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On Efficient Distribution With Private Information

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This paper is a study of the dynamics of the efficient distribution of consumption in an exchange economy with many consumers, each of whom is subject to private, idiosyncratic taste shocks. We propose a recursive method for finding feasible allocations that are incentive-compatible and that are Pareto optimal within this set. The method is applied to several parametric examples. We find that in an efficient allocation the degree of inequality continually increases, with a diminishing fraction of the population receiving an increasing fraction of the resources. We discuss the extent to which these allocations can be decentralized via market arrangements.

1. INTRODUCTION

This paper is a study of the dynamics of the efficient distribution of consumption in an exchange economy with private information. The economy we study has a constant endowment flow of a single, non-storable consumption good which is to be allocated each period among a large number of consumers. Each period, these consumers experience unpredictable, idiosyncratic, privately observed taste shocks affecting their marginal utility of current consumption. Efficiency dictates that more resources be allocated to those consumers who, in any given period, have a high marginal utility of current consumption, due to a high value of their taste shock. But since individual shocks are not observable, the efficient allocation of resources in this environment is impeded by the problem of incentive compatibility: if consumers who report a high value of the taste shock receive more current consumption, then all other consumers have an incentive to misreport their current taste shock to receive the same treatment.

The problem of incentive compatibility is solved in this environment by conditioning each consumer's consumption allocation not only on his current report of his taste shock, but also on the history of his past reports. In particular, it is possible to induce consumers to report their taste shocks truthfully by promising agents who report that they have a high marginal utility of consumption in the current period that they will receive more current consumption at the expense of less consumption in future periods, and promising agents who report a low marginal utility of current consumption that they will receive higher consumption in future periods at the expense of less current consumption. Thus, incentive compatible allocations in this environment induce dynamics in the distribution of consumption that are absent in a full-information environment. Our concern in this paper is to examine the dynamics of the distribution of consumption that may be induced by the need for incentive compatibility.

It is possible to construct a wide variety of command or market mechanisms to implement incentive compatible allocations. From previous studies, we know that the dynamics of the distribution of consumption that are induced by different incentive compatible mechanisms or market allocations can be remarkably different. For instance, Lucas (1978) shows that the allocation of resources that results from the use of money as the sole asset for intertemporal exchange yields an invariant distribution of consumption. The current consumption of any given individual varies over time as that consumer spends more or less of his inventory of money each period to accommodate the fluctuations in his marginal utility of current consumption, but these fluctuations in individual consumption remain confined to a stable cross-sectional distribution of consumption. On the other hand, the allocation of resources based on the use of shares of the endowment or pure credit arrangements, as studied, for example, in Taub (1990), typically result in an ever increasing disparity in the cross-sectional distribution of consumption.

In this paper, we characterize efficient allocations in this informationally constrained environment and examine the dynamics of the distribution of consumption that are implied on normative as opposed to positive grounds. To solve our model, we reformulate our problem as a recursive problem and establish a Bellman equation which characterizes the efficient allocation of resources. We then solve this Bellman equation for two classes of current utility functions. We find in our examples that the efficient allocation of consumption induces spreading of the cross sectional distribution of consumption over time.

Our approach in this paper builds on the partial equilibrium analyses of dynamic incentive problems carried out by Spear and Srivastava (1987), Green (1987), Taub (1990a), Phelan and Townsend (1991), Marimon and Marcet (1990), Abreu, Pearce, and Stacchetti (1990), Thomas and Worrall (1990), and Atkeson (1991).[1] In these earlier papers, a single principal is assumed to choose an incentive compatible allocation designed to minimize the discounted value of resources needed to provide a single agent with a given level of expected utility, where the value of resources is evaluated at some given set of prices. In such a formulation, no period-by-period resource constraint is imposed upon the principal. In this paper, the principal—or planner—chooses the incentive compatible allocation for all agents subject to a constraint that the total consumption handed out each period to the population of agents cannot exceed some constant endowment level. Though many specific results from these other papers can be adapted to this new context, the basic Bellman equation that we study is quite different from those studied by earlier writers. In the last section of the paper, we discuss the sense in which the one-on-one principal-agent problem studied by Green and others can be viewed as a component of a decentralized version of the efficient allocation that we construct.

2. A MODEL

In this section we set out the model informally, describe the allocation problem in more detail, and provide a plan for the rest of the paper. We consider an economy in which there is a constant endowment of a single, non-storable consumption good available at each date. There is a continuum of consumers, each with the preferences

$$E\{\textstyle\sum_{t=0}^{\infty}(1-\beta)\beta^t V(c_t)\theta_t\}$$

where c_t is consumption of the good at date t and θ_t is an idiosyncratic, serially independent taste shock realized at date t with the distribution μ.

1. These papers consider various forms of private information including unobserved effort, investment, income shocks, and taste shocks. The taste shock model is similar to the income shock model in that current income shocks cause consumers to have different marginal utilities of given transfers in the current period. In particular, when current utility takes on the negative exponential form, the two models are identical.

We identify each consumer with a number w, which we interpret as his initial entitlement to expected, discounted utility. We assume that all agents identified with the same w receive the same treatment. Let ψ denote a distribution of utilities w across the population of agents: $\psi(A)$ is the fraction of consumers who will receive expected discounted utility equal to a number w in the set $A \subseteq R$. We take the distribution ψ implied by a given way of allocating resources as a complete description of the welfare consequences of this allocation, thus treating individual consumers as anonymous.

One way to think of a resource allocation in this setting is to think of a social planner as choosing a sequence of functions $\{c_t\}$, where $c_t(w, \theta^t)$ is the consumption agent w gets at date t if he reports the shock history $(\theta_0, \theta_1, \ldots, \theta_t) = \theta^t$ up to that date. Since the shocks are assumed to be private, we will also want to restrict attention to allocations that are incentive-compatible: sequences $\{c_t\}$ that induce each consumer to reveal his shock history θ^t truthfully at each date. For any given initial distribution ψ of entitlements w, we say that the allocation described by the sequence $\{c_t\}$ attains ψ with resources y if (i) it is incentive-compatible, (ii) it delivers expected utility w to all consumers initially entitled to w, and (iii) the total consumption of all consumers does not exceed y in any period.

The efficiency problem that we address in this paper is the following. We define a function φ^* mapping distributions of utility ψ to the real line, where $\varphi^*(\psi)$ is defined to be the greatest lower bound on constant endowments y such that there exists an allocation that attains ψ with resources y. We call $\varphi^*(\psi)$ the minimum cost of attaining distribution ψ. Our objective is to characterize the function φ^* and to find allocations that attain ψ with resources $\varphi^*(\psi)$. This dual approach to the efficiency problem is similar to Green's.

To begin to solve for efficient allocations, we reformulate our problem in the following recursive manner. Instead of having the planner choose an agent's consumption each period as function of the agent's initial entitlement and the entire history of his taste shock reports, let the planner choose a function c_0 that assigns initial consumption $c_0(w, \theta)$ to any consumer with the initial entitlement w who gets the initial shock θ, and let the planner choose a second function g_0 that specifies this consumers' expected utility entitlement w_1 from tomorrow on as a function of the same two variables: $w_1 = g_0(w, \theta)$. That is, on the basis of his w-value and his announced shock, a consumer receives an immediate quantity of goods and an expected utility from next period on. With goods so allocated and consumers' entitlements so respecified, the planner is faced with a problem of the same form next period, except that due to the reassignment of expected utilities, the initial utility distribution ψ has been replaced with a new one, ψ_1. Accordingly, he chooses a new pair of functions (c_1, g_1), and so on, *ad infinitum.*

We call such a sequence of functions, suitably restricted, an allocation *rule* (to distinguish it from an allocation). We say that an allocation rule attains a utility distribution ψ with resources y if (i) it is incentive-compatible in an appropriate (one-period) sense, (ii) if at each date t it delivers expected utility w to all consumers entitled to w at the beginning of that period, and (iii) the total consumption of all consumers does not exceed y in any period.

