exogeneous stochastic i.i.d. return $\tilde{R}$. His budget constraint is thus

$$A_{t+1} = (A_t - c_t)\,\tilde{R}_{t+1}$$

We can now write a Bellman equation

$$W(A_t) = \max_{c_t} V\left(u(c_t), \mu\left(W\left((A_t - c_t)\,\tilde{R}_{t+1}\right)\right)\right)$$

where as above

$$\begin{aligned} u(c) &= c \\ \mu(W) &= \left[EW^{\alpha}\right]^{\frac{1}{\alpha}} \end{aligned}$$

and

$$V(u,\mu) = \left[(1-\beta)\, u^{\rho} + \beta \mu^{\rho}\right]^{\frac{1}{\rho}}. \tag{3}$$

Conjecture that

$$W(A) = \psi_1 A$$

and

$$c = \psi_2 A$$

for yet undetermined coefficients ψ_1 and ψ_2.

Using this, we find that

$$\begin{aligned} W(A_{t+1}) &= W\left((A_t - c_t)\,\tilde{R}_{t+1}\right) = \psi_1 (A_t - c_t)\,\tilde{R}_{t+1} = \psi_1 (1-\psi_2)\, A_t \tilde{R}_{t+1} \\ \mu(W(A_{t+1})) &= \left(E\left(\psi_1(1-\psi_2)\, A_t \tilde{R}_{t+1}\right)^{\alpha}\right)^{\frac{1}{\alpha}} = \psi_1(1-\psi_2)\, A_t \left(E\left(\tilde{R}_{t+1}\right)^{\alpha}\right)^{\frac{1}{\alpha}} \end{aligned}$$

Using this in the Bellman equation;

$$\begin{aligned} \psi_1 A_t &= \left((1-\beta)(\psi_2 A_t)^{\rho} + \beta\left(\psi_1(1-\psi_2)\, A_t \left(E\left(\tilde{R}_{t+1}\right)^{\alpha}\right)^{\frac{1}{\alpha}}\right)^{\rho}\right)^{\frac{1}{\rho}} \\ &= \left((1-\beta)(\psi_2)^{\rho} + \beta\left(\psi_1(1-\psi_2)\left(E\left(\tilde{R}_{t+1}\right)^{\alpha}\right)^{\frac{1}{\alpha}}\right)^{\rho}\right)^{\frac{1}{\rho}} A_t \end{aligned}$$

which is satisfied, provided

$$\psi_1 = \left((1-\beta)(\psi_2)^{\rho} + \beta\left(\psi_1(1-\psi_2)\left(E\left(\tilde{R}_{t+1}\right)^{\alpha}\right)^{\frac{1}{\alpha}}\right)^{\rho}\right)^{\frac{1}{\rho}}. \tag{4}$$

The first-order condition for c_t is the same as choosing ψ_2 optimally,

$$\frac{\partial\left(\left((1-\beta)(\psi_2)^{\rho} + \beta\left(\psi_1(1-\psi_2)\left(E\left(\tilde{R}_{t+1}\right)^{\alpha}\right)^{\frac{1}{\alpha}}\right)^{\rho}\right)\right)}{\partial \psi_2} = 0 \tag{5}$$

Solving these equations gives us the solution to the problem. Assuming a specific form for the distribution of $\tilde{R}$, makes it possible to calculate $\left(E\left(\tilde{R}\right)^{\alpha}\right)^{\frac{1}{\alpha}}$. If, for example, if $\ln\left(\tilde{R}\right) \equiv r$ is normal with mean m and standard deviation σ, i.e., $\ln R \sim n(R; m, \sigma)$ we have

$$\begin{aligned} E\left(\tilde{R}^{\alpha}\right) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (e^{r})^{\alpha}\, e^{-\frac{(r-m)^2}{2\sigma^2}}\, dr \\ &= e^{\alpha m + \frac{\sigma^2 \alpha^2}{2}}, \\ E\left(\tilde{R}^{\alpha}\right)^{\frac{1}{\alpha}} &= e^{m + \frac{\sigma^2 \alpha}{2}} \end{aligned}$$

Note here that increasing σ has a direct effect on the expected return, so it is not a mean preserving spread, i.e.,

$$E\left(\tilde{R}\right) = e^{m+\frac{\sigma^2}{2}}.$$

We can define a mean-preserving spread as increasing σ by letting $m = \bar{m} - \frac{\sigma^2}{2}$, then

$$E\left(\tilde{R}\right) = e^{\left(\bar{m}-\frac{\sigma^2}{2}\right)+\frac{\sigma^2}{2}}.$$

That is, if for any σ, the mean of r is $\bar{m} - \frac{\sigma^2}{2}$, and the standard deviation σ, an increase in σ, is a mean preserving spread. In this case,

$$E\left(\tilde{R}^{\alpha}\right)^{\frac{1}{\alpha}} = e^{\bar{m}-\frac{\sigma^2}{2}+\frac{\sigma^2\alpha}{2}} = e^{\bar{m}-\frac{\sigma^2}{2}(1-\alpha)}$$

Clearly, we here see that an increase in σ, while keeping the mean of the return constant, reduces the certainty equivalent, since $\alpha \leq 1$ and more so the smaller is α.

Denoting $R \equiv \left(E\left(\tilde{R}\right)^{\alpha}\right)^{\frac{1}{\alpha}}$, the choice of ψ_2 solves (5), implying that ψ_2 is a root of

$$\psi_2^{\rho}\left(1-\psi_2^{-1}\right)(1-\beta) + \beta\left(\psi_1\left(1-\psi_2\right)R\right)^{\rho} = 0.$$

In the simplest case of unitary intertemporal elasticity of substitution, i.e., $\rho = 0$, this is

$$\left(1-\psi_2^{-1}\right)(1-\beta) + \beta = 0$$

with the solution $\psi_2 = 1-\beta$.

We should then substitute our optimized value of ψ_2 back into the Bellman equation and find ψ_2 in (4). Note, however, that the formulation in (3) is not valid if $\rho = 0$.We can however, look at the limit as $\rho \to 0$;

$$\begin{aligned} V(u,\mu) &= \lim_{\rho\to 0}\left((1-\beta)u^{\rho} + \beta\mu^{\rho}\right)^{\frac{1}{\rho}} \\ &= u^{1-\beta}\mu^{\beta} \end{aligned}$$

Using this is formalution plus $u_t = c_t = \psi_2 A_t$ and $\mu\left(W\left(A_{t+1}\right)\right) = \psi_1\left(1-\psi_2\right)A_t R$ with $\psi_2 = 1-\beta$ in the Bellman equation gives

$$\begin{aligned} \psi_1 A_t &= u_t^{1-\beta}\mu_t^{\beta} \\ &= \left((1-\beta)A_t\right)^{1-\beta}\left(\psi_1\beta A_t R\right)^{\beta} \\ &= A_t\left((1-\beta)\right)^{1-\beta}\left(\psi_1\beta R\right)^{\beta} \\ \psi_1 &= \psi_1^{\beta}(1-\beta)^{1-\beta}(\beta R)^{\beta} \\ &= (1-\beta)(\beta R)^{\frac{\beta}{1-\beta}} > 0. \end{aligned}$$

In which case we conclude that if $\tilde{R}$ is log-normal

$$W(A) = (1-\beta)\left(\beta e^{m+\frac{\alpha\sigma^2}{2}}\right)^{\frac{\beta}{1-\beta}} A$$

and

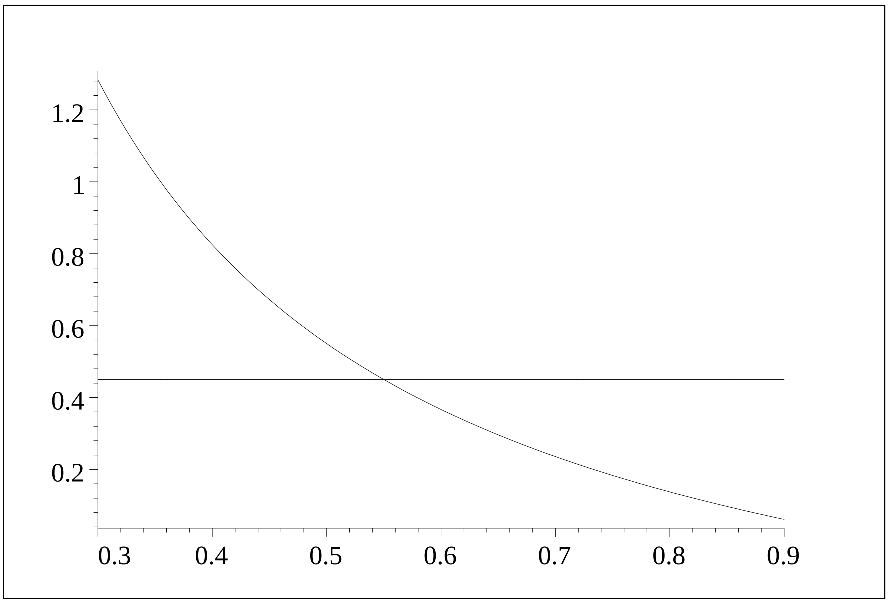
$$c = (1-\beta) A.$$

In this case, high riskaversion or higher risk reduces welfare by reducing the certainty equivalent return (high riskaversion means low α) but saving is unaffected. What happens if $\rho > 0$? First we note that higher risk reduces R and therefore tends to reduce ψ_1, then, in (**??**) a lower R and lower ψ_1 reduces (increases) the term $\beta(\psi_1(1-\psi_2)R)^\rho$ if $\rho > (<)\ 0$. In optimum, this term must equal

$$-\psi_2^\rho\left(1-\psi_2^{-1}\right)(1-\beta)$$

which is downward sloping since $\rho < 1$

$$\frac{d\left(-\psi_2^\rho\left(1-\psi_2^{-1}\right)(1-\beta)\right)}{d\psi_2} = -(1-\beta)\psi_2^{\rho-1}\left(\frac{1-\rho}{\psi_2}+\rho\right) < 0.$$



Therefore, an increase in risk leads to a higher (lower) ψ_2 if $\rho > (<)\ 0$. I.e., consumption increases and savings decreases if the certainty equivalent return decreases iff the intertemporal elasticity of substitution is higher than unity. This is very much along the results with standard expected utility – if the intertemporal elasticity of substitution is higher than unity, the substitution effect dominates after an decrease in interest rates so consumption increases. The converse is true if the elasticity is lower than unity.

3 Ambiguity

Aversion to unknown odds, as is demonstrated in labs, e.g., the Ellsberg paradox, is given axiomatic foundations by Gilboa and Schmeidler. They show that reasonable axioms capturing such ambiguity averse behavior can be represented by a sort of $max - min$ preferences. Suppose it is known that there is a set of possible realizations of the state, $z \in \Gamma$ with associated consumption levels $c(z)$. Then, assume that the individual does not know the probability distribution over these states, but he knows that this probability distribution belongs to a set Π. Call the elements of this set p being particular possible probability distributions. An element p is thus a vector of probabilities $p(z)$. Then, preferences are given by

$$U(\{c(z)\}) = \min_{p \in \Pi} \sum_{z \in \Gamma} p(z)\, c(z) = \min_{p \in \Pi} E_p c(z).$$

As with standard preferences, it is not necessary to take this literally in the sense the individuals actually maximize these minimized preferences. It may, for example, not be possible to ask individuals to tell us directly about Π.

A simple example, consider again the Ellsberg paradox.

There are two urns each with 100 balls. In urn 1, there are 50 red and 50 black. In urn 2, there are only red and black balls but the proportions are unknown. The subject is given a color and can pick one ball. If a ball with the given color comes up, the gain is 50$, if not the gain is zero. The subject is asked to rank lotteries. Typically the following response comes up.

1. Red from urn 1 $\sim$ Black from urn 1.

2. Red from urn 2 $\sim$ Black from urn 2.

3. Red from urn 1 $\succ$ Red from urn 2.

4. Black from urn 1 $\succ$ Black from urn 2.

Suppose we now ask individuals about the "ambiguity premium", i.e., we find value r, such that if we let urn 1 contain $50 - r$ red balls and $50 + r$ black ones.

- Red from urn 1 $\sim$ Red from urn 2.
  and hopefully if we let urn 1 contain $50 + r$ red balls and $50 - r$ black ones

- Black from urn 1 $\sim$ Black from urn 2.

In this case, the relevant $p's$ are the shares of red balls in urn 2. The value of r, pins down Π,being the interval $\left[\frac{50-r}{100}, \frac{50+r}{100}\right]$.

Given this, the value of urn 2 in a bet on red is

$$\begin{aligned} & \min_{p \in \left[\frac{50-r}{100}, \frac{50+r}{100}\right]} pU(50) + (1-p) U(0) \\ = & \frac{50-r}{100} U(50) + \frac{50+r}{100} U(0) \end{aligned}$$

with the same value in a bet on black since in this case, we have

$$\begin{aligned} & \min_{p \in \left[\frac{50-r}{100}, \frac{50+r}{100}\right]} pU(0) + (1-p) U(50) \\ & \frac{50+r}{100} U(0) + \frac{50-r}{100} U(0). \end{aligned}$$

Suppose now that there are two individuals, one owns an asset that gives 50 dollars with a probability p he knows is 50% but that he cannot verify this knowledge to the other person. Otherwise it pays zero. The other person is in the same position, he owns an asset that gives 50 dollars with probability of $p = .5$ that he can not credibly verify. Suppose individuals consume 1+the payoff from the asset and have log utility.

Under standard assumption including that both individuals assign a 50% probability to the other's urn. Individuals should share the risk and get a payoff of

$$\frac{1}{4}\ln(51) + \frac{1}{2}\ln(26) + \frac{1}{4}\ln 1 = 2.612$$

If instead individuals have ambiguity aversion the probability of winning in the "foreign" asset is $p' \in \Pi$. How big share ω should he choose to invest in the other persons asset? Given a $p \in \Pi$, the four states of the world, $\{high, high\}, \{low, high\}, \{high, low\}, \{low, low\}$ happen with probabilities, $\frac{1}{2}p, \frac{1}{2}p, \frac{1}{2}(1-p)$,and $\frac{1}{2}(1-p)$. Given the symmetric nature of the game, we focus on the case when the price of the "foreign" asset in terms of the "domestic" is unity. Consumption, given ω is then $51, .1 + \omega 50, 1 + (1-\omega) 50, 1$. The utility of the optimal portfolio is therefore

$$\min_{p \in P} \max_{\omega} \left(\frac{1}{2} p \ln(1+50) + \frac{1}{2} p \ln(1+\omega 50) + \frac{1}{2}(1-p)\ln(1+(1-\omega)50) + \frac{1}{2}(1-p)\ln 1 \right)$$

The first-order condition for the maximization problem is

$$\frac{d\left(\frac{1}{2}p\ln(1+51) + \frac{1}{2}p\ln(1+\omega 50) + \frac{1}{2}(1-p)\ln(1+(1-\omega)50)\right)}{d\omega}$$

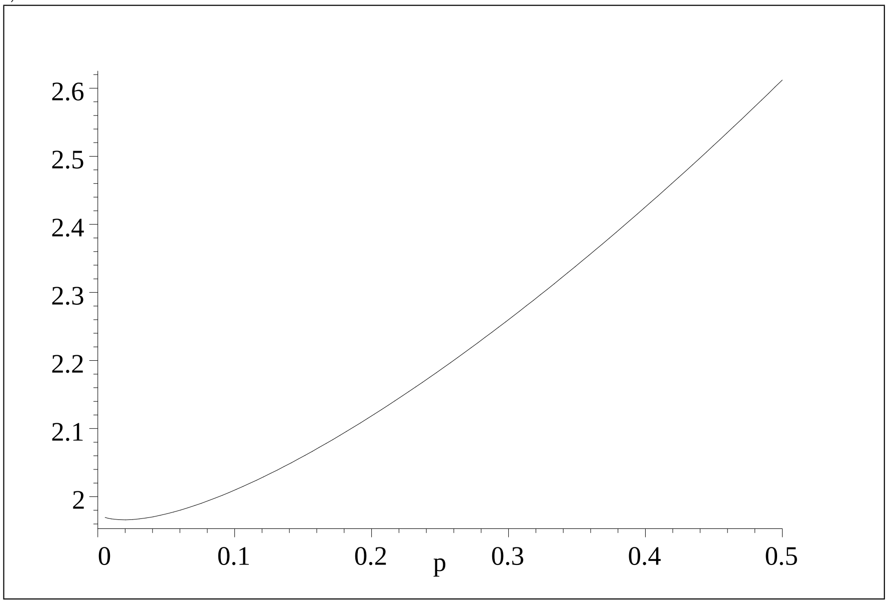
$$\omega = \frac{52p - 1}{50}$$

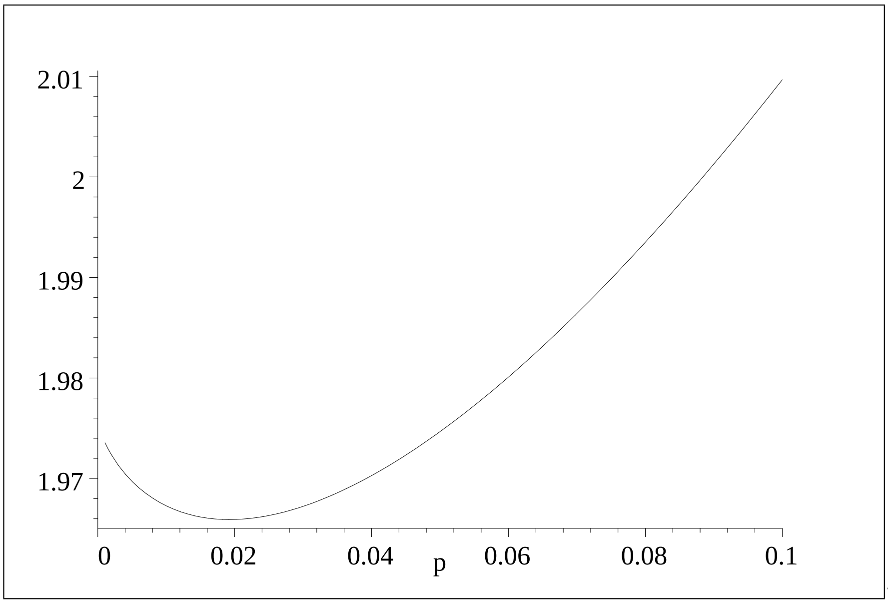
Of course, this is increasing in p. Note also that for $p = 1/52$, $\omega = 0$ and for $p < 1/52, \omega < 0$. For example, if $p = 0.01$, $\omega \approx -0.01$,i.e., a short position in the foreign asset. The short position implies that if the foreign asset pays 50, a payment from hom to abroad takes place.

Now, we have to pick p.To do this, we consider the maximized value, i.e.,

$$\max_{\omega}\left(\frac{1}{2}p\ln(1+51) + \frac{1}{2}p\ln(1+\omega 50) + \frac{1}{2}(1-p)\ln(1+(1-\omega)50)\right)$$
$$= \frac{1}{2}(p\ln(51) + \ln(52) + p\ln(p) + (1-p)\ln(1-p))$$

Let's plot this, what we see at the this is increasing in p for p close to 0.5. But, it is not monotone.





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In fact,

$$\arg\min_p \frac{1}{2}\left(p\ln(51)+\ln(52)+p\ln(p)+(1-p)\ln(1-p)\right)=\frac{1}{52}$$

which corresponds to a share $\omega = 0$. Why is this? The answer is that if the individual would be short in the foreign asset, what is a bad realization has flipped. It is now the state when the foreign asset pays 50, happening with probability p. Therefore, in the range where $\omega < 0$, a higher p means lower utility.

In general, since trade can only increase utility, the worst p, is the one that implies autarky.

The situation would be quite different under asymmetric information, in which case we would expect to sometimes see agents with more information (domestic) go short in home securities.

Let's take another example, suppose production in each country is 2 or 1, so the state space is $\{1,2,3,4\}$, implying output $\{y_1, y_2\}$ is $\{2,2\}, \{2,1\}, \{1,2\}$ or $\{1,1\}$. Utility is

$$U_i = \min_{p\in\Pi} \sum p(z)\ln c_i(z)$$

Suppose also individual the knows that the probability of there own production being high is 0.5, while the set of possible probabilities for the other countries production being high is $0.5 - 2\gamma$, for $\gamma \in [-a, a]$ and that these events are independent.

The agent decides how much to invest abroad ω. Due to symmetry, the relative price of the two assets, foreign and domestic production should be one. The budget constraint is therefore for agent 1.

$$c_1(1) = 2, c_1(2) = (1-\omega)2+\omega, c_1(3) = (1-\omega)+2\omega, c_1(4) = 1$$

Given γ, the maximization problem is

$$\begin{aligned} & \max_{\omega} \sum p(z) \ln c_i(z) \\ = & \max_{\omega} \frac{1}{2}\left(\frac{1}{2}-2\gamma\right)\ln 2 + \frac{1}{2}\left(\frac{1}{2}+2\gamma\right)\ln((1-\omega)2+\omega) \\ & +\frac{1}{2}\left(\frac{1}{2}-2\gamma\right)\ln((1-\omega)+2\omega) + \frac{1}{2}\left(\frac{1}{2}+2\gamma\right)\ln 1 \end{aligned}$$

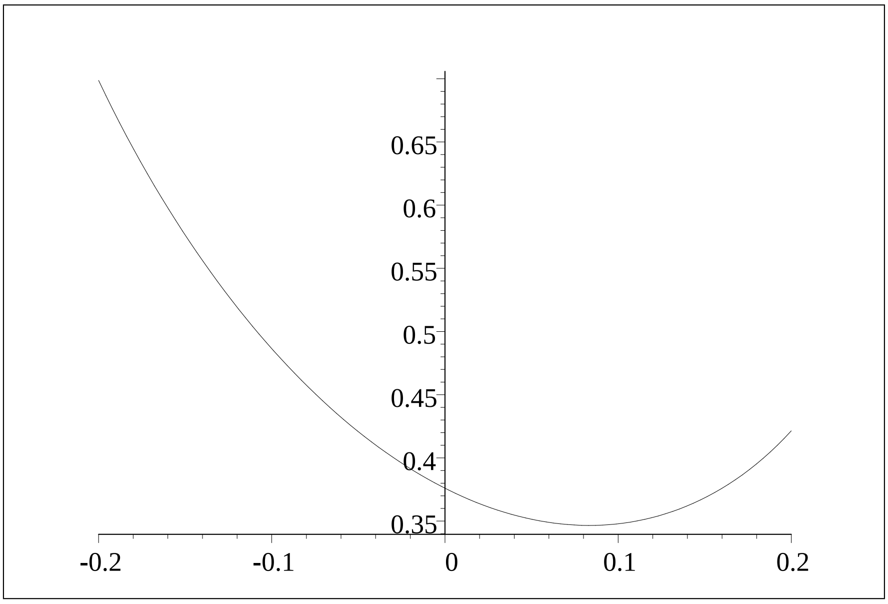
The first order condition is

$$\begin{aligned} \frac{d\left(\frac{1}{2}\left(\frac{1}{2}+2\gamma\right)\ln((1-\omega)2+\omega)+\frac{1}{2}\left(\frac{1}{2}-2\gamma\right)\ln((1-\omega)+2\omega)\right)}{d\omega} &= 0 \\ \omega &= \frac{1}{2}-6\gamma \end{aligned}$$

Substituting this into the utility function, yields

$$\begin{aligned} & \max_{\omega} \sum p(z) \ln c \\ = & \left[\frac{1}{2}\left(\frac{1}{2}-2\gamma\right)\ln 2 + \frac{1}{2}\left(\frac{1}{2}+2\gamma\right)\ln((1-\omega)2+\omega) + \frac{1}{2}\left(\frac{1}{2}-2\gamma\right)\ln((1-\omega)+2\omega)\right]_{\omega=\frac{1}{2}-6\gamma} \\ = & \frac{-1}{4}\ln 2 + \frac{1}{2}\ln 3 - \gamma(\ln 2) + \left(\frac{1}{4}+\gamma\right)\ln(1+4\gamma) + \left(\frac{1}{4}-\gamma\right)\ln(1-4\gamma). \end{aligned}$$

In the following graph, this is plotted against γ



For small a,i.e., γ close to zero. The maximized value is falling in γ in which case γ is "picked" at the corner a. However, for sufficiently large a, this is not true, since the maximized value is non-monotone in γ.

The first order condition for minimizing the maimized value over γ

$$\frac{d\left(\frac{-1}{4}\ln 2+\frac{1}{2}\ln 3-\gamma\left(\ln 2\right)+\left(\frac{1}{4}+\gamma\right)\ln\left(1+4\gamma\right)+\left(\frac{1}{4}-\gamma\right)\ln\left(1-4\gamma\right)\right)}{d\gamma}=0$$

giving $\gamma_{\min}=\frac{1}{12}$. This means that γ will never be chosen larger than $\frac{1}{12}$ and in particular that

$$\omega_{\min}=\frac{1}{2}-6\gamma_{\min}=\frac{1}{2}-\frac{6}{12}=0.$$

:

This is implies the important result that shortsales cannot occur. This is in line with empirical evidence and is not the prediction of models with asymmetric information.

Why is this? Note first that if $\omega=0$, the probability of high production abroad has no effect on utility. If ω is negative, a reduction in p actually increases welfare. Why, the stream of payment from the foreign asset is

$$\omega\left(2p+(1-p)\right)=\omega(1+p),$$

which decreases in p if ω is negative. In this case, high production abroad is the bad state!

4 Time-inconsistency and temptation

Lab evidence discussed in the introduction shows *preference* reversal, quicker discounting in for close time periods than for distant. Also preference for commitment. People sometimes prefer to restrict their future behavior – force themselves to save, hide the jar of cookies, not bring to much money to the bar, and so on.

Evidence that hyperbolic discount factor represents time preference better than geometric.