The advantage of this recursive reformulation of the problem of finding the efficient allocation of resources is that it delivers a Bellman equation that the cost function $\varphi^*(\psi)$ must satisfy. For several example utility functions (logarithmic utility, utility displaying constant relative risk aversion, and utility displaying constant absolute risk aversion) we are able to use this Bellman equation to show the existence of a solution to our original resource allocation problem and characterize the solution in some detail.

In Section 3, we spell out the details of the formulation of the resource allocation problem, and then establish that a utility distribution can be attained by an allocation with the constant endowment y if and only if it can be attained by an allocation rule with the same constant endowment y. This result will justify our focus on allocation rules in the rest of the paper. In Section 4, we define the function φ^* taking utility distributions into endowment levels and provide a Bellman equation for this function φ^*. We describe an iterative process that, if it converges, produces a solution to this Bellman equation.

In Section 5, we apply these results to particular parametric families of utility functions. We show that with log utility, constant relative risk averse preferences, and constant absolute risk averse preferences, the solution to the Bellman equation can be constructed from a static incentive problem and we develop some facts about the latter problem. Section 6 then characterizes the solution to the static problem for all three preference assumptions and develops the implications of this solution for the originally posed, dynamic allocation problem.

In Section 7, we consider the possibility of decentralizing efficient allocations through exchange at competitively determined prices. We consider first the problem of finding prices which decentralize our planning problem of finding the least-cost method of attaining a given distribution of utility into component planning problems of minimizing the cost of attaining each individual utility level w within that distribution of utilities. An analogue to the first welfare theorem is proved. Our results here are used to relate our work to Green's work and other earlier work in this area.

We then ask whether the allocations that solve these component problems can be obtained through competitive trading in securities. We find that if unmonitored trading in certain, one-period real bonds is admitted, efficient allocations cannot be so supported. Section 8 then concludes the paper. The proofs of various lemmas are contained in the Appendix.

3. PROBLEM STATEMENT AND PRELIMINARIES

In this section we state our assumptions on preferences and the distribution of taste shocks, provide definitions of allocations and allocation rules, and state two results that justify our subsequent focus on allocation rules.

The taste shocks θ take values in a finite set $\Theta = \{\theta^1, \dots, \theta^n\}$, $\theta^1 > \cdots > \theta^n > 0$, with the fixed probability distribution μ that assigns positive probability to all θ values. We adopt the normalization $E(\theta) = 1$. The current period utility function $V: R_+ \to D \subseteq R$ (where D is an interval) is assumed to be continuous, strictly increasing and strictly concave. We denote the inverse function of V by $C: D \to R_+$, and refer to the value $C(x)$ of this function as the resources required by the utility level x (even though the utility level is really $(1-\beta)x\theta$, not x).[2]

Each consumer is identified with a point $w \in D$ (which we interpret below as the expected discounted utility this consumer obtains). If two consumers have the same w we allocate the same discounted expected utility to each of them. Let Θ^{t+1} be the $(t+1)$-fold product space and let μ^{t+1}, the product measure, be the distribution of the

2. Among the examples considered in Section 5, we include constant absolute risk aversion preferences of the form $V(c) = -\exp(-\gamma c)$. In this case, we allow consumption c to take on both positive and negative values, so the functions $C(u)$ and $\varphi^*(\psi)$ can take on values over the whole real line. This example is instructive, but we do not modify our general theory to accommodate this case with negative consumption.

shock history θ^t in this space. Let Θ^∞ and μ^∞ be the corresponding infinite product space and probability measure.

The individual knows his own history θ^t at date t, but the planner's only sources of information about this history are the reports provided by the agent himself. We use $z_t(\theta^t)$ to denote the report an agent plans (at date 0) to give about his date t shock if he has actually experienced θ^t, and refer to $z=\{z_t(\theta^t)\}_{t=0}^\infty$, where for all t, $z_t:\Theta^{t+1}\to\Theta$, as a *reporting strategy*. Let Z be the set of all reporting strategies. The truthful reporting strategy is denoted by $z^*=\{z_t^*(\theta^t)\}_{t=0}^\infty$ where $z_t^*(\theta^t)=\theta_t$ for all t and $\theta^t\in\Theta^{t+1}$.

Let $c_t(w, z^t)$ denote the consumption that individual w receives at date t on the basis of the reporting history z^t, and let $(1-\beta)u_t(w, z^t)\theta_t$ be the utility he receives from this consumption. We will find it convenient to think of a social planner as assigning the sequence $u=\{u_t(w, z^t)\}_{t=0}^\infty$ to a consumer. Let S be the set of such sequences—we call them *plans*—such that for all $t\geqq 0$ and all $z^t\in\Theta^{t+1}$, $u_t(\cdot, z^t)$ is a Borel-measurable function on D, and such that:

$$\lim_{t\to\infty}\beta^t\textstyle\sum_{s=0}^\infty\beta^s u_{t+s}(w, \theta^{t+s})\theta_{t+s}=0 \tag{3.1}$$

for all $w\in D$ and $\{\theta_t\}\in\Theta^\infty$.

Define the total expected utility function $U: D\times S\times Z\to D$ by:

$$U(w, u, z)=(1-\beta)\textstyle\sum_{t=0}^\infty\beta^t\displaystyle\int_{\Theta^{t+1}}u_t[w, z^t(\theta^t)]\theta_t d\mu^{t+1}.$$

Thus $U(w, u, z)$ is the expected discounted utility agent w receives if the social planner chooses the plan $u\in S$ and the agent chooses the reporting strategy $z\in Z$.

We define an *allocation* as a plan $u\in S$ that induces each agent to adopt the truthful reporting strategy z^* and that delivers expected discounted utility w to each agent w. These requirements are, in turn:

$$w=U(w, u, z^*) \tag{3.2}$$

for all $w\in D$ and

$$U(w, u, z^*)\geqq U(w, u, z) \tag{3.3}$$

for all $w\in D$ and all $z\in Z$.

By a *utility distribution* we will mean any element ψ of the set M of all probability measures on $(D, \boldsymbol{D})$, where $\boldsymbol{D}$ are the Borel subsets of D. We say that an allocation u *attains* ψ *with resources* y if:

$$\int_{D\times\Theta^{t+1}}C[u_t(w, \theta^t)]d\mu^{t+1}d\psi\leqq y \tag{3.4}$$

for all t. Since u_t is Borel measureable and C is continuous and non-negative, the integral on the left in (3.4) is well-defined, though for some allocations u it can be $+\infty$. Our objective is to characterize the allocations that attain given utility distributions $\psi\in M$.

We next formulate a recursive description of resource allocation, which we refer to as an *allocation rule* (to distinguish it from the allocation we have just defined). In this description, think of the planner as choosing a pair (f_t, g_t) of functions of (w, θ) at each date, where $f_t(w, \theta)$ is interpreted as the current utility an agent receives if his expected utility entitlement from t on is w and if he announces the shock θ, and $g_t(w, \theta)$ is the expected utility entitlement this same consumer is assigned from tomorrow on. That is, we think of the planner as summarizing a consumer's entire report history and his initial entitlement w_0 in a single number w that represents his expected utility entitlement from the current period on. For each t, $(f_t, g_t): D\times\Theta\to D\times D$, and we require $f_t(\cdot,\theta)$ and $g_t(\cdot,\theta)$ to be Borel-measurable. Let $\sigma=\{f_t, g_t\}_{t=0}^{\infty}$ denote a sequence of such functions.