Two approaches:

- Quasi geometric preferences.
- Preferences over sets (menus), allows modelling of temptation, cost of employing self control and welfare analysis.

4.1 Quasi-geometric preferences

Between current and next period, an "extra" discount factor δ, is introduced (the $\beta\delta$ model).

Self 0 $\quad U_0\left({}_0c\right) = u\left(c_0\right) + \delta\left(\beta u\left(c_1\right) + \beta^2 u\left(c_2\right) + \beta^3 u\left(c_3\right) + \cdots\right)$

Self 1 $\quad U\left({}_1c\right) = u\left(c_1\right) + \delta\left(\beta u\left(c_2\right) + \beta^2 u\left(c_3\right) + \beta^3 u\left(c_3\right) + \cdots\right)$

Self 2 $\quad U\left({}_2c\right) = u\left(c_2\right) + \delta\left(\beta u\left(c_3\right) + \cdots\right)$

This implies that preferences are changing over time.

$$\begin{aligned} \text{Self 0 } MRS_{1,2} &= \frac{\beta^2 u'\left(c_2\right)}{\beta u'\left(c_1\right)} = \beta \frac{u'\left(c_2\right)}{u'\left(c_1\right)} \\ \text{Self 1 } MRS_{1,2} &= \delta\beta \frac{u'\left(c_2\right)}{u'\left(c_1\right)} \end{aligned}$$

Self 1 cares relatively less of period 2 utility than self 1.

<u>Assumptions about behavior</u>:

- The consumer cannot commit to future actions.
- The consumer is "sophisticated": he realizes that his preferences will change and makes the current decision taking this into account.
- The decision-making process is viewed as a dynamic game, with the agent's current and future selves as players. (Alternative: "naive" behavior. The agent thinks that he will not change preferences.)

- Focus is on Markov equilibria, but other equilibria with trigger strategies also exists. For example, a self "behaves" and does not overconsume as long as previous selves has behaved well.
- Markov equilibria can be strange or non-existent. Standard existence theorems not applicable.

Example: Consumption and savings problem.
Suppose the agent has log period utility

$$U_t\left({}_tc\right) = \ln\left(c_t\right) + \delta \sum_{s=1}^{\infty} \beta^s \ln\left(c_{t+s}\right)$$

and face a constant return r. The budget constraint is

$$a_{t+1} = r\left(a_t - c_t\right)$$

The state variable is a_t and let us implicitly define a continuation value from

$$J\left(a_t\right) = \ln c_t^* + \beta J\left(r\left(a_t - c_t^*\right)\right), \tag{6}$$

for some value c_t^*.

If $\delta = 1$, J is also the value function if c_t^* is the *argmax* to the RHS (the whole equation is then the Bellman equation). Under quasi geometric discounting, c_t^* is NOT $\arg\max_c \ln c_t + \beta J\left(r\left(a_t - c_t\right)\right)$ instead

$$c_t^* = \arg\max_{c_t} \ln c_t + \beta\delta J\left(r\left(a_t - c_t\right)\right)$$

and the utility of self t is

$$W\left(a_t\right) = \max_{c_t} \ln c_t + \beta\delta J\left(r\left(a_t - c_t\right)\right)$$

Note that the value of giving assets to the next self is depreciated by the fact that self 0 knows that self 1 is going to overconsume in the eyes of self 0.

Now, we can guess that J has the form

$$J\left(a_t\right) = A + b \ln a_t$$

Given this, c_t^* is the solution to the first order condition

$$\begin{aligned} 0 &= \frac{1}{c_t^*} - \beta\delta J'\left(r\left(a_t - c_t^*\right)\right) r \\ &= \frac{1}{c_t^*} - \beta\delta \frac{b}{r\left(a_t - c_t^*\right)} r \\ &\Rightarrow\ c_t^* = \frac{a_t}{1 + \beta\delta b} \end{aligned}$$

Substituting this in (6) gives

$$\begin{aligned} A + b\ln a_t &= \ln \frac{a_t}{1+\beta\delta b} + \beta\left(A + b\ln r\left(a_t - \frac{a_t}{1+\beta\delta b}\right)\right) \\ &= (1+\beta b)\ln a_t + \beta A + \beta b \ln \frac{r\beta\delta b}{1+\beta\delta b} - \ln(1+\beta\delta b) \end{aligned}$$

This is satisfied for all a_t iif

$$\begin{aligned} (1+\beta b) &= b \\ &\Rightarrow b = \frac{1}{1-\beta} \end{aligned}$$

implying

$$c_t^* = \frac{a_t}{1+\beta\delta\frac{1}{1-\beta}} = \frac{1-\beta}{1-\beta(1-\delta)}a_t$$

and A is

$$A = \frac{\frac{\beta}{1-\beta}\ln\frac{r\beta\delta}{1-\beta(1-\delta)} + \ln\left(\frac{1-\beta}{1-\beta(1-\delta)}\right)}{1-\beta}.$$

As we see, if $\delta < 1$, consumption is higher and savings are lower than in the time-consistent case, when the consumption rate is $1-\beta$.

Let's now find the commitment solution if self 0 determines all consumption values. In this case, we first calculate $J_c(a_t)$,which is the continuation value when everything is determined by self 0. Note, however, that future selves will agree with self zero on this. The difference between the no-commitment case is that now the continuation value maximizes the standard Bellman equation without any δ. Thus, $J_c$must satisfy,

$$J_c(a_t) = \max_{c_t} \ln c_t + \beta J_c(r(a_t - c_t))$$

If we don't remember the solution to this, we do as usual. We take the first order condition

$$\frac{1}{c_t} = \beta r J'(r(a_t - c_t)).$$

Guessing

$$J_c(a_t) = A_c + b_c \ln a_t$$

implies

$$\begin{aligned} \frac{1}{c_t} &= \beta b_c r \frac{1}{r(a_t - c_t)} \\ c_t &= \frac{a_t}{1+\beta b_c}. \end{aligned}$$

Substituting,

$$
\begin{aligned}
A_c + b_c \ln a_t &= \ln \frac{a_t}{1+\beta b_c} + \beta A_c + \beta b_c \ln r a_t \left(1 - \frac{1}{1+\beta b_c}\right) \\
A_c + b_c \ln a_t &= \ln a_t + \ln \frac{1}{1+\beta b_c} + \beta A_c + \beta b_c \ln a_t + \beta b_c \ln r \left(\frac{\beta b_c}{1+\beta b_c}\right)
\end{aligned}
$$

Giving

$$
\begin{aligned}
&\Rightarrow \; b_c = (1+\beta b_c)\,, b_c = \frac{1}{1-\beta} \\
c_t &= \frac{a_t}{1+\beta b_c} = (1-\beta)\, a_t \\
A_c &= \ln \frac{1}{1+\beta \frac{1}{1-\beta}} + \beta A_c + \beta \frac{1}{1-\beta} \ln r \left(\frac{\beta \frac{1}{1-\beta}}{1+\beta \frac{1}{1-\beta}}\right) \\
&= \ln (1-\beta) + \beta A_c + \frac{\beta}{1-\beta} \ln r\beta \\
&\Rightarrow \; A_c = \frac{\ln(1-\beta) + \frac{\beta}{1-\beta} \ln r\beta}{1-\beta}
\end{aligned}
$$

As we see, the coefficient on $\ln a_{t+1}$ is the same in both cases, commitment and no commitment. The difference between the constants under no commitment and commitment is negative, I think. The fact that the coefficient on $\ln a_{t+1}$ is the same in J_c and J, implies that the marginal value of leaving assets to self 1 is the same in both cases. Thus, consumption in period 0 is independent of whether there is commitment or not. Note that two forces here are affecting the results. On the one hand, giving assets to self 1 under no commitment has a lower value since he consumers too much in the eyes of self 0. This reduces the incentive to save for self 0. On the other hand, this leaves self 2, 3,... with too little consumption and the way only way self 0 can increase consumption of self 2,3,... is to save. This increases the value of saving. Apparently, this two effects cancel in the log utility case.

From period 1 and onwards, savings is higher under commitment.

$$
\beta > \frac{\beta\delta}{1-\beta(1-\delta)}
$$

Commitment would of course increase welfare for self 0, it can never reduce it. What about later selves?

In the no commitment case, self 1 gets

$$\begin{aligned}
W_{nc} &= \ln c_1^* + \beta\delta J\left(r\left(a_1 - c_{t+1}^*\right)\right) \\
&= \ln \frac{1-\beta}{1-\beta(1-\delta)} a_1 \\
&\quad + \beta\delta\left(A + \frac{1}{1-\beta}\ln r a_1\left(\frac{\beta\delta}{1-\beta(1-\delta)}\right)\right) \\
&= \frac{1-\beta(1-\delta)}{1-\beta}\ln a_1 + \ln\frac{1-\beta}{1-\beta(1-\delta)} \\
&\quad + \beta\delta\left(A + \frac{1}{1-\beta}\ln r\left(\frac{\beta\delta}{1-\beta(1-\delta)}\right)\right)
\end{aligned}$$

With commitment, the continuation value is different since now self 0 determines everything.

Self 1 gets under commitment

$$\begin{aligned}
& \ln(1-\beta)a_1 + \beta\delta J_c(r\beta a_1) \\
=\ & \ln(1-\beta)a_1 + \beta\delta b_c\ln(r\beta a_1) + \beta\delta A_c \\
=\ & \left(\frac{1-\beta(1-\delta)}{1-\beta}\right)\ln a_1 + \ln(1-\beta) + \frac{\beta\delta}{1-\beta}\ln r\beta + \beta\delta A_c
\end{aligned}$$

The difference between no comittment and commitment is.

$$\frac{1}{(1-\beta)^2}\left(\left(\ln\left(1-\beta(1-\delta)\right)\right)\left(\beta(2-\beta)(1-\delta)-1\right) + \beta\delta\ln\delta\right) \equiv D(\delta)$$

This expression is a little ugly, but we can evaluated it by a Taylor approximation around $\delta = 1$, using

$$\begin{aligned}
\left[D(\delta)\right]_{\delta=1} &= 0 \\
\left[D'(\delta)\right]_{\delta=1} &= 0 \\
\left[D''(\delta)\right]_{\delta=1} &= (1-2\beta)\frac{\beta}{1-\beta}
\end{aligned}$$

which is negative if $\beta > \frac{1}{2}$.Therefore, a reduction in δ from unity makes the no commitment better (worse) if $\beta > \frac{1}{2}(\beta < \frac{1}{2})$.

Commitment gives extra value which is good also for self 1, but she cannot control her consumption which reduces the value. For self 1, commitment can therefore be better than no commitment, also if it is done by self 1. For later individuals, it may be even better with previous commitment since asset levels are higher.

4.2 Preferences over choice sets, Gul and Pesendorfer

An alternative approach. Does not assume multiple selves, no game thus no multiplicity. Also allows resistance to temptation and to model costs of resisting temptation.

- Two subperiods.
- Second subperiod preferences defined over ordered pairs (A, x), where A is a choice set and $x \in A$ is a choice (consumed).
- Definition: y tempts x if $(\{x\}, x)$ is preferred to $(\{x, y\}, x)$. That is, individuals are better of getting x *without* having y in the choice set.
- Assumptions:
  1. Eliminating temptations cannot make the consumer worse off.
  2. If y tempts x, then x does not tempt y.
  3. The utility of a fixed choice is affected by the choice set only through its most tempting element.
- Second-period preferences induce first-period preferences over choice sets themselves: $A \succeq B$ if and only if there is an $x \in A$ such that (A, x) is preferred to (B, y) for all $y \in B$.
- The above assumptions imply what is labelled *set betweenness*:

  $$A \succeq B \Rightarrow A \succeq A \cup B \succeq B.$$

  Choice sets cannot be compared simply by looking at their "best" or chosen elements. Instead, the utility of a fixed choice depends on the choice set (through its most "tempting" element). Note that this violates one of the axioms in standard theory. Removing a non-choosen element from a choice set cannot change utility or behavior (independence of irrelevant alternatives).

  Set betweenness allows three possibilities:

  1. Standard decision maker: $A \sim A \cup B \succeq B$.
  2. *Preference for commitment* and *self-control*: $A \succ A \cup B \succ B$. Interpretation: there is an element in B that tempts me. Nevertheless, I choose the same element in A and $A \cup B$, but if faced with only A, I don't have to take the effort of controlling myself. Thus $A \succ A \cup B$. Furthermore, $A \cup B \succ B$ since the choice in $A \cup B$ provides higher utility,than the tempting choice.

3. *Preference for commitment* and *succumbing to temptation*: $A \succ A \cup B \sim B$.
   Interpretation: there is an element in B that tempts me. $A \succ A \cup B$ since it provides higher utility. Faced with the tempting choice, however. I cannot resist. I choose the same element in $A \cup B$ and B and there is no cost of controlling myself.Thus, $A \cup B \sim B$

4.3 The representation theorem

The assumptions implies that preference over sets in the first period can be written

$$W(A) = \max_{x \in A} \{U(x) + V(x)\} - \max_{\tilde{x} \in A} V(\tilde{x})$$

Second period, preference are represented by

$$W^*(A, x) = \{U(x) + V(x)\} - \max_{\tilde{x} \in A} V(\tilde{x})$$

Interpretation:

- U determines the *commitment* ranking (i.e., the utility of singleton sets, no temptation).
- V determines the *temptation* ranking (i.e., V gives higher values to more tempting elements).
- $\arg\max_{\tilde{x} \in A} V(\tilde{x})$ is the most tempting element in A.
- The second-period choice picks x as

  $$\arg\max_{x \in A} \{U(x) + V(x)\}$$

  giving utility

  $$\max_{x \in A} \{U(x) + V(x)\} - \max_{\tilde{x} \in A} V(\tilde{x}).$$

- If a person is given $x \in A$, *without* anything else to choose from, there is no cost of self control. The utility is

  $$U(x) + V(x) - V(x) = U(x).$$

- If a person *chooses* $x \in A$, the disutility of self-control is $V(x) - \max_{\tilde{x} \in A} V(\tilde{x}) \geq 0$,so utility is

  $$U(x) + V(x) - \max_{\tilde{x} \in A} V(\tilde{x}).$$

- If a person *chooses* $\tilde{x} = \arg\max_{\tilde{x}\in A} V(\tilde{x})$, he gives in to temptation, there is no cost of self-control, and the utility is

$$U(\tilde{x}) + \max_{\tilde{x}\in A} V(x) - \max_{\tilde{x}\in A} V(\tilde{x}) = U(\tilde{x}).$$

4.4 A 2-period consumption-savings model

- Consumption today and tomorrow.
- Neoclassical production.
- Standard budget set (borrowing and lending at r).
- General equilibrium.
- With $U(c_1, c_2)$ playing the role of U and $V(c_1, c_2)$ the role of V, let the temptation function V have a stronger preference for present consumption. For example, let

  $$U(c_1, c_2) = u(c_1) + \beta u(c_2)$$

  and

  $$V(c_1, c_2) = \gamma \left(u(c_1) + \beta\delta u(c_2)\right),$$

  with $\delta, \beta < 1$.

  Aggregate resource constraint given by

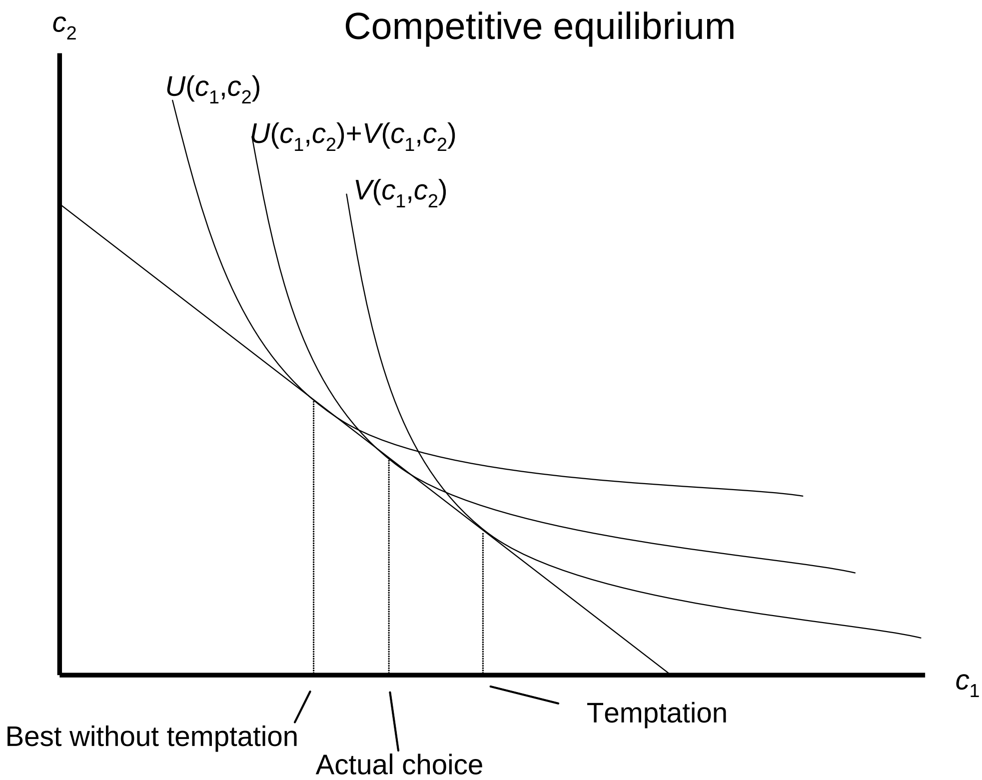
  $$\begin{aligned} c_1 + k_2 &= f(k_1) \\ c_2 &= f(k_2) \end{aligned}$$

- Strength of temptation determined by γ. Standard model when $\gamma = 0$. As $\gamma \to \infty$, Laibson model.

  In equilibrium choices are made to maximize

  $$U(c_1, c_2) + V(c_1, c_2)$$

- In competitive general equilibrium individuals take prices (here interest rate) as given, provides a linear budget set for the individual.



Policy implications. Command optimum can achieve $\arg\max U$, without any temptation cost. Nothing is better than this. A subsidy to investments (tax on first period consumption) can improve upon *laissez faire.* Does so by reducing temptation.

For example, let $u(x) = \ln(x)$. Then, choices are governed by solving

$$\begin{aligned} & \max\left(\ln c_1) + \beta \ln(c_2)\right) + \gamma\left(\ln(c_1) + \beta\delta \ln(c_2)\right) \\ \text{s.t. } c_1 + \left(\frac{c_2}{1+r}\right) &= w \\ c_1 &= \arg\max\left((1+\gamma)\ln c_1 + \beta\left(1+\gamma\delta\right)\ln((1+r)(w-c_1))\right) \\ &= w\frac{1+\gamma}{1+\beta+\gamma\left(1+\beta\delta\right)} \end{aligned}$$

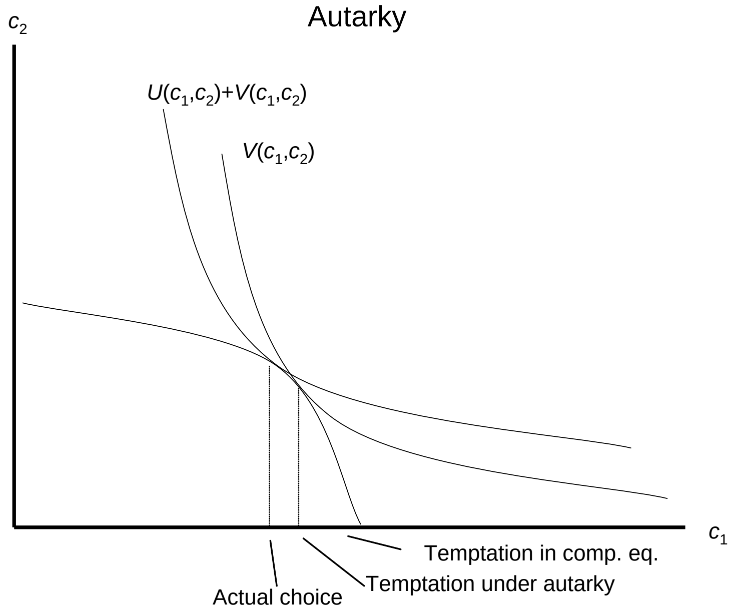
which increases in γ.

The maximum temptation is

$$\begin{aligned} c_1^t &= \arg\max \gamma\left(\ln(c_1) + \beta\delta \ln((1+r)(w-c_1))\right) \\ &= \frac{w}{1+\beta\delta} \end{aligned}$$

Interesting implication

Compare autarky, i.e., each individual runs his own machine. Then the interest rate is not exogenous.



Result: Autarky delivers the same allocation, but at higher welfare. Why?

The choice sets shrinks, the temptation to overconsume is reduced and the cost of resisting temptation falls.

4.5 Macroeconomic applications

Krusell, Kuruşçu, Smith "Temptation and Taxation"

- Consider long horizons: the limit of the finite-horizon problems.
- Study competitive equilibrium under two kinds of parametric restrictions:

    1. *Logarithmic utility, Cobb-Douglas production, and full depreciation*: full analytical solution of recursive competitive equilibria.
    2. *Iso-elastic utility and no restrictions on technology*: analytical characterization of steady state and computational analysis of dynamics.

Analysis: Vary δ and γ, while adjusting β to keep the steady-state interest rate constant.

Results:

- Almost observationally equivalence (like in Barro "Laibson meets Ramsey", when the utility is log, the speed of adjustment to the steady state does not depend on δ (as in Barro).
- With more (less) curvature in utility, the speed of adjustment is decreasing (increasing) in δ.
- The effects of δ on the speed of adjustment is quantitatively small: observational equivalence found in Barro "almost" carries over.
- Savings should be subsidized. But not much, and the welfare gains are small (little reduction in temptation costs).

5 Habits

We have previously assumed *history independence,* meant to mean that the marginal rate of substitution, evaluated at t between:

- goods consumed at t and $t+1$, and
- different goods consumed in the same time is independent of the consumption history prior to t.

This is not necessarily a good assumption.
There are cases in which we might want to relax history independence.

1. The case when previous consumption leads to higher aspiration. To live on a small budget may be easier if you are used to it than if your are used to the good life.

2. The relative taste for some goods is affected by previous consumption of them, e.g., food, culture goods and drugs.

   Another case, perhaps conceptually different but analytically similar is when relative consumption matters for utility. "Catching up with the Joneses or "Poverty is more easily accepted if it is shared by everyone" Ernst Wigfors, Social Democratic Finance Minister 1932-49.

In the literature, the first case and the case of "Catching up with the Joneses" have been used to try to explain asset market puzzles, i.e., why standard models have great problems explaining the co-movements of prices and consumption. The second case is used to explain, for example, cultural diversity.

To simplify, in particular in order to be able to specify recursive preferences, utility is assumed to be

$$U_t = \sum_{j=0}^{\infty} \beta^j u\left(c_{t+j}, \nu_{t+j}\right)$$

where

$$\nu_t = v\left(\tilde{c}_{t-1}, \tilde{c}_{t-2}, ...\tilde{c}_{t-n}\right),$$

is denoted the habit. A couple of things to note,

1. We maintain time additivity here, although this should be straightforward to generalize to the constant elasticity aggregator.

2. $\tilde{c}_{t-s}$ can denote the own previous consumption of the household, the consumption of some reference group or some combination. If $\tilde{c}_{t-s} = c_{t-s}$ (own consumption), we have what is called "internal habits" while if it is the consumption of some reference group, it is called "external" habit. Abel uses a geometric average

$$\nu_t \equiv \left(c_{t-1}^D C_{t-1}^{1-D}\right)^\gamma,$$

where C_{t-1} is aggregate consumption. Here $\gamma = 0$ gives the standard model, $\gamma \neq 0$, and $D = 1$,gives the internal habit case and $D = 0$ the external case.

3. History matters only though the habit function, i.e., through

$$v\left(\tilde{c}_{t-1}, \tilde{c}_{t-2}, ...\tilde{c}_{t-n}\right)$$

4. We assume either a finite value of n or at least that

$$\lim_{n\to\infty} \frac{\partial v\left(\tilde{c}_{t-1}, \tilde{c}_{t-2}, ...\tilde{c}_{t-n}\right)}{\partial \tilde{c}_{t-n}} = 0.$$

so that we can hope to find stationary decision rules.

5.1 Optimal consumption under external vs. internal habits.

Suppose the representative agent solves

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t u\left(c_t, \nu_t\right) \\ \text{s.t. } a_{t+1} &= \left(a_t - c_t\right) r \\ v_t &= \left(c_{t-1}^D C_{t-1}^{1-D}\right)^\gamma \end{aligned}$$

and a no-Ponzi condition.

Let's look at the Bellman equation. Note that now, v_t is a state variable. Furthermore, aggregate assets A_t is a state variable, since aggregate assets tell us what aggregate consumption will be in the future. Therefore

$$\begin{aligned} V\left(a_t, A_t, v_t\right) &= \max_{c_t} u\left(c_t, \nu_t\right) + \beta V\left(a_{t+1}, A_{t+1}, v_{t+1}\right) \\ \text{s.t. } A_{t+1} &= \left(A_t - C_t\right) r, \\ v_t &= \left(c_{t-1}^D C_{t-1}^{1-D}\right)^\gamma, \\ a_{t+1} &= \left(a_t - C_t\right) r \end{aligned}$$

The first order condition is

$$\begin{aligned} u_1(c_t, v_t) &= \beta V_1(a_{t+1}, A_{t+1}, v_{t+1}) r - \beta V_3(a_{t+1}, A_{t+1}, v_{t+1}) \frac{\partial v_{t+1}}{\partial c_t} \\ &= \beta V_1(a_{t+1}, A_{t+1}, v_{t+1}) r - \beta \gamma D \frac{v_{t+1}}{c_t} V_3(a_{t+1}, A_{t+1}, v_{t+1}) \end{aligned}$$

Clearly, the first order condition is not affected if $\gamma D = 0$. Suppose instead $\gamma D > 0$. Then an increase in today's consumption increases the habit. Suppose this reduces utility, then this implies that there is a negative dynamic effect of consumption which will show up in a negative V_3. Therefore, the term $-\beta\gamma D \frac{v_{t+1}}{c_t} V_3(a_{t+1}, A_{t+1}, v_{t+1})$ is positive and marginal utility of consumption should be set higher in period t.