A sequence σ defines a plan u as follows. Let $\{w_t\}$ solve the difference equation $w_{t+1}=g_t(w_t, z_t)$ with initial value w_0, so that $w_t: D\times\Theta^t\to D$, and define u by $u_t(w_0, z^t)=f_t[w_t(w_0, z^{t-1}), z_t]$. We say this plan u is *generated by* $\sigma=\{(f_t, g_t)\}$. Clearly the plan so defined has the requisite measurability properties. We call the sequence σ an *allocation rule* if the plan it generates satisfies the boundedness condition (3.1) (if it is in S); if the sequence $\{w_t\}$ it implies satisfies the boundedness condition that for all $w_0\in D$ and all $\{\theta_t\}\in\Theta^{\infty}$

$$\lim_{t\to\infty}\beta^t w_t(w_0, \theta^{t-1})=0; \tag{3.5}$$

and if it satisfies two conditions that are analogous to the restrictions (3.2) and (3.3) that an allocation must satisfy:

For all $t\geqq 0$ and all $w\in D$,

$$w=\int_{\Theta}[(1-\beta)f_t(w,\theta)\theta+\beta g_t(w,\theta)]d\mu. \tag{3.6}$$

For all $t\geq 0$, all $w\in D$ and all $\theta, \hat{\theta}\in\Theta$,

$$(1-\beta)f_t(w,\theta)\theta+\beta g_t(w,\theta)\geqq(1-\beta)f_t(w,\hat{\theta})\theta+\beta g(w,\hat{\theta}). \tag{3.7}$$

The requirement (3.7) is Green's (1987) temporary incentive compatibility.

Given an initial utility distribution ψ, the functions $\{g_t\}_{t=0}^{\infty}$ of an allocation rule σ define a sequence $\{\psi_t\}_{t=0}^{\infty}$ of distributions as follows. For any Borel measurable $g: D\times\Theta\to D$ and any $D_0\in\boldsymbol{D}$, let:

$$(S_g\psi)(D_0)=\int_{B_g(D_0)}d\mu d\psi,$$

where $B_g(D_0)=\{(w,\theta)\in D\times\Theta\text{: } g(w,\theta)\in D_0\}$. This defines an operator $S_g: M\to M$. That is, if ψ is today's utility distribution and if utilities from tomorrow on are determined by g, then tomorrow's utility distribution is $S_g\psi$. Given an initial utility distribution $\psi\in M$, an allocation rule σ defines a sequence of distributions $\{\psi_t\}_{t=0}^{\infty}$ in M by $\psi_0=\psi$ and $\psi_{t+1}=S_{g_t}\psi_t$, $t\geqq 0$. We will say that the allocation rule σ *attains* ψ *with resources* y if:

$$\int_{D\times\Theta}C[f_t(w,\theta)]d\psi_t d\mu\leqq y, \tag{3.8}$$

for all $t\geqq 0$.

In the rest of the paper we focus exclusively on allocation rules. The next two results, which state that there exists an allocation u that attains $\psi\in M$ with resources y if and only if there exists an allocation rule σ that attains ψ with resources y, are presented to justify this focus.

Lemma 3.1. *Let $\psi\in M$ and suppose the allocation u attains ψ with resources y. Then there is an allocation rule σ that attains ψ with resources y.*

Lemma 3.2. *Let $\psi\in M$. Suppose the allocation rule σ attains ψ with resources y and that u is the utility plan generated by σ. Then u is an allocation, and u attains ψ with resources y.*

The proofs of these two lemmas are given in the Appendix. The proof of Lemma 3.1 uses the fact that the set of utility plans satisfying the constraints (3.2) and (3.3) is convex, which in turn depends on our assumption that the taste shocks affect consumer utility multiplicatively. The proof of Lemma 3.2 involves proving that temporary incentive compatibility (3.7) implies incentive compatibility in the sense of (3.3) and that the constraint (3.6) requiring that the allocation rule deliver to every consumer w the expected utility he is entitled to is equivalent to same corresponding constraint (3.3) for allocations.

4. A BELLMAN EQUATION FOR EFFICIENT ALLOCATIONS

For any utility distribution $\psi \in M$, define the value $\varphi^*(\psi)$ to be the infimum of the set of endowment levels y that have the property that there exists an allocation u that attains ψ with resources y. Roughly, $\varphi^*(\psi)$ is the minimum *cost* (as a constant, perpetual, endowment flow) of attaining ψ. If the distribution ψ cannot be attained with any finite resource level, we say that $\varphi^*(\psi) = +\infty$. Hence $\varphi^*: M \to R_+ \cup \{+\infty\}$. We call an allocation *efficient* if it attains a distribution ψ with resources $\varphi^*(\psi)$.

Recall from (3.8) that an allocation rule $\sigma = \{f_t, g_t\}_{t=0}^{\infty}$ attains a given distribution of expected utilities ψ with resources y only if the current resource cost is less than or equal to y in all periods t. We can restate this condition in a recursive fashion by saying that an allocation rule σ attains a given distribution of expected utilities ψ with resources y only if both the total consumption $\int_{D\times\Theta} C[f_0(w, \theta)]d\mu d\psi$ allocated in time $t=0$ is less than or equal to y *and* the allocation rule σ' that is the continuation of σ from period $t=1$ on also attains, with resources y, the distribution of utilities $\psi_1 = S_{g_0}\psi$ that arises at the beginning of period of $t=1$ from the reassignment of entitlements w according to g_0. Then, from the definition of the function φ^*, the minimum resources needed to attain a distibution ψ *contingent* on choices of functions f_0 and g_0 assigning utilities in the current period and entitlements to utility in the future is given by

$$\max\left\{\int_{D\times\Theta} C[f_0(w, \theta)]d\mu d\psi, \quad \varphi^*(S_{g_0}\psi)\right\}.$$

Thus, the problem of finding the minimum resources required to attain the distribution $\psi \in M$ is simply the problem of minimizing this quantity over choices of functions f and g subject to constraints (3.6) and (3.7).

This line of argument suggests that φ^* should satisfy the functional equation:

For all $\psi \in M$,

$$\varphi(\psi) = \inf_{f,g\in B} \max\left\{\int_{D\times\Theta} C[f(w, \theta)]d\mu d\psi, \quad \varphi(S_g\psi)\right\} \tag{4.1}$$

where B is the set of functions $f, g: D\times\Theta \to D\times D$ such that $f(\cdot, \theta)$and $g(\cdot, \theta)$ are Borel measurable and such that for all $w \in D$,

$$w = \int_{\Theta} [(1-\beta)f(w, \theta)\theta + \beta g(w, \theta)]d\mu; \tag{4.2}$$

and, for all $w \in D$ and $\theta, z \in \Theta$,

$$(1-\beta)f(w, \theta)\theta + \beta g(w, \theta) \geqq (1-\beta)f(w, z)\theta + \beta g(w, z). \tag{4.3}$$

In the remainder of this section, we demonstrate that φ^* satisfies this Bellman equation (Lemma 4.1) and propose an iterative method for solving this equation (Lemma 4.2).

It is useful to define an operator T on the space of candidate resource cost functions. Let X be the set of all functions $\varphi : M \to R_+ \cup \{+\infty\}$, and define the operator $T: X \to X$ by:

Problem T:

$$(T\varphi)(\psi) = \inf_{f,g\in B} \max\left\{\int\int_{D\times\Theta} C[f(w,\theta)]d\mu d\psi, \quad \varphi(S_g\psi)\right\}. \tag{4.4}$$

Then solutions to the Bellman equation (4.1) are fixed points of the operator T.

Lemma 4.1. *φ^* is a fixed point of T.*

Proof. We show first that $\varphi^* \leqq T\varphi^*$, then that $\varphi^* \geqq T\varphi^*$.

Suppose that for some $\psi \in M$, $\varphi^*(\psi) > (T\varphi^*)(\psi)$. Then there is some $(f^0, g^0) \in B$ and a $\delta > 0$ such that

$$\varphi^*(\psi) - \max\left\{\int\int_{D\times\Theta} C[f^0(w,\theta)]d\mu d\psi, \quad \varphi^*(S_{g^0}\psi)\right\} > \delta.$$

Let $\psi_1 = S_{g^0}\psi$. Since $\varphi^*(\psi) > \varphi^*(\psi_1)$, $\varphi^*(\psi_1)$ must be finite. Then by the definition of φ^* and Lemma 3.2, there is an allocation rule $\sigma^1 = \{f_t^1, g_t^1\}_{t=0}^\infty$ that attains ψ_1 with resources $\varphi^*(\psi_1) + (\delta/2)$. Define the allocation rule $\sigma^0 = \{f_t^0, g_t^0\}_{t=0}^\infty$ by setting $(f_0^0, g_0^0) = (f^0, g^0)$ and $(f_{t+1}^0, g_{t+1}^0) = (f_t^1, g_t^1)$ for $t \geqq 0$. Then σ^0 attains ψ with resources $\varphi^*(\psi) - (\delta/2)$, a contradiction. This proves $\varphi^* \leqq T\varphi^*$.