Consider the case of external habits, $D = 0$. We have seen that in this case, the FOC is the same as under no habits. Suppose first for simplicity that

$$u(c_t, \nu_t) = \ln c_t - \ln v_t$$

Recall the solution strategy in the case when we expect that there can be an analytical solution.

1. Write Bellman equation.

2. Guess a functional form of the value function with unknown parameters. From the Bellman equation we see that it has to be of the same functional form as the per-period utility function.

3. Solve for the choice variable that maximizes the RHS of the Bellman equation given our guess on the value function.

4. Substitute your optimal choice variable into the RHS of the Bellman equation to express the maximized RHS.

5. Verify that the Bellman equation is satisfied for *all* values of the state variables by finding the unknown parameters.

If step 5 fails you have made an incorrect guess and must start with another. However, most problems do not admit closed form solutions for the value function in which case this approach is useless.

Now, we guess that the value function is

$$V(a_t, A_t, v_t) = B_1 \ln a_t + B_2 \ln A_t + B_3 \ln v_t + B_4$$

for the unknown coefficients B_1, B_2, B_3 and B_4.

The FOC is

$$\begin{aligned}
\frac{1}{c_t} &= \frac{\beta B_1 r}{a_{t+1}} \\
\frac{1}{c_t} &= \frac{\beta B_1}{a_t - c_t} \\
&\Rightarrow \; c_t = \frac{1}{1+\beta B_1} a_t, \\
a_{t+1} &= (a_t - c_t)\, r \\
&= \frac{\beta B_1}{1+\beta B_1} r a_t
\end{aligned}$$

Which we recognize well.

Substituting this into our guess gives

$$B_1 \ln a_t + B_2 \ln A_t + B_3 \ln v_t + B_4 = \ln \frac{a_t}{1+\beta B_1} - \ln v_t$$
$$+\beta \left(B_1 \ln a_{t+1} + B_2 \ln A_{t+1} + B_3 \ln v_{t+1} + B_4\right)$$
$$= \ln \frac{a_t}{1+\beta B_1} - \ln v_t + \beta \left(B_1 \ln \frac{\beta B_1}{1+\beta B_1} r a_t + B_2 \ln A_{t+1} + B_3 \ln v_{t+1} + B_4 \right)$$

This cannot work unless we get rid of v_{t+1} and A_{t+1} in the RHS. To do this we note that

$$v_{t+1} = C_t^{\gamma}$$

and since everybody does the same thing, $C_t = c_t$ and therefore,

$$\begin{aligned}
C_t &= \frac{1}{1+\beta B_1} A_t \\
v_{t+1} &= \left(\frac{1}{1+\beta B_1} A_t \right)^{\gamma}.
\end{aligned}$$

Furthermore, for the same reason $a_{t+1} = \frac{\beta B_1}{1+\beta B_1} r a_t$ implies $A_{t+1} = \frac{\beta B_1}{1+\beta B_1} r A_t$
Therefore,

$$\begin{aligned}
& B_1 \ln a_t + B_2 \ln A_t + B_3 \ln v_t + B_4 \\
= \;& \ln \frac{a_t}{1+\beta B_1} - \ln v_t + \beta \left(B_1 \ln \frac{\beta B_1}{1+\beta B_1} r a_t + B_2 \ln \frac{\beta B_1}{1+\beta B_1} r A_t + B_3 \gamma \ln \frac{1}{1+\beta B_1} A_t + B_4 \right)
\end{aligned}$$

We solve this by equalizing the coefficients on the different terms

$$\begin{aligned}
B_1 &= (1+\beta B_1) \\
B_2 &= \beta B_2 + \beta \gamma B_3 \\
B_3 &= -1 \\
B_3 &= \beta B_1 \ln \beta B_1 r - (1 + \beta B_1 + \beta B_2 + \beta \gamma B_3) \ln (1+\beta B_1) + \beta B_4
\end{aligned}$$

Giving

$$\begin{aligned} B_1 &= \frac{1}{1-\beta} \\ B_2 &= -\frac{\beta\gamma}{1-\beta} \\ B_3 &= -1 \end{aligned}$$

Again using the FOC, we get

$$\begin{aligned} c_t &= \frac{1}{1+\beta\frac{1}{1-\beta}}A_t \\ &= (1-\beta)\,A_t \end{aligned}$$

As we see, consumption is independent of γ. This is due to the fact marginal utility is not affected by the habit. This is due to the log utility. Marginal utility of consumption is $\frac{1}{c_t}$regardless of v_t since u is separable in c and v.

Let us therefore consider a generalization. Instead of solving the full problem, we can at least characterize consumption dynamics.

Suppose

$$\begin{aligned} u\,(c_t, v_t) &= \frac{1}{1-\alpha}\left(\frac{c_t}{v_t}\right)^{1-\alpha} \\ u_1\,(c_t, v_t) &= \frac{1}{c_t}\left(\frac{c_t}{v_t}\right)^{1-\alpha} \end{aligned}$$

The Euler condition under purely external habits is the usual

$$1 = \beta\frac{u_1\,(c_{t+1}, v_{t+1})}{u_1\,(c_t, v_t)}r$$

In general equilibrium $v_t = c_{t-1}^{\gamma}$, giving

$$1 = \beta\frac{\frac{1}{c_{t+1}}\left(\frac{c_{t+1}}{v_{t+1}}\right)^{1-\alpha}}{\frac{1}{c_t}\left(\frac{c_t}{v_t}\right)^{1-\alpha}}r = \beta\frac{\frac{1}{c_{t+1}}\left(\frac{c_{t+1}}{c_t^{\gamma}}\right)^{1-\alpha}}{\frac{1}{c_t}\left(\frac{c_t}{c_{t-1}^{\gamma}}\right)^{1-\alpha}}r = \beta c_{t+1}^{-\alpha}c_t^{\gamma(\alpha-1)+\alpha}c_{t-1}^{\gamma(1-\alpha)}r$$

Taking logs

$$\begin{aligned} 0 &= \ln\beta r - \alpha\ln c_{t+1} + (\alpha - \gamma\,(1-\alpha))\ln c_t + \gamma\,(1-\alpha)\ln c_{t-1} \\ \alpha\,(\ln c_{t+1} - \ln c_t) &= \ln\beta r - \gamma\,(1-\alpha)\,(\ln c_t - \ln c_{t-1}) \end{aligned}$$

In the case $\gamma = 0$,the standard case, the growth rate of consumption is constant at

$$\ln c_{t+1} - \ln c_t = \frac{\ln \beta r}{\alpha},$$

as we should now from standard models.

We have also seen that with $\alpha = 1$,

$$\ln c_{t+1} - \ln c_t = \ln \beta r$$

Dynamics becomes interesting now under $\alpha \neq 1$ and $\gamma > 0$. Define

$$\ln c_{t+1} - \ln c_t \equiv g_{t+1}$$

Then,

$$g_{t+1} = \frac{\ln \beta r}{\alpha} - \gamma \frac{(1-\alpha)}{\alpha} g_t \tag{7}$$

When riskaversion is low (IES high), that is $a > 0$, we get oscillations! A low growth rate is followed by a high and vice versa. The oscillations may even be unstable if $\gamma \frac{(1-\alpha)}{\alpha} > 1$. If instead $\alpha < 0$,we get a monotone path, that is stable if $-\gamma \frac{(1-\alpha)}{\alpha} < 1$.

Note that we have assumed a constant interest rate r, this is quite easy to relax. With a varying interest rate, for example if we include capital accumulation, we still have

$$g_{t+1} = \frac{\ln \beta r_{t+1}}{\alpha} - \gamma \frac{(1-\alpha)}{\alpha} g_t.$$

5.2 Adding stochastics

With stochastics, we have

$$\begin{aligned}
\beta E_t \frac{1}{c_{t+1}} \left(\frac{c_{t+1}}{c_t^{\gamma}} \right)^{1-\alpha} r_{t+1} &= \frac{1}{c_t} \left(\frac{c_t}{c_{t-1}^{\gamma}} \right)^{1-\alpha} \\
\beta E_t c_{t+1}^{-\alpha} c_t^{\gamma(\alpha-1)} r_{t+1} &= c_t^{-\alpha} c_{t-1}^{\gamma(\alpha-1)} \\
E_t \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\alpha} r_{t+1} &= \left(\frac{c_t}{c_{t-1}} \right)^{\gamma(1-\alpha)} \\
E_t \beta \left(e^{g_{t+1}} \right)^{-\alpha} r_{t+1} &= \left(e^{g_t} \right)^{\gamma(1-\alpha)}.
\end{aligned}$$

: : :

This can be analyzed by linearization. Suppose

$$r_{t+1} = e^{z_{t+1}} r$$

where $\beta r = 1$, giving a steady state of the economy.

Approximating around $g = 0, z = 0$,

$$\begin{aligned} \beta\left(e^{g_{t+1}}\right)^{-\alpha} e^{z_{t+1}} r &\approx \beta r - \alpha g_{t+1} \beta r + \beta r z_{t+1} \\ \left(e^{g_t}\right)^{\gamma(1-\alpha)} &\approx 1 + \gamma(1-\alpha) g_t \end{aligned}$$

Giving

$$\begin{aligned} E_t\left(\beta r - \alpha g_{t+1} \beta r + \beta r z_{t+1}\right) &= 1 + \gamma(1-\alpha) g_t \\ E_t g_{t+1} &= -\frac{\gamma(1-\alpha)}{\alpha} g_t + \frac{E_t z_{t+1}}{\alpha}, \end{aligned}$$

which can provide interesting dynamics.

To understand the reason for the oscillatory behavior, it may be of help to note that individuals with habits might prefer variations over time. Clearly, when $\gamma = 0$, individuals are risk averse for $\alpha > 0$ and also averse to variations over time. Consider instead the case when $\gamma = 1$. Assume that (individual and aggregate) consumption is

$$c_t = \begin{cases} c(1+\varepsilon) & \text{if } t \text{ odd} \\ c(1-\varepsilon) & \text{else.} \end{cases}$$

In this case, utility in even periods (multiplied by $(1-\alpha)$ for convenience) is

$$u_t = \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{1-\alpha}$$

and in odd

$$u_{t+1} = \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{1-\alpha}$$

we have

$$\begin{aligned} \frac{u_t + u_{t+1}}{2} &= \frac{\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{1-\alpha} + \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{1-\alpha}}{2} \\ &= \frac{\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{1-\alpha} + \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{1-\alpha}}{2} \\ &\approx 1 + \varepsilon \left[\frac{d\left(\frac{\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{1-\alpha} + \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{1-\alpha}}{2}\right)}{d\varepsilon}\right]_{\varepsilon=0} + \frac{\varepsilon^2}{2}\left[\frac{d^2\left(\frac{\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{1-\alpha} + \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{1-\alpha}}{2}\right)}{d\varepsilon^2}\right]_{\varepsilon=0} \\ &= 1 + 2(1-\alpha)^2 \varepsilon^2 \end{aligned}$$

which is increasing in ε^2.

5.3 Asset market implications.

Abel has shown that habits also may have an ability to explain asset market puzzles. Let us finally go over this.

Using

$$\begin{aligned} u(c_t, v_t) &= \frac{1}{1-\alpha}\left(\frac{c_t}{v_t}\right)^{1-\alpha} \\ u_1(c_t, v_t) &= \frac{1}{c_t}\left(\frac{c_t}{v_t}\right)^{1-\alpha} \end{aligned}$$

utility in period t can be written

$$\begin{aligned} U_t &= \frac{1}{1-\alpha}\frac{c_t}{v_t}^{1-\alpha} + \beta\left(\frac{1}{1-\alpha}\frac{c_{t+1}}{v_{t+1}}^{1-\alpha} + \beta V(A_{t+2}, v_{t+2})\right) \\ \text{s.t. } A_{t+1} &= (A_t - c_t)\, r_{t+1}, v_t = \left(c_{t-1}^D C_{t-1}^{1-D}\right)^\gamma \\ A_{t+2} &= (A_{t+1} - c_{t+1})\, r_{t+2} \end{aligned}$$

Using the definition of habits, we get

$$U_t = \frac{1}{1-\alpha}\left(\frac{c_t}{v_t}\right)^{1-\alpha} + \beta\left(\frac{1}{1-\alpha}\left(\frac{c_{t+1}}{\left(c_{t-1}^D C_{t-1}^{1-D}\right)^\gamma}\right)^{1-\alpha} + \beta V(A_{t+2}, v_{t+2})\right)$$

Therefore

$$\begin{aligned} \frac{\partial U_t}{\partial c_t} &= \frac{\partial\left(\frac{1}{1-\alpha}\left(\frac{c_t}{v_t}\right)^{1-\alpha} + \beta\left(\frac{1}{1-\alpha}\left(\frac{c_{t+1}}{\left(c_t^D C_t^{1-D}\right)^\gamma}\right)^{1-\alpha}\right)\right)}{\partial c_t} \\ &= \frac{1}{c_t}\left(\left(\frac{c_t}{v_t}\right)^{1-\alpha} - \beta\gamma D\left(\frac{c_{t+1}}{\left(c_t^D C_t^{(1-D)}\right)^\gamma}\right)^{1-\alpha}\right) \\ &= \frac{1}{c_t}\left(\left(\frac{c_t}{v_t}\right)^{1-\alpha} - \beta\gamma D\left(\frac{c_{t+1}}{\nu_{t+1}}\right)^{1-\alpha}\right) \\ &= \frac{1}{c_t}\left(\left(\frac{c_t}{\nu_t}\right)^{1-\alpha} - \beta\gamma D\left(\frac{c_{t+1}}{\nu_{t+1}}\right)^{1-\alpha}\right) \\ &= \frac{1}{c_t}\left(\left(\frac{c_t}{\nu_t}\right)^{1-\alpha} - \beta\gamma D\left(\frac{c_{t+1}\nu_t}{c_t\nu_{t+1}}\right)^{1-\alpha}\left(\frac{c_t}{\nu_t}\right)^{1-\alpha}\right) \\ &= \frac{1}{c_t}\left(\frac{c_t}{\nu_t}\right)^{1-\alpha}\left(1 - \beta\gamma D\left(\frac{c_{t+1}}{c_t}\right)^{1-\alpha}\left(\frac{\nu_t}{\nu_{t+1}}\right)^{1-\alpha}\right) \end{aligned}$$

Now, define gross output growth

$$x_{t+1} \equiv \frac{y_{t+1}}{y_t}$$

and since the economy is closed

$$x_{t+1} = \frac{c_{t+1}}{c_t} = \frac{C_{t+1}}{C_t}$$

and thus

$$\frac{\nu_{t+1}}{v_t} = \frac{\left(c_t^D C_t^{1-D}\right)^\gamma}{\left(c_{t-1}^D C_{t-1}^{1-D}\right)^\gamma} = x_t^\gamma,$$

implying

$$\begin{aligned} \frac{\partial U_t}{\partial c_t} &= \frac{1}{c_t}\left(\frac{c_t}{\nu_t}\right)^{1-\alpha}\left(1-\beta\gamma D\left(\frac{c_{t+1}}{c_t}\right)^{1-\alpha}\left(\frac{\nu_t}{\nu_{t+1}}\right)^{1-\alpha}\right) \qquad (8)\\ &= c_t^{-\alpha}\nu_t^{\alpha-1}\left(1-\beta\gamma D\left(\frac{x_{t+1}}{x_t^\gamma}\right)^{1-\alpha}\right)\\ &= c_t^{-\alpha}\nu_t^{\alpha-1}H_{t+1} \end{aligned}$$

where

$$H_{t+1} \equiv 1-\beta\gamma D\left(\frac{x_{t+1}}{x_t^\gamma}\right)^{1-\alpha}$$

Clearly the marginal utility of consumption in t is increasing in H_{t+1}. Furthermore, $\frac{\partial H_{t+1}}{\partial x_{t+1}}$ is positive if $\alpha > 1$, and negative otherwise. So if growth is expected to be high between t and $t+1$, and IES is small ($\alpha > 1$), this boosts the marginal utility of consumption. In other words, an expectation of high growth has a *negative* effect on savings. Note that this strngthens the standard smoothing results that if you expect high income in the future, the savings motive falls. Furthermore, in asset market equilibrium, this effect tends to *reduce the* price of assets, i.e., increasing the expected return. The opposite is true if $\alpha < 1$, i.e., the intertemporal elasticity is high.

Let us now first consider the case when $D = 0$. We will see that we can get some interesting results for bond and asset returns.

When $D = 0$, $H_t = 1 \forall t$ and $\frac{\partial U_t}{\partial c_t} = c_t^{-\alpha}\nu_t^{\alpha-1}$.The Euler equation implies as usual

$$\begin{aligned} \frac{\partial U_t}{\partial c_t} &= E_t \beta r_{t+1}\frac{\partial U_{t+1}}{\partial c_{t+1}}\\ 1 &= E_t \beta r_{t+1}\frac{c_{t+1}^{-\alpha}\nu_{t+1}^{\alpha-1}}{c_t^{-\alpha}\nu_t^{\alpha-1}} \end{aligned}$$

Now consider the endowment economy, where $c_t = y_t$ and we define

$$x_{t+1} \equiv \frac{y_{t+1}}{y_t}.$$

Then, we have

$$1 = E_t \beta r_{t+1} \frac{c_{t+1}^{-\alpha} \nu_{t+1}^{\alpha-1}}{c_t^{-\alpha} \nu_t^{\alpha-1}} = E_t \beta r_{t+1} x_{t+1}^{-\alpha} x_t^{\gamma(\alpha-1)}$$

Now, consider a risky share that pays y_t as dividend (the apple trees) with price $p_{r,t}$. The price of this asset must satisfy

$$r_{r,t+1} = \frac{p_{r,t+1} + y_{t+1}}{p_{r,t}}.$$

Denoting the price-dividend ratio

$$w_t \equiv \frac{p_{r,t}}{y_t}$$

we can write

$$\begin{aligned} r_{r,t+1} &= \frac{w_{t+1} y_{t+1} + y_{t+1}}{w_t y_t}, \\ &= \frac{1 + w_{t+1}}{w_t} x_{t+1}. \end{aligned}$$

Now, using this last expression for the return on stocks into the Euler equation yields

$$\begin{aligned} 1 &= E_t \beta r_{t+1} x_{t+1}^{-\alpha} x_t^{\gamma(\alpha-1)} \\ &= E_t \beta \frac{1 + w_{t+1}}{w_t} x_{t+1} x_{t+1}^{-\alpha} x_t^{\gamma(\alpha-1)} \\ w_t &= x_t^{\gamma(\alpha-1)} E_t \beta \left(1 + w_{t+1}\right) x_{t+1}^{1-\alpha}, \end{aligned}$$

since w_t is known in t.

When growth rates are i.i.d., this can be calculated quite easily. In particular, we will show that the expression

$$E_t \left(\beta \left(1 + w_{t+1}\right) x_{t+1}^{1-\alpha}\right)$$

is constant at some value A, so that we can write

$$w_t = A x_t^{\gamma(\alpha-1)}$$

for some A and verify that this satisfies the pricing equation (11).

Using our "guess", we have

$$\begin{aligned} Ax_t^{\gamma(\alpha-1)} &= x_t^{\gamma(\alpha-1)} E_t \beta \left(1 + A x_{t+1}^{\gamma(\alpha-1)}\right) x_{t+1}^{1-\alpha}, \\ A &= E_t \beta \left(1 + A x_{t+1}^{\gamma(\alpha-1)}\right) x_{t+1}^{1-\alpha} \\ &= \beta \left(E_t x_{t+1}^{1-\alpha} + E_t A x_{t+1}^{\gamma(\alpha-1)+(1-\alpha)}\right) \\ &= \beta E_t x_{t+1}^{1-\alpha} + \beta E_t A x_{t+1}^{(1-\gamma)(1-\alpha)} \end{aligned}$$

Clearly, the RHS is a constant if growth rates are i.i.d. and we have established that

$$w_t = A x_t^{\gamma(\alpha-1)}$$

where

$$\begin{aligned} A &= \beta E x^{1-\alpha} + \beta E A x^{(1-\gamma)(1-\alpha)} \\ A &= \frac{\beta E x^{1-\alpha}}{1 - \beta E x^{(1-\gamma)(1-\alpha)}} \end{aligned}$$

To calculate the expected stock market return, we use

$$\begin{aligned} r_{r,t+1} &= \frac{1 + w_{t+1}}{w_t} x_{t+1} \\ &= \frac{1 + A x_{t+1}^{\gamma(\alpha-1)}}{A x_t^{\gamma(\alpha-1)}} x_{t+1} \end{aligned}$$

and

$$\begin{aligned} E_t r_{r,t+1} &= \frac{1 + A E x^{\gamma(\alpha-1)}}{A x_t^{\gamma(\alpha-1)}} E x \\ &= \frac{1 + A E x^{\gamma(\alpha-1)}}{A x_t^{\gamma(\alpha-1)}} E x \end{aligned}$$

As we see, the expected return is time dependent, despite the i.i.d. assumption, provided $\gamma \neq 0$. Why?

Furthermore, if $\gamma > 0$, and $\alpha < 1$, the denominator decreases in x_t and the expected return is thus higher when $x_t \equiv \frac{y_t}{y_{t-1}}$ is high. If $\alpha > 1$, the opposite is true.

We can easily calculate the unconditional return, i.e., the average return over time

$$E r_r = E\left[\frac{1 + A E x^{\gamma(\alpha-1)}}{A x^{\gamma(\alpha-1)}} E x\right].$$

In a similar fashion, using the Euler equation

$$1 = \beta r_b \frac{E_t \frac{\partial U_{t+1}}{\partial c_{t+1}}}{\frac{\partial U_t}{\partial c_t}} = \beta r_b x_t^{\gamma(\alpha-1)} E_t x_{t+1}^{-\alpha}$$

the unconditional return on bonds is

$$Er_b = \frac{Ex^{-\gamma(\alpha-1)}}{\beta\left(Ex^{-\alpha}\right)}$$

Consider the special case when output growth x is lognormal, with mean g and standard deviation σ.

Recall that then

$$Ex = e^{g+\frac{1}{2}\sigma^2}.$$

Furthermore, if $\ln x$ is normal, $\ln x^\xi = \xi \ln x$ is normal with mean ξg and standard deviation $\xi\sigma$.Thus,

$$Ex^\xi = e^{\xi g + \frac{\xi^2\sigma^2}{2}}.$$

Now, let $\ln x$ be normally distributed with mean g and standard deviation σ. Then,

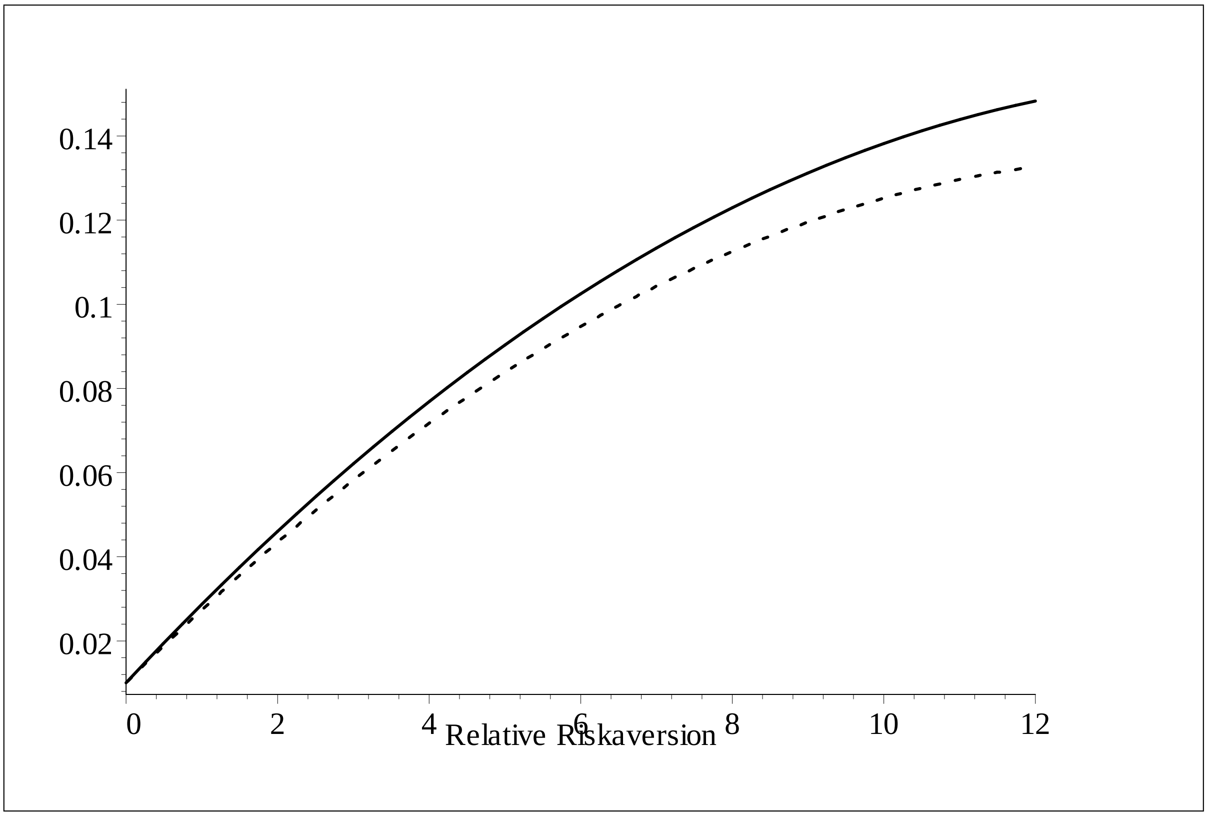
$$\beta Ex^{1-\alpha} + \beta AEx^{(1-\gamma)(1-\alpha)}$$