Now suppose that for some $\psi \in M$ there is a $\delta > 0$ such that for all $(f, g) \in B$

$$\max\left\{\int\int_{D\times\Theta} C[f(w,\theta)]d\mu d\psi, \quad \varphi^*(S_g\psi)\right\} - \varphi^*(\psi) > \delta.$$

If $\varphi^*(\psi) = +\infty$, this is impossible. If $\varphi^*(\psi)$ is finite, then by the definition of φ^* and Lemma 3.2 there is an allocation rule $\sigma^0 = \{f_t^0, g_t^0\}_{t=0}^\infty$ that attains ψ with resources less than $\varphi^*(\psi) + (\delta/2)$. Let $\psi_1 = S_{g_0^0}\psi$. Construct an allocation rule $\sigma^1 = \{f_t^1, g_t^1\}_{t=0}^\infty$ by setting $(f_t^1, g_t^1) = (f_{t+1}^0, g_{t+1}^0)$ for all $t \geqq 0$. The rule σ^1 attains ψ_1 with resources $\varphi^*(\psi) + (\delta/2)$. Thus $\varphi^*(\psi_1) \leqq \varphi^*(\psi) + (\delta/2)$, from which it follows that

$$\max\left\{\int\int_{D\times\Theta} C[f_0^0(w,\theta)]d\mu d\psi, \quad \varphi^*(S_{g_0^0}\psi)\right\} \leqq \varphi^*(\psi) + \frac{\delta}{2},$$

a contradiction. This proves $\varphi^* \geqq T\varphi^*$, and completes the proof of the Lemma. ||

The next result is our main tool for constructing φ^*.

Lemma 4.2. *Suppose there are functions φ_a, φ_c and φ such that for all $\psi \in M$,* (i) *$\varphi_c \leqq \varphi^* \leqq \varphi_a$, and* (ii) *$\lim_{n\to\infty} T^n\varphi_a = \lim_{n\to\infty} T^n\varphi_c = \varphi$. Then $\varphi = \varphi^*$.*

Proof. The operator T is monotone, so by (i),

$$T^n\varphi_c \leqq T^n\varphi^* \leqq T^n\varphi_a$$

for all n. By Lemma 4.1, $T^n\varphi^* = \varphi^*$. Then the result follows from (ii). ||

In applying Lemma 4.2, economically natural candidates for the bounding functions φ_a and φ_c are suggested by the work of Thomas and Worrall (1990). For the upper bound, consider the autarkic allocation that provides each consumer with the *constant* consumption level that yields him the expected utility w. Recalling the normalization $E(\theta)=1$, this requires $C(w)$ units of goods each period. Then integrating with respect to a utility distribution $\psi \in M$ gives the total resource cost:

$$\varphi_a(\psi)=\int_D C(w)d\psi. \tag{4.5}$$

Since an autarky allocation is incentive compatible, $\varphi_a(\psi) \geqq \varphi^*(\psi)$ for all $\psi \in M$.

For the lower bound, consider utility plans that use a constant level of total resources and attain a given distribution ψ by *completely* insuring each agent w against idiosyncratic risk. Such an allocation will evidently be constant with respect to time: $u_t(w, \theta^t)=u(w, \theta_t)$ for some fixed function u. Call this function $u_c(w, \theta)$. This function must solve the problem:

$$\min_u \int_{D\times\Theta} C[u(w, \theta)]d\psi d\mu$$

subject to:

$$\int_\Theta \theta u(w, \theta)d\mu = w \tag{4.6}$$

for all $w \in D$.

The first-order conditions for this convex problem are (4.6) and:

$$C'[u(w, \theta)]=\lambda(w)\theta \tag{4.7}$$

for all $w \in D$, $\theta \in \Theta$. (That is, for each w, equate marginal utilities of consumption across all shock values.) Solving (4.6) and (4.7) for $\lambda(w)$ and $u_c(w, \theta)$ and integrating with respect to μ and ψ gives:

$$\varphi_c(\psi)=\int_{D\times\Theta} C[u_c(w, \theta)]d\psi d\mu. \tag{4.8}$$

(Note that if the variance of θ is zero, the functions φ_a and φ_c defined by (4.5) and (4.8) are the same.) Since any feasible, incentive compatible allocation is feasible for the problem solved by the allocation $u_c(w, \theta)$, $\varphi_c(\psi) \leqq \varphi^*(\psi)$ for all ψ. In the next section, we apply Lemma 4.2 to three examples with these choices of the bounding functions φ_c and φ_a.

Of course, our objective is to construct efficient allocation rules—allocation rules that attain the infimum in (4.1). Finding a solution φ^* to (4.1) is only a means to that end. We cannot show the existence of efficient allocation rules at the level of generality of this section. We address and resolve the issue of existence of efficient allocation rules in the examples that we consider in Section 5.

5. SOME EXAMPLES SOLVED

In this section, the results of Sections 3 and 4 and will be used to construct the efficient allocations in three cases in which current period utility takes either the logarithmic form $V(c)=\log(c)$, the constant relative risk aversion (CRRA) form $V(c)=\gamma^{-1}c^\gamma$, $\gamma<1$, $\gamma\neq 0$, or the constant absolute risk aversion (CARA) form $V(c)=-\exp(-\gamma c)$, $\gamma>0$. In all of these cases, we use Lemma 4.2 to construct the efficient allocation of resources. The outline of our solution method is the same in all three cases: the procedure in each case differs only in the details. The method is best explained in the case of logarithmic utility first, with proofs and the details of the other cases left to the appendix.

In the case of logarithmic utility, the set D of utility values is all of R, and the inverse, resource requirement function is $C(u)=\exp(u)$. In this case, the bounding functions φ_a and φ_c defined in (4.5) and (4.8) are given by:

$$\varphi_a(\psi)=\int_D \exp(w)d\psi. \tag{5.1}$$

and:

$$\varphi_c(\psi)=\exp\{-E[\theta\log(\theta)]\}\int_D \exp(w)d\psi. \tag{5.2}$$

To apply Lemma 4.2, we will repeatedly apply the operator T defined in section 4 by (4.4) to these two functions. This application is facilitated by noticing that both φ_a and φ_c take the form of a constant α (say) times the moment $\int_D C(w)d\psi$ of the distribution ψ. We will refer to these constants later, so define $\alpha_a=1$ and $\alpha_c=\exp\{-E[\theta\log(\theta)]\}$. It turns out in the logarithmic utility case that when T is applied to a function φ of the form $\alpha\int_D C(w)d\psi$, the functions f and g that minimize the right-hand side of (4.4) take the form $f(w,\theta)=r(\theta;\alpha)+w$, $g(w,\theta)=h(\theta;\alpha)+w$ and that $T(\varphi)$ takes the same form as φ—a constant $\phi(\alpha)$, say, times the *same* moment $\int_D C(w)d\psi$. In this case then, the problem of finding the fixed point of the operator T is reduced to the problem of finding the fixed point of the function $\phi: R_+\to R_+$ and the form of the functions r and h. This reduced problem is greatly simplified by the fact that the definition of $\phi(\alpha)$ and the choices of r and h are *independent* of both the individual entitlement w and the distribution of entitlements ψ. Thus, in the case of logarithmic utility, the problem of providing incentives (choosing r and h and finding $\alpha^*=\phi(\alpha^*)$) is separable from the problem of delivering entitlements w to each consumer and can be solved as a relatively simple static incentive problem.

To be more specific, we show, in the logarithmic utility case, that the functions $r(\theta;\alpha)$, $h(\theta;\alpha)$, and $\phi(\alpha)$ are defined by the following problem.

Problem P:

$$\phi(\alpha)=\min_{r,h}\max\left\{\int_\Theta \exp(r(\theta))d\mu,\quad \alpha\int_\Theta \exp(h(\theta))d\mu\right\},$$

where r and h are functions (n-vectors) $r, h: \Theta\to R$ such that:

$$\int_\Theta [(1-\beta)\theta r(\theta)+\beta h(\theta)]d\mu=0, \tag{5.3}$$

and:

$$(1-\beta)\theta r(\theta)+\beta h(\theta)\geqq(1-\beta)\theta r(z)+\beta h(z),\quad \text{all } \theta, z\in\Theta. \tag{5.4}$$

We develop the connection between Problem P and the Bellman equation in a series of lemmas, the proofs of which are given in the Appendix, part 2.

Lemma 5.1. *For any $\alpha>0$, the minimum in Problem P is attained by a unique $(r^0(\theta;\alpha), h^0(\theta,\alpha))$.*

The next result shows that Problem P has the desired relationship to Problem T.

Lemma 5.2. *Let ϕ be defined by Problem P and for any $\alpha > 0$ let the minimum be attained by $r(\theta; \alpha), h(\theta; \alpha)$. Let $\varphi(\psi) = \alpha \int_D \exp(w) d\psi$ and let T be the operator defined by* (4.4). *Then*

$$(T\varphi)(\psi) = \phi(a) \int_D \exp(w) d\psi,$$

and the right-hand side of (4.4) *is attained by the pair* $(f, g) = (r(\theta; \alpha) + w, h(\theta; \alpha) + w)$.

The essential aspects of the proof of this lemma are as follows. First, it is clear that when functions f and g take the form $f(w, \theta) = r(\theta) + w$, $g(w, \theta) = h(\theta) + w$, then f and g satisfy the incentive constraints (4.2) and (4.3) if and only if r and h satisfy the incentive constraints (5.3) and (5.4). Next, observe that when f and g take the form $f(w, \theta) = r(\theta; \alpha) + w$, $g(w, \theta) = h(\theta; \alpha) + w$ and φ takes the form $\alpha \int_D C(w) d\psi$, then the right-hand side of equation (4.4) evaluated at the proposed form for f and g can be written as

$$\max\left\{\int_\Theta \exp(r(\theta)) d\mu, \quad \alpha \int_\Theta \exp(h(\theta)) d\mu\right\}\left(\int_D C(w) d\psi\right)$$

When this expression is evaluated at the solution to Problem P $(r^0(\theta; \alpha), h^0(\theta; \alpha))$, it is clearly equal to $\phi(\alpha) \int_D C(w) d\psi$. To show that $T(\varphi) = \phi(\alpha) \int_D C(w) d\psi$ when $\varphi(\psi) = \alpha \int_D C(w) d\psi$, all that remains to be proved is that the functions $f^0(w, \theta) = r^0(\theta; \alpha) + w$ and $g^0(w, \theta) = h^0(\theta; \alpha) + w$ attain the infimum on the right-hand side of the operator T as defined in (4.4). Details are given in the Appendix.

Once it is established that $T(\varphi) = \phi(\alpha) \int_D C(w) d\psi$ when $\varphi = \alpha \int_D C(w) \psi$ the evaluation of $\lim_{n \to} T^n \varphi_i$, $i = a, c$ is straightforward. We do this in Lemma 5.3.

Lemma 5.3. *The function ϕ defined in Problem P has a unique fixed point $\alpha^* \in [\alpha_c, \alpha_a]$ and $\lim_{n \to \infty} \phi^n(\alpha_i) = \alpha^*$ for $i = c, a$.*

Once we find the fixed point α^* of the function ϕ and the corresponding solution to Problem P $(r^0(\theta; \alpha^*), h^0(\theta; \alpha^*))$, we construct the efficient allocation rule $\sigma^* = \{f_t^*, g_t^*\}$ by $f_t^*(w, \theta) = r^0(\theta; \alpha^*) + w$, $g_t^*(w, \theta) = h^0(\theta; \alpha^*) + w$. As a final technical matter, we verify that σ^* satisfies the boundedness condition (3.5). This condition is verified in Section 6.

This same approach can be used to solve (4.4) with either CRRA or CARA utility. In the CRRA case, utility is given by $V(c) = \gamma^{-1} c^\gamma$, $\gamma < 1$, $\gamma \neq 0$, and the inverse, resource requirement function is given by $C(u) = (\gamma u)^{1/\gamma}$. The bounding functions take the form:

$$\varphi_a(\psi) \int_D (\gamma w)^{1/\gamma} d\psi$$

$$\varphi_c(\psi) = [E(\theta^{1/(1-\gamma)})]^{-(1-\gamma)/\gamma} \int_D (\gamma w)^{1/\gamma} d\psi,$$

so again they take the form $\alpha \int_D C(w) d\psi$. As in the log case, we exploit the fact that, when applying T to functions of the form α times a moment of ψ, the functions that minimize the right-hand side of the operator T take a simple form. In this case, $f^0(w, \theta) = r^0(\theta; \alpha)|w|$, and $g^0(w, \theta) = h^0(\theta; \alpha)|w|$, where (r^0, h^0) solve a static Problem P, and $T(\varphi)$ is a function equal to $\phi(\alpha)$ times the same moment $\int_D C(w) d\psi$. The Problem P that defines the function ϕ in this case is:

Problem P:

$$\phi(\alpha) = \min_{r,h} \max\left\{\int_\Theta \left(\frac{\gamma}{|\gamma|} r(\theta)\right)^{1/\gamma} d\mu, \quad \alpha \int_\Theta \left(\frac{\gamma}{|\gamma|} h(\theta)\right)^{1/\gamma} d\mu\right\},$$

subject to:

$$\int_{\Theta}[(1-\beta)\theta r(\theta)+\beta h(\theta)]d\mu=\frac{\gamma}{|\gamma|},$$

and:

$$(1-\beta)\theta r(\theta)+\beta h(\theta)\geqq(1-\beta)\theta r(z)+\beta h(z),\quad \text{all } \theta, z\in\Theta.$$

In the CARA case, with utility given by $V(c)=-\exp(-\gamma c)$, $\gamma>0$, $D=(-\infty,0)$, and the inverse, resource requirement function by $C(u)=-\gamma^{-1}\log(-u)$, the bounding functions φ_a and φ_c defined in (4.5) and (4.8) are given by:

$$\varphi_a(\psi)=-\gamma^{-1}\int_D \log(-w)d\psi,$$

$$\varphi_c(\psi)=\gamma^{-1}E[\log(\theta)]-\gamma^{-1}\int_D \log(-w)d\psi,$$

so that these functions are now additive in α and the moment $\int_D C(w)d\psi$. (It is a familiar fact that in order to exploit the conveniences of this particular functional form one must let consumption assume any real value—negative as well as positive.)

The resource-cost-minimizing choices of f and g take the form $f^0(w,\theta)=-r(\theta;\alpha)w$, $g^0(w,\theta)=-h(\theta;\alpha)w$, and the static problem that defines the candidate function ϕ in this case is:

Problem P:

$$\phi(\alpha)=\min_{r,h}\max\left\{-\gamma^{-1}\int_{\Theta}\log[-r(\theta)]d\mu,\quad \gamma^{-1}\alpha-\gamma^{-1}\int_{\Theta}\log[-h(\theta)]d\mu\right\},$$

subject to:

$$\int_{\Theta}[(1-\beta)\theta r(\theta)+\beta h(\theta)]d\mu=-1,$$

and:

$$(1-\beta)\theta r(\theta)+\beta h(\theta)\geqq(1-\beta)\theta r(z)+\beta h(z),\quad \text{all } \theta, z\in\Theta.$$

In the Appendix, we adapt the statements and the proofs of Lemmas 5.1-5.3, stated above for the case of logarithmic utility, to cover CRRA and CARA cases.