$$\begin{aligned}
A &= \frac{\beta Ex^{1-\alpha}}{1-\beta Ex^{(1-\gamma)(1-\alpha)}} = \frac{\beta e^{(1-\alpha)g+\frac{(1-\alpha)^2\sigma^2}{2}}}{1-\beta e^{(1-\gamma)(1-\alpha)g+\frac{((1-\gamma)(1-\alpha))^2\sigma^2}{2}}} \\
\ln Er_r &= \ln\left(Ex^{-\gamma(\alpha-1)}\frac{1+AEx^{\gamma(\alpha-1)}}{AEx^{\gamma(\alpha-1)}}Ex\right) \\
&= \ln\left(e^{-\gamma(\alpha-1)g+(\gamma(\alpha-1))^2\frac{\sigma^2}{2}}\frac{e^{g+\frac{\sigma^2}{2}}+Ae^{(1+\gamma(\alpha-1))g+(1+\gamma(\alpha-1))^2\frac{\sigma^2}{2}}}{A}\right)
\end{aligned}$$

and

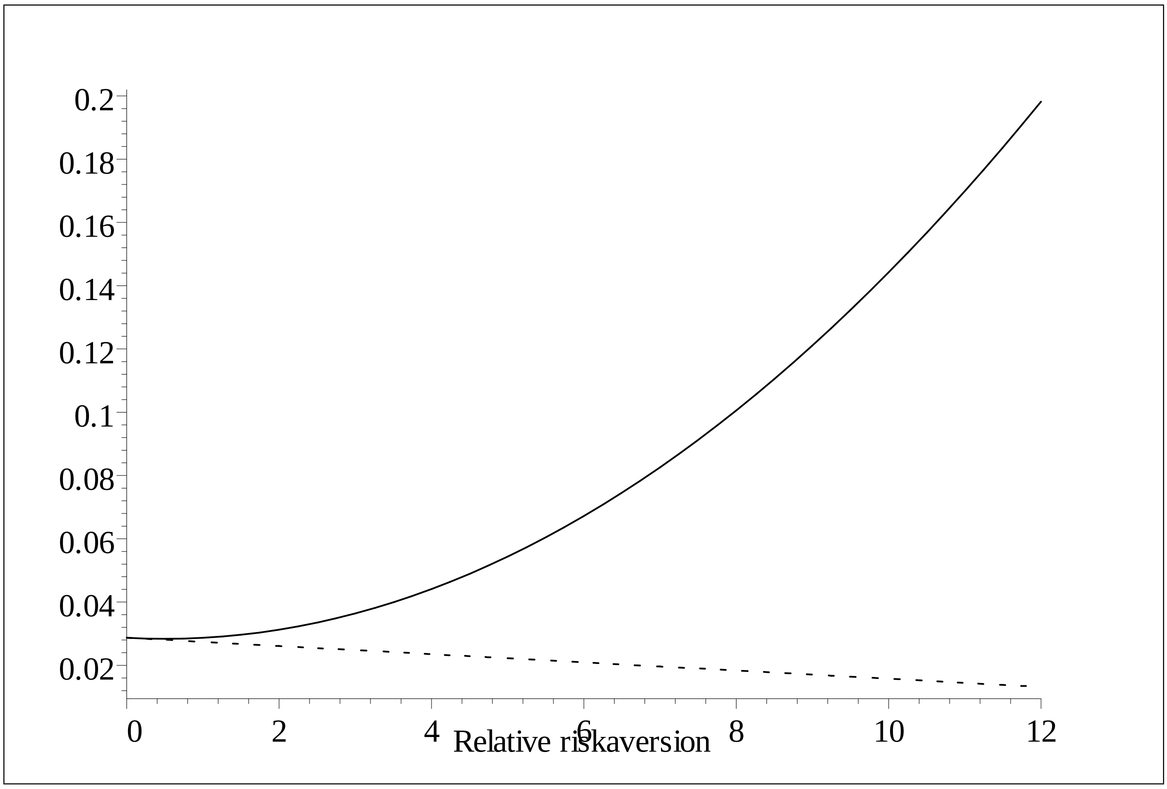
$$\begin{aligned}
\ln ER^B &\equiv r_B = \ln\frac{Ex^{-\gamma(\alpha-1)}}{\beta\left(Ex^{-\alpha}\right)} \\
&= \ln\left(\frac{e^{-\gamma(\alpha-1)g+\frac{(\gamma(\alpha-1))^2\sigma^2}{2}}}{\beta e^{-\alpha g+\frac{\alpha^2\sigma^2}{2}}}\right) \\
&= \left(\alpha-\gamma\left(\alpha-1\right)\right)g + \frac{\left(\left(\gamma\left(\alpha-1\right)\right)^2-\alpha^2\right)\sigma^2}{2} - \ln\beta
\end{aligned}$$

Setting, $\gamma = 0, \sigma = 0.036, g = 0.018, \beta = .99$ and plotting against r_s and r_B against $1-\alpha$, we have the no habit case.



Average stockmarket return r_S (solid line) and safe return r_B against risk aversion (α). Standard utility ($\gamma = 0$).

Keeping the other parameters, but now introducing external habits by setting $\gamma = 1$, the returns are given in the second figure.



Average stockmarket return r_S (solid line) and safe return r_B against risk aversion (α). Extrenal Habit ($\gamma = 1$).

As we see, the stock market return quickly becomes very high as we reduce α.

5.4 Appendix: The case when $D > 0$.

The stochastic Euler equation

$$E_t \frac{\partial U_t}{\partial c_t} = E_{t+1} \beta r_{t+1} \frac{\partial U_{t+1}}{\partial c_{t+1}}$$

Note that is $\frac{\partial U_t}{\partial c_t}$ is actually not realized at t, thus the expectations operator.

$$\begin{aligned} E_t \frac{\partial U_t}{\partial c_t} &= \beta E_t r_{t+1} \frac{\partial U_{t+1}}{\partial c_{t+1}} \\ 1 &= \frac{\beta E_t \left(r_{t+1} \frac{\partial U_{t+1}}{\partial c_{t+1}} \right)}{E_t \frac{\partial U_t}{\partial c_t}}. \end{aligned} \tag{9}$$

Now, it is convenient to find an expression for

$$\frac{\frac{\partial U_{t+1}}{\partial c_{t+1}}}{E_t\left(\frac{\partial U_t}{\partial c_t}\right)}$$

Shifting (8) forward, yields

$$\frac{\partial U_{t+1}}{\partial c_{t+1}} = H_{t+2}\nu_{t+1}^{\alpha-1}c_{t+1}^{-\alpha}.$$

Using, $\frac{\nu_{t+1}}{\nu_t} = x_t^{\gamma}$and $\frac{c_{t+1}}{c_t} = x_{t+1}$,yields

$$\begin{aligned}
\frac{\frac{\partial U_{t+1}}{\partial c_{t+1}}}{E_t\left(\frac{\partial U_t}{\partial c_t}\right)} &= \frac{H_{t+2}\nu_{t+1}^{\alpha-1}c_{t+1}^{-\alpha}}{E_t H_{t+1}\nu_t^{\alpha-1}c_t^{-\alpha}} \\
&= \frac{H_{t+2}}{E_t H_{t+1}}\left(\frac{\nu_{t+1}}{\nu_t}\right)^{\alpha-1}\left(\frac{c_{t+1}}{c_t}\right)^{-\alpha} \\
&= \frac{H_{t+2}}{E_t H_{t+1}}x_t^{\gamma(\alpha-1)}x_{t+1}^{-\alpha}.
\end{aligned}$$

Now, consider a risky share that pays y_t as dividend (the apple trees) with price $p_{r,t}$. The price of this asset must satisfy

$$r_{r,t+1} = \frac{p_{r,t+1} + y_{t+1}}{p_{r,t}}.$$

Denoting the price-dividend ratio

$$w_t \equiv \frac{p_{r,t}}{y_t}$$

we can write

$$\begin{aligned}
r_{r,t+1} &= \frac{w_{t+1}y_{t+1} + y_{t+1}}{w_t y_t}, \\
&= \frac{1 + w_{t+1}}{w_t}x_{t+1}.
\end{aligned}$$

Now, using this last expression for the return on stocks into the Euler equation yields

$$\begin{aligned}
1 &= \frac{\beta E_t r_{r,t+1}\frac{\partial U_{t+1}}{\partial c_{t+1}}}{E_t\frac{\partial U_t}{\partial c_t}}, \\
w_t &= \frac{\beta E_t\left(1 + w_{t+1}\right)x_{t+1}\frac{\partial U_{t+1}}{\partial c_{t+1}}}{E_t\frac{\partial U_t}{\partial c_t}}, \qquad (10)
\end{aligned}$$

since w_t is known in t.

From (10), we have

$$\begin{aligned}
w_t &= \frac{\beta E_t\left((1+w_{t+1})\,x_{t+1}\frac{\partial U_{t+1}}{\partial c_{t+1}}\right)}{E_t\left(\frac{\partial U_t}{\partial c_t}\right)} \\
&= \frac{\beta E_t\left((1+w_{t+1})\,x_{t+1}\frac{\partial U_{t+1}}{\partial c_{t+1}}\right)}{E_t\left(\frac{\partial U_t}{\partial c_t}\right)} \\
&= \frac{\beta E_t\left((1+w_{t+1})\,x_{t+1}c_{t+1}^{-\alpha}\nu_{t+1}^{\alpha-1}H_{t+2}\right)}{E_t\left(c_t^{-\alpha}\nu_t^{\alpha-1}H_{t+1}\right)} \\
&= \frac{\beta E_t\left((1+w_{t+1})\,x_{t+1}\left(\frac{c_{t+1}}{c_t}\right)^{-\alpha}\left(\frac{\nu_{t+1}}{\nu_t}\right)^{\alpha-1}H_{t+2}\right)}{E_t\left(H_{t+1}\right)} \\
&= \frac{\beta E_t\left((1+w_{t+1})\,x_{t+1}x_{t+1}^{-\alpha}\left(x_t^{\gamma}\right)^{\alpha-1}H_{t+2}\right)}{E_t\left(H_{t+1}\right)} \\
&= \frac{\beta E_t\left((1+w_{t+1})\,x_t^{\gamma(\alpha-1)}H_{t+2}x_{t+1}^{1-\alpha}\right)}{E_tH_{t+1}} \\
&= \frac{\beta x_t^{\gamma(\alpha-1)}E_t\left((1+w_{t+1})\,H_{t+2}x_{t+1}^{1-\alpha}\right)}{E_tH_{t+1}}.
\end{aligned}$$

Using the law of iterated expectations

$$E_t\left(1+w_{t+1})H_{t+2}x_{t+1}^{1-\alpha}\right) = E_t\left((1+w_{t+1})\,x_{t+1}^{1-\alpha}E_{t+1}\left(H_{t+2}\right)\right)$$

Define

$$J_t \equiv E_t\left(H_{t+1}\right) = 1-\beta\gamma D x_t^{\gamma(\alpha-1)}E_t x_{t+1}^{1-\alpha}.$$

Then we have

$$w_t = \frac{\beta x_t^{\gamma(\alpha-1)}E_t\left((1+w_{t+1})\,J_{t+1}x_{t+1}^{1-\alpha}\right)}{J_t}. \tag{11}$$

We now need to find w_t as a function of state variables (which are they?) that satisfies (11).

When growth rates are i.i.d., this can be calculated quite easily. In particular, we will show that the expression

$$\beta E_t\left((1+w_{t+1})\,J_{t+1}x_{t+1}^{1-\alpha}\right)$$

is constant at some value A, so that we can write

$$w_t = \frac{Ax_t^{\gamma(\alpha-1)}}{J_t}$$

for some A and verify that this satisfies the pricing equation (11).