There are two features of all of these examples that make possible the factorization of the solution to our Bellman equation into a resource cost associated with providing incentives (α^*) and a resource cost associated with delivering the utility entitlements ($\int_D C(w)d\psi$). The first feature of the problem that we use is the linearity of the incentive constraints (4.2) and (4.3). Because the incentive constraints are linear in current and future utility, utility assignments $f(w,\theta)$, $g(w,\theta)$ that are incentive compatible for any given entitlement w can be scaled up or down in an additive or multiplicative fashion to be made incentive compatible for any other entitlement w'. The second, and special, feature of the examples that we study is that the resource requirement function C satisfies a separability property that can be described as follows. In each example that we study, there exists a function $F: D\to R$ such that the resource cost ofproviding utility $uw\in D$ (or $u+w\in D$) can be factored into parts $F(u)C(w)$ (or $F(u)+C(w)$) for any choice of u and w. Thus when we scale utility assignments $f(w,\theta)$, $g(w,\theta)$ up or down in an additive or multiplicative fashion, we also scale resource requirements up or down in an additive or multiplicative fashion. By assuming that the taste shocks are i.i.d., we assume that w and θ are independent, so we can integrate over the cost of the basic utility assignment $F(u)$ and the cost of the scale factors $C(w)$ separately. These conditions are also satisfied in the case of linear utility (so C is linear), and in the case in which utility is given by $V(c)=\log(c-\delta)$ or $V(c)=\gamma^{-1}(c-\delta)^{\gamma}$ with fixed γ (so $C(u)=\exp(u)+\delta$ or $C(u)=\gamma u^{1/\gamma}+\delta$), so we suspect that the list of examples that we have supplied is not complete. On the other hand, it is clear that there are utility functions such that (4.4) cannot be solved by this method.

6. EFFICIENT ALLOCATIONS WITH TWO SHOCK VALUES

In the previous section we showed that when utility takes suitable parametric forms, solutions to the Bellman equation defined by (4.1) can be constructed from solutions to the static incentive problem we called Problem P. In this section, we verify that the efficient allocation rules we have constructed satisfy the boundedness condition (3.5) and develop some other properties of these rules.

In the case of logarithmic utility, we demonstrated that the minimum on the right-hand side of the Bellman equation (4.1) is attained by a pair of functions of the form $f(w,\theta)=r(\theta)+w$ and $g(w,\theta)=h(\theta)+w$ that assign current and future utility to agents entitled to w and reporting θ. These functions are independent of the distribution of entitlements to expected utilities ψ. From these functions, we construct the efficient allocation rule $\sigma=\{f_t(w,\theta), g_t(w,\theta)\}_{t=0}^{\infty}$, where $f_t=f$ and $g_t=g$ for all $t\geqq 0$. Using this form for g_t, the solution of the difference equation describing the evolution of individual entitlements to utility is

$$w_t(w_0,\theta^{t-1})=w_0+\textstyle\sum_{s=0}^{t-1} h(\theta_s).$$

The utility allocated to the individual consumer under the efficient allocation is thus given by

$$u_t(w_0,\theta^t)=f_t(w_t(w_0,\theta^{t-1}),\theta_t)=w_0+\textstyle\sum_{s=0}^{t-1} h(\theta_s)+r(\theta_t), \tag{6.1}$$

and consumption allocated to that individual is given by

$$c_t(w_0,\theta^t)=C[u_t(w_0,\theta^t)]=\exp(w_0)\textstyle\prod_{s=0}^{t-1}\exp(h(\theta_s))\exp(r(\theta_t)). \tag{6.2}$$

It is clear from these expressions that the allocation obtained in this way from the solution to Problem P in the logarithmic utility case satisfies the boundedness condition (3.5). We will prove that (3.5) is satisfied in the cases of CRRA and CARA utility in Lemma 6.1 later in this section after we impose the assumption that there are only two possible values of the taste shock θ.

Equation (6.1) implies that the variance of the cross-sectional distribution of utility (the logarithm of consumption) is given by

$$\operatorname{Var}(u_t(w_0,\theta^t))=\operatorname{Var}(w_o)+(t-1)\operatorname{var}(h(\theta))+\operatorname{Var}(r(\theta)).$$

Hence, the degree of inequality in the cross sectional distribution of wealth—measured in utils—grows without bound when resources are efficiently allocated. This finding is striking but certainly not unexpected in view of the results of Green (1987), Taub (1990*a*), Phelan and Townsend (1991), and Thomas and Worrall (1990).

The equivalent expression for the efficient evolution of individual utility (6.1) in the cases of CRRA and CARA utility is given by

$$u_t(w_0, \theta^t) = |w_0 \prod_{s=0}^{t-1} h(\theta_s)| \, r(\theta_t).$$

In these cases, it is the variance of the distribution of the logarithm of utility that grows without bound.

One consequence of the fact that agents have concave utility functions and that their expected utility w_t (or the log of w_t) follows a random walk is that each consumer's expected utility level converges to the minimum level in D with probability one. In the logarithmic utility case, this can be seen from the definition of the fixed point $\alpha^* = \phi(\alpha^*)$. For α^* to be a fixed point of ϕ, it must be the case that $\int_\Theta C[h(\theta; \alpha^*)]d\mu = 1$, and since $C(x) = \exp(x)$ is convex, $\int_\Theta h(\theta; \alpha^*)d\mu < 0$, so that $\{w_t\}$ drifts toward $-\infty$. The same conclusion holds for the other examples. Thomas and Worrall (1990), Proposition 3, prove this result more generally in a closely related context.

In the remainder of this section, we characterize the solutions to Problem P for the case of two shock values, $\theta_1 > \theta_2 > 0$, with probabilities μ_1 and μ_2. We verify the boundedness condition (3.5) for CRRA and CARA utility only in this case. We also explore the nature of the solution to Problem P in the case of logarithmic utility. Since the notation for Problem P differs slightly between the CRRA and the CARA cases (α enters multiplicatively in the former and additively in the latter) we deal here with the CRRA case, treating the essentially identical CARA case in side remarks.

We begin by reformulating Problem P (in the logarithmic utility case) as a problem of choosing (r, h) to minimize $\int_\Theta \exp\{r(\theta)\}d\mu$, subject to incentive constraints (5.3), (5.4), and the constraint that

$$\int_\Theta \exp(r(\theta))d\mu \geqq \alpha \int_\Theta \exp(h(\theta))d\mu.$$

This reformulation is justified by the fact that the minimizing choices of r and h in Problem P equate $\int_\Theta \exp\{r(\theta)\}d\mu$ and $\alpha \int_\Theta \exp\{h(\theta)\}d\mu$. This fact can be proved for all the example utility functions we consider using the line of proof for Lemma A.3 in the appendix. Thus Problem P, as specialized to the two shock case, can be restated:

Problem P:

$$\phi(\alpha) = \min_{r,h} \textstyle\sum_{i=1,2} \mu_i C(r_i)$$

subject to:

$$\textstyle\sum_{i=1,2} \mu_i[C(r_i) - \alpha C(h_i)] = 0, \tag{6.3}$$

$$\textstyle\sum_{i=1,2} \mu_i[(1-\beta)\theta_i r_i + \beta h_i] = K, \tag{6.4}$$

$$(1-\beta)\theta_1 r_1 + \beta h_1 \geqq (1-\beta)\theta_1 r_2 + \beta h_2, \tag{6.5}$$

$$(1-\beta)\theta_2 r_2 + \beta h_2 \geqq (1-\beta)\theta_2 r_1 + \beta h_1, \tag{6.6}$$

where $r_i = r(\theta_i)$ and $h_i = h(\theta_i)$, $i = 1, 2$, and where the function C and the constant K varies with the case under study.

We associate the multipliers λ ξ, δ and ν with the constraints (6.3)-(6.6) respectively, requiring the latter two to be non-negative. The first-order conditions are then:

$$(1-\lambda)\mu_1 C'(r_1)=\xi(1-\beta)\mu_1\theta_1+\delta(1-\beta)\theta_1-\nu(1-\beta)\theta_2, \tag{6.7}$$

$$(1-\lambda)\mu_2 C'(r_2)=\xi(1-\beta)\mu_2\theta_2-\delta(1-\beta)\theta_1+\nu(1-\beta)\theta_2, \tag{6.8}$$

$$\lambda\alpha\mu_1 C'(h_1)=\xi\beta\mu_1+\delta\beta-\nu\beta, \tag{6.9}$$

$$\lambda\alpha\mu_2 C'(h_2)=\xi\beta\mu_2-\delta\beta+\nu\beta, \tag{6.10}$$

together with the complementary slackness conditions. (For the CARA case, conditions (6.7)-(6.10) hold with $\alpha=1$.)

In all cases, we have extablished that Problem P is solved by unique values (r_1, r_2, h_1, h_2), given α, and that $\alpha=\phi(\alpha)$ has a unique solution. We can construct this solution by calculating the unique solution $(r_1, r_2, h_1, h_2, \lambda, \xi, \delta, \nu, \alpha)$ to the nine equations given by (6.4), (6.7)-(6.10), the complementary slackness conditions associated with (6.5)-(6.6), and the equation $\alpha=\phi(\alpha)$. Numerically, this is an inexpensive enterprise, and indeed would be so with many more than two shock values. Here, we summarize the main qualitative features of the solution in a lemma.

Lemma 6.1. *Let* (r_1, r_2, h_1, h_2) *solve Problem* P. *Then* $r_1>r_2$ and $h_1<h_2$. *The constraint* (6.6) *is binding, and* (6.5) *is slack. In the* CRRA *case with* $\gamma\neq 0$,

$$-\beta^{-1}<h_1<h_2<\beta^{-1}.$$

In the CARA *case,*

$$-\beta^{-1}<h_1.$$

Remark. In the CRRA and CARA utility cases, the boundedness condition (3.5) is an immediate consequence of the bounds on h_i given above.

Proof. The two incentive constraints together imply $(\theta_1-\theta_2)(r_1-r_2)\geqq 0$, with equality if and only if both are binding. Since $\theta_1>\theta_2$, this implies $r_1\geqq r_2$ and hence $h_1\leqq h_2$, with equality if and only if both constraints bind.

Suppose, contrary to the lemma, that both incentive constraints bind, so the multipliers δ and ν are both positive, $r_1=r_2$, and $h_1=h_2$. Then (6.9), (6.10) and the fact that C' is strictly increasing imply that $\nu=\delta>0$. Then (6.7), (6.8) and the fact that $\theta_1>\theta_2$ imply that $C'(r_1)>C'(r_2)$ contradicting the fact that $r_1=r_2$. Hence at most one of these multipliers is positive, which proves that $r_1>r_2$ and $h_1<h_2$.

Now if $h_1<h_2$, (6.9) and (6.10) imply that $(\delta-\nu)<0$. Since only one can be positive, we conclude that $\delta=0$ and $\nu>0$. Thus

$$(1-\beta)\theta_2 r_2+\beta h_2=(1-\beta)\theta_2 r_1+\beta h_1 \tag{6.11}$$

must also hold at the optimum.

To verify the bounds on the h_i in the CRRA case, note that (6.11) and the total utility constraint together imply:

$$(1-\beta)r_1+\beta h_1=\frac{\gamma}{|\gamma|}. \tag{6.12}$$

When $\gamma>0$, the lower bound obviously holds. Then (6.11) and (6.12) imply:

$$\beta h_2=1-(1-\beta)(\theta_1 r_1+\theta_2 r_2)<1,$$

where the strict inequality is implied by $r_1 > r_2 \geqq 0$. When $\gamma < 0$, the upper bound is obvious and (6.12) implies

$$\beta h_1 = -1 - (1-\beta) r_1 > -1,$$

where the inequality is implied by the fact that $r_1 < 0$.

In the CARA case, (6.12) is replaced by:

$$(1-\beta) r_1 + \beta h_1 = -1. \tag{6.13}$$

Since the minimum in Problem P cannot be attained with $r_1 = 0$, the condition $\beta h_1 > -1$ clearly holds. ||

The conclusion in Lemma 6.1 that the only incentive constraints that bind at the optimum are those comparing an agent's utility when he reports his true shock to the utility he would get if he reported the next higher value can be proved for the general case of N shock values. See Thomas and Worrall (1990), Lemma 4.

In all of the cases considered, then, efficient allocation rules, and hence efficient allocations, can be constructed using the Bellman equation defined in Section 4. The solutions all have the property that the consumer with the low θ value—a low urgency to consume—is just indifferent between revealing his true situation and pretending he is more eager to consume than he really is. That is to say, if the efficient allocation were to provide any more insurance against a high taste shock, people with low shocks would submit false claims. Put yet a third way, following Green, a constraint must be placed on consumers' ability to borrow to finance high current consumption.

7. DECENTRALIZING EFFICIENT ALLOCATIONS

To what extent is it possible to use competitive exchange to achieve the efficient allocations we have calculated in the examples of Section 5, or the allocations more generally defined in Sections 3 and 4? In this section, we show how prices can be used to decentralize the overall planning problem into component planning problems. This decentralization provides a connection between the efficiency problem addressed in this paper and the principal-agent problem studied by Green and others. Then we discuss the possibilities for decentralizing the efficient allocation using unmonitored trading of securities among individual consumers.

To define what we mean by the "component planning problem," consider a planner responsible for allocating resources *only* to those who are initially entitled to expected utility w_0. He assigns an allocation of utility (specific to w_0) $u(w_0) = \{u_t(w_0, \theta^t)\}_{t=0}^{\infty}$, $u_t : \Theta^{t+1} \to D$. He does so in such a way as to minimize the value of the total resources he allocates, with resources at each date valued at prices determined by the sequence $q = \{q_t\}_{t=0}^{\infty}$, $q_t \in (0, 1)$. The objective for this planner is:

$$v(w_0) = \min_{\{u_t(w_0, \cdot)\}} (1-q_0) \int_{\Theta} C[u_0(w_0, \theta)] d\mu$$

$$+ \sum_{t=1}^{\infty} (1-q_t) \prod_{s=0}^{t-1} q_s \int_{\Theta^{t+1}} C[u_t(w_0, \theta^t)] d\mu^{t+1} \tag{7.1}$$

subject to the incentive constraints (3.2) and (3.3). It is as if consumers are grouped by their initial w_0 values, with each group represented by its own social planner or principal, and then these planners trade in claims to current and future resources among themselves at prices $\{q_t\}$.

The next result provides one connection between these component planning problems (7.1) and the problem of finding efficient allocations.

Theorem 1. *Suppose there exists an allocation* $u=\{u_t(w_0,\theta^t)\}_{t=0}^{\infty}$, *prices* $q=\{q_t\}_{t=0}^{\infty}$, *a distribution of utility* ψ_0, *and resources* y *such that*

(i) *at prices* q, *for all* $w_0 \in D$, $u(w_0)$ *solves* (7.1) *subject to* (3.2) *and* (3.3);

(ii) *for all* t, $\int_{D\times\Theta^{t+1}} C[u_t(w_0,\theta^t)]d\mu^{t+1}d\psi_0 = y$.

(iii) $\sum_{t=1}^{\infty}(1-q_t)\prod_{s=0}^{t-1} q_s < +\infty$.

Then the allocation u *attains* ψ_0 *with resources* y *and* $y=\varphi^*(\psi_0)$.

Proof. That u attains ψ_0 with resources y is immediate. We prove that u is efficient by contradiction. Suppose that $y>\varphi^*(\psi_0)$. Then there exists some other allocation $\tilde{u}$ which attains ψ_0 with resources $\tilde{y}<y$. Thus, by (ii),

$$\int_{D\times\Theta^{t+1}} C[\tilde{u}_t(w_0,\theta^t)]d\mu^{t+1}d\psi_0 < \int_{D\times\Theta^{t+1}} C[u_t(w_0,\theta^t)]d\mu^{t+1}d\psi_0$$

for all t, with the difference between these two quantities being at least $y-\tilde{y}$. Then,

$$(1-q_0)\int_{D\times\Theta} C[\tilde{u}_0(w_0,\theta)]d\mu\psi_0 + \textstyle\sum_{t=1}^{\infty}(1-q_t)\prod_{s=0}^{t-1} q_s \displaystyle\int_{\Theta^{t+1}} C[\tilde{u}_t(w_0,\theta^t)]d\mu^{t+1}d\psi_0 <$$

$$(1-q_0)\int_{D\times\Theta} C[u_0(w_0,\theta)]d\mu d\psi_0 + \textstyle\sum_{t=1}^{\infty}(1-q_t)\prod_{s=0}^{t-1} q_s \displaystyle\int_{\Theta^{t+1}} C[u_t(w_0,\theta^t)]d\mu^{t+1}d\psi_0 < +\infty$$

with the last inequality following from (iii). This contradicts the fact (i) that $u(w_0)$ solves (7.1) for each w_0. ||

Theorem 1 is an analogue to the first theorem of welfare economics, with conditions (i)-(iii) defining the counterpart to a competitive equilibrium. Condition (i) requires quantities to be optimal (cost-minimizing) for each w, given prices; condition (ii) is market clearing; and condition (iii) is a boundedness condition on prices.