Using our "guess", we have

$$\begin{aligned}
\frac{Ax_t^{\gamma(\alpha-1)}}{J_t} &= \frac{\beta x_t^{\gamma(\alpha-1)}}{J_t}E_t\left(\left(1+\frac{Ax_{t+1}^{\gamma(\alpha-1)}}{J_{t+1}}\right)J_{t+1}x_{t+1}^{1-\alpha}\right)\\
A &= \beta E_t\left(\left(1+\frac{Ax_{t+1}^{\gamma(\alpha-1)}}{J_{t+1}}\right)J_{t+1}x_{t+1}^{1-\alpha}\right)\\
&= \beta\left(E_tJ_{t+1}x_{t+1}^{1-\alpha}+E_tAx_{t+1}^{\gamma(\alpha-1)+(1-\alpha)}\right)\\
&= \beta E_tJ_{t+1}x_{t+1}^{1-\alpha}+\beta E_tAx_{t+1}^{(1-\gamma)(1-\alpha)}
\end{aligned}$$

So

$$A\left(1-\beta E_tx_{t+1}^{(1-\gamma)(1-\alpha)}\right) = \beta E_tJ_{t+1}x_{t+1}^{1-\alpha} \tag{12}$$

Now, under the assumption of i.i.d. output (consumption) shocks,

$$E_tx_{t+1}^{1-\alpha} = E_tx_{t+2}^{1-\alpha} = Ex^{1-\alpha}$$

we have

$$\begin{aligned}
J_t &= E_tH_{t+1}\\
&= 1-\beta\gamma Dx_t^{\gamma(\alpha-1)}E_tx_{t+1}^{1-\alpha}\\
&= 1-\beta\gamma Dx_t^{\gamma(\alpha-1)}Ex^{1-\alpha}
\end{aligned}$$

so

$$\begin{aligned}
J_{t+1} &= 1-\beta\gamma Dx_{t+1}^{\gamma(\alpha-1)}Ex^{1-\alpha}\\
J_{t+1}x_{t+1}^{1-\alpha} &= x_{t+1}^{1-\alpha}\left(1-\beta\gamma Dx_{t+1}^{\gamma(\alpha-1)}Ex^{1-\alpha}\right)
\end{aligned}$$

Take conditional expectation at t

$$\begin{aligned}
E_t\left(J_{t+1}x_{t+1}^{1-\alpha}\right) &= E_t\left(x_{t+1}^{1-\alpha}\left(1-\beta\gamma Dx_{t+1}^{\theta}Ex^{1-\alpha}\right)\right)\\
&= E_tx_{t+1}^{1-\alpha}-\beta\gamma DEx^{1-\alpha}E_t\left(x_{t+1}^{1-\alpha}x_{t+1}^{\gamma(\alpha-1)}\right)\\
&= Ex^{1-\alpha}-\beta\gamma DEx^{1-\alpha}E\left(x^{(1-\gamma)(1-\alpha)}\right)\\
&= Ex^{1-\alpha}\left(1-\beta\gamma DE\left(x^{(1-\gamma)(1-\alpha)}\right)\right)
\end{aligned}$$

Finally, use this in (12), and use the i.i.d. assumption to replace conditional expectations

$$\begin{aligned}
A\left(1-\beta E_t x_{t+1}^{(1-\gamma)(1-\alpha)}\right) &= \beta E x^{1-\alpha}\left(1-\beta\gamma D E\left(x^{(1-\gamma)(1-\alpha)}\right)\right) \\
A &= \frac{\beta E x^{1-\alpha}\left(1-\beta\gamma D E\left(x^{(1-\gamma)(1-\alpha)}\right)\right)}{\left(1-\beta E x^{(1-\gamma)(1-\alpha)}\right)},
\end{aligned}$$

which is clearly a constant under the i.i.d. assumption.

To calculate the expected stock market return, we use

$$\begin{aligned}
R_{t+1}^S &= \frac{p_{t+1}^S + y_{t+1}}{p_{t+}^S} \\
&= \frac{(1+w_{t+1})\, x_{t+1}}{w_t}
\end{aligned}$$

and

$$\begin{aligned}
E_t R_{t+1}^S &= \frac{E_t\left(\left(1+\frac{A x_{t+1}^{\gamma(\alpha-1)}}{J_{t+1}}\right) x_{t+1}\right)}{w_t} \\
&= \frac{Ex + E_t \frac{A x_{t+1}^{1+\gamma(\alpha-1)}}{J_{t+1}}}{\frac{A x_t^{\gamma(\alpha-1)}}{J_t}}
\end{aligned}$$

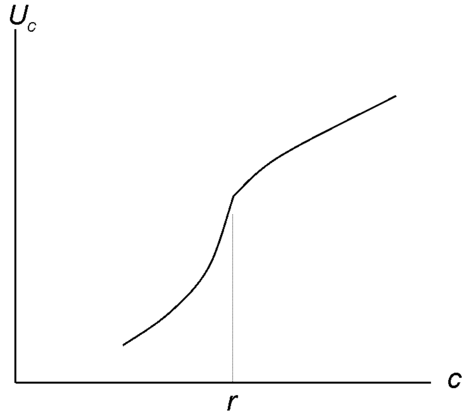
Which we at least can simulate.

6 Loss-Aversion

Substantial amounts of lab-evidence suggests that individuals behave like if the formed reference levels for consumption. Preferences over actual consumption then depend in a particular way of consumption relative to this reference level. Specifically, preferences are consistent with lab evidence if

1. utility is concave in consumption if consumption is above the reference level,
2. utility is convex in consumption if consumption is below the reference level, and
3. marginal utility is discretely larger below the reference level than what it is above.

The utility function can then be depicted as follows:



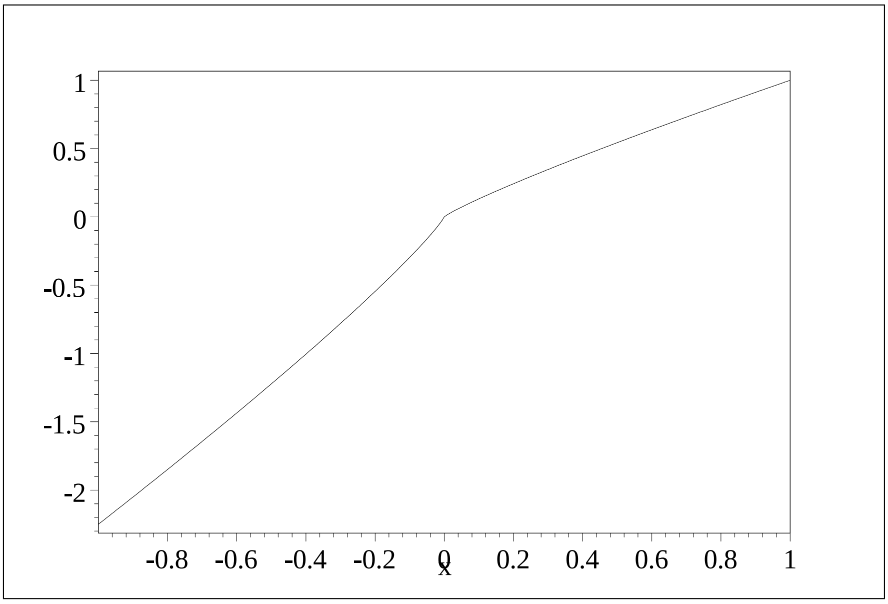
Two important implications of this utility function that is supported by (at least) expermental evidence is that

1. Individuals have a strict distaste also for abitrarily small gambles (because of the discontinuity).
2. Individuals are risk-lovers for losses. This means that they may prefer a 50/50 bet of loosing x or nothing over a sure loss of $x/2$.

   Kahneman & Tversky proposes

$$u(c-r) = \begin{cases} (c-r)^{\alpha} \text{ if } c \geq r \\ -\lambda(-(c-r))^{\beta} \text{ else} \end{cases}$$

and in lab-experiments finds that $\alpha = \beta = 0.88$, and $\lambda = 2.25$as in the following graph[1]



In the paper by Bowman et al., this implication is expanded into a dynamic setting. One implication is then that consumption may respond asymmetrically to positive and negative news about future income. To get the intuition, consider a two period setting and suppose that income at the outset is expected to be w in both periods. Suppose also that the reference point for consumption is $r = w$. For simplicty suppose that the interest rate equals the subjective discount rate. Clearly, the optimal consumption is now $c = w$ in both periods.

Consider now a positive but uncertain signal about period 2 income. Say that income is $w + 2x$ with probability $1/2$ and w with probability $1/2$. Expected lifetime income is then $2w + x$ and unless there is some precautionary savings, consumption in period 1 will be $w + \frac{1}{2}x$. This is the permanent income hypothesis. In any case, consumption will certainly increase when this positive signal comes. Behavior is "standard" for gains.

Consider now instead a negative signal saying that second period income is $w - 2x$ with probability $\frac{1}{2}$ and w otherwise. Now, it may very well pay for the household to continue to consume w in period 1 and then with probability $1/2$ consume w also in the final period and with probability $\frac{1}{2}$ consume $w - 2x$, rather than consuming the permanent income $w - \frac{1}{2}x$ in the first period, in which case second period consumption is either $w + \frac{1}{2}x$ or $w - \frac{3}{2}x$. Why?

[1] A better formulation, allowing $\alpha, \beta < 0$, is $u(c - r) = \begin{cases} \frac{(c-r)^{\alpha}}{\alpha} \text{if } c \geq r \\ -\frac{\lambda(-(c-r))^{\beta}}{\beta} \text{ else} \end{cases}$

Let r denote the reference level for consumption. Then, the expected utility of the first strategy is

$$u(w-r)+\frac{1}{2}u(w-r)+\frac{1}{2}u(w-2x-r)$$
$$0+0-\frac{1}{2}\lambda(2x)^{\beta}$$

and for the second strategy is it

$$\begin{aligned} & u\left(w-r-\frac{1}{2}x\right)+\frac{1}{2}u\left(w+\frac{1}{2}x-r\right)+\frac{1}{2}u\left(w-\frac{3}{2}x-r\right) \\ = & -\lambda\left(\frac{1}{2}x\right)^{\beta}+\frac{1}{2}\left(\frac{1}{2}x\right)^{\alpha}-\lambda\left(\frac{3}{2}x\right)^{\beta} \end{aligned}$$

The difference is

$$\begin{aligned} & -\frac{1}{2}\lambda(2x)^{\beta}-\left(-\lambda\left(\frac{1}{2}x\right)^{\beta}+\frac{1}{2}\left(\frac{1}{2}x\right)^{\alpha}-\lambda\left(\frac{3}{2}x\right)^{\beta}\right) \\ = & \lambda\left(-\frac{1}{2}(2)^{\beta}+\left(\frac{1}{2}\right)^{\beta}+\left(\frac{3}{2}\right)^{\beta}\right)x^{\beta}-\left(\frac{1}{2}\right)^{1+\alpha}x^{\alpha} \end{aligned}$$

Under the assumption $a=\beta$, this is positive if

$$\begin{aligned} \lambda\left(-\frac{1}{2}(2)^{\beta}+\left(\frac{1}{2}\right)^{\beta}+\left(\frac{3}{2}\right)^{\beta}\right)-\left(\frac{1}{2}\right)^{1+\beta} & > 0 \\ \lambda & > \frac{2\left(\frac{1}{2}\right)^{1+\beta}}{-2^{\beta}+2\left(\frac{1}{2}\right)^{\beta}+2\left(\frac{3}{2}\right)^{\beta}}\in\left[\frac{1}{4},\frac{1}{2}\right] \end{aligned}$$

Due to the convexity of utility in losses, its better to take a chance that consumption might not need to be reduced below the reference point. In other words, there is a tendency that consumption does not fall "unless it is clear that it has to". Bowman et al documents such an asymmetry in U.S. consumption data.

In a dynamic setting, a key issue is how reference points are formed. Unfortunately, not much empirical evidence is collected regarding this issue. Bowman et al assume r_1, the reference point for period 1, is given and that

$$r_2=(1-\alpha)r_1+\alpha c_1$$

If $\alpha=0$, we have the case discussed above – static reference points. For $\alpha=1$, next periods reference points is completely determined by the previous periods consumption. Here, we could think both of the case when the agent internalize the effect her consumption has on the reference point and the case when she doesn't (due to naivite or external reference points).

6.1 Loss-aversion as commitment (Hassler&Rodriguez Mora)

- Two types of rational and non-altruistic individuals, (poor) *workers* and *entrepreneurs*, living in a two period OLG-setup.

- The workers make no private choices, having a fixed wage normalized to zero, consuming in the second period of life, having high marginal utility since they are poor.

- Young entrepreneurs in t choose investments i_t at a utility cost $\frac{i_t^2}{2}$, returning $i_t(1-\tau_{t+1})$ in second period of life when consumption takes place and the capital fully depreciates.

- Old workers get a transfer G_t, financed by taxes on installed capital.

- Taxes, $\tau_t \in [0,1]$, are determined without commitment by probabilistic voting with equal weight on all living individuals. Alternative interpretation, a benevolent planner that cares equally of all living individuals.

- Without commitment, the only Markov equilibrium is one with 100 percent taxation since temptation to tax installed capital is too high.

- As Köszegi and Rabin, we consider the case when reference points for consumption are *forward-looking*. We can refrase reference points for cosumption in terms of the corresponding tax-level, τ^r.

- We require τ_{t+1}^r to be in the set of equilibrium tax rates for $t+1$.

- We allow politicians to affect reference points by making "promises" about the future. But remember that the promise is empty – the politician does not remain in office nor runs again and he has no formal commitment power.

- The promise can affect the future if it is believed, in which case it becomes the the reference point.

- It is believed if it is done by the winning candidate and is in the set of equilibria for next period. If the promise is not an equilibrium, τ_{t+1}^r is some element of the set of equilibrium tax rates.

- As a variation, we consider the opposite case of history dependence. Reference tax-levels are *backward-looking*, $\tau_{t+1}^r = \tau_t$

6.1.1 Results

Under both backward and forward-looking reference points, there is a Markov equilibrium with limited amounts of taxation. Dynamics differ between the two cases.

The level of taxation in equilibrium depends inversely on on the degree of loss-aversion.

Intuition: If people have reference points, implying that they feel "entitled" to some return on their investments – if becomes politically costly to go against this. If the entitlements are not too large, they will be satisfied in equilibrium.