Theorem 1 is helpful in relating our approach to that taken by Green, (1987), Taub (1990*b*), Phelan and Townsend (1991), and others who have formulated the allocation problem as one involving a single, risk-neutral principal dealing one-on-one with a continuum of agents. In these formulations, the principal is given an objective function that corresponds to (7.1), but with a constant price $q_t=q$ for all t that is simply assigned to him. These authors then show that (7.1) (with constant prices) can be solved with a Bellman equation of the form

$$v(w)=\min_{r,h}\int_{\Theta}\{(1-q)C(r(\theta))+qv(h(\theta))\}d\mu$$

subject to:

$$w=\int_{\Theta}[(1-\beta)r(\theta)\theta+\beta h(\theta)]d\mu;$$

and

$$(1-\beta)r(\theta)\theta+\beta h(\theta)\geqq(1-\beta)r(z)\theta+\beta h(z),$$

for all $\theta, z\in\Theta$. Green, and Townsend and Phelan assign $q=\beta$ to the principal. For some preferences (for example, logarithmic) this will clear markets in the sense of condition (ii) of Theorem 1; for others (for example, the CARA preferences used in Green's paper) it will not. Taub leaves q free, but constant, and then varies it to satisfy (ii). He uses linear utility, an instance of CRRA preferences, so the price associated with efficiency is in fact constant and this procedure works. This procedure of varying the constant price to clear markets would also work in Green's framework (since his income shock model with CARA utility maps exactly into the model with multiplicative taste shocks). In general, though, this procedure will not work because it fails to recognize the potential dependence of q on the distribution ψ_0.

In the examples that we have studied in Section 5 and 6, the prices that decentralize the efficient allocation can be found from the solution to Problem P. In particular, we show that this problem is equivalent to the problem of minimizing current resource use ($\int_\Theta \exp(r(\theta))d\mu$ in the case of logarithmic utility) subject to the incentive constraints (5.3), (5.4) and the resource constraint

$$\int_\Theta \exp(r(\theta))d\mu \geqq \alpha \int_\Theta \exp(h(\theta))d\mu.$$

It is straightforward to show that the Lagrange multiplier λ on this constraint serves to define prices $q=\{q_t\}_{t=0}^{\infty}$, $q_t=\lambda$ for all t, that decentralize the efficient allocation into component planning problems in the sense of (7.1).

The foregoing discussion involved the possibilities for decentralizing the economy's overall planning problem into component planning problems, one for each level of expected utility entitlements. It leaves open the question of whether decentralization can proceed further, with efficient allocations exhibited as some kind of market equilibrium with a particular set of securities or intermediation arrangements. We address this issue next.

One sense in which it must be possible to decentralize further is the following. Let titles to current and future endowment streams be allocated across agents, and imagine that every agent deposits all of his claims with a zero-profit intermediary that thereafter acts exactly as the central planner of earlier sections. Since there are many households, we can imagine that there are many such intermediaries, so this arrangement could be viewed as competitive. The difficulty with using such an equilibrium as a model of observed market arrangements stems from the capability to monitor individual wealth positions granted to this intermediary, relative to the capabilities of actual financial institutions. The question we ask next, then, is whether the efficient allocation can be supported by private intermediation if households are permitted to engage in unmonitored trading of ordinary securities.

The specific security we introduce is a one-period real bond, entitling the holder to one unit of goods tomorrow. Let the equilibrium price, today, of such a bond be $Q(\psi)$. We consider the situation of an individual household that is entitled to w units of expected utility under its arrangement with the intermediary (planner), that has realized the shock θ, and that holds a real bond when the economy is in the state ψ. Now if (f, g) are efficient and if this allocation is consistent with unmonitored bond trading, it must be the case that at the market clearing price $Q(\psi)$ every household chooses to report θ truthfully, and that no household chooses to use the bond market to trade away from the efficient allocation. If so, then the familiar condition:

$$Q(\psi)V'[C(f(w,\theta,\psi))]\theta=\beta\int_\Theta V'[C(f(g(w,\theta,\psi),\theta',S_g\psi))]\theta' d\mu' \tag{7.4}$$

must hold for all (ψ, w, θ). That is to say, all households' marginal rates of substitution would be equated to the equilibrium bond price.

In the last section we presented first-order conditions (6.7)–(6.10) that characterize the functions f and g, so to see if (7.4) holds we can just solve these equations and inspect the result. We do this for the logarithmic case only. In this case, neither f nor g depends on the distribution ψ. Consumption is given by $c(w, \theta) = C[f(w, \theta)]$. The marginal utility function is:

$$V'[c(w, \theta)] = \frac{\theta}{\exp(w)\exp(r(\theta))}.$$

The expected marginal utility of tomorrow's consumption is:

$$E\{V'[c(g(w, \theta), \theta']\} = \frac{1}{\exp(w)\exp(h(\theta))} E\left\{\frac{\theta'}{\exp(r(\theta'))}\right\}.$$

Hence for (7.4) to hold, it must be the case that

$$\frac{\theta_1 \exp(h(\theta_1))}{\exp(r(\theta_1))} = \frac{\theta_2 \exp(h(\theta_2))}{\exp(r(\theta_2))}. \tag{7.5}$$

To check (7.5) in this logarithmic case, we solve the first-order conditions (6.7)–(6.10). When $C(x) = \exp(x)$, the multipliers λ and ξ on the constraints (6.3) and (6.4) are equal to β and α respectively, and (6.7)–(6.10) can be simplified to:

$$r_1 = \log(\alpha) + \log(\theta_1) + \log\left[1 - \left(\frac{\theta_2}{u_1\theta_1}\right)\frac{\nu}{\alpha}\right], \tag{7.6}$$

$$r_2 = \log(\alpha) + \log(\theta_2) + \log\left[1 + \left(\frac{1}{\mu_2}\right)\frac{\nu}{\alpha}\right], \tag{7.7}$$

$$h_1 = \log\left[1 - \left(\frac{1}{\mu_1}\right)\frac{\nu}{\alpha}\right], \tag{7.8}$$

$$h_2 = \log\left[1 + \left(\frac{1}{\mu_2}\right)\frac{\nu}{\alpha}\right]. \tag{7.9}$$

From these equations (7.6)–(7.9), we see that (7.5) can hold only if the multiplier $\frac{\nu}{\alpha}$ equals zero, which is to say, only at the full-information allocation.

Allen (1985) addresses the related question of whether any insurance is possible if, ex ante, consumers can trade sure claims to future consumption before their unobserved idiosyncratic shocks are realized. He argues that, in this instance, no insurance is possible since the incomplete information insurance problem here reduces to a static incentive problem in which consumers, independently of their taste shock realizations, all choose the shock reporting strategy that maximizes the discounted present value of transfers received and then trade consumption claims to achieve the desired time path for consumption, much as they would in a pure credit decentralization of this economy.

8. CONCLUSIONS

Within a specific dynamic setting with private information we have defined efficiency in a way that respects the information structure, proved that if efficiency can be achieved it can be achieved through a recursive allocation rule, developed a Bellman equation for efficient allocation rules, and used this Bellman equation to construct such rules under specific parametric assumptions on preferences. In all of the cases we study, efficient allocations have the features found by Green (1987) for the CARA case, by Taub (1990b) for the case of linear utility, and by Phelan and Townsend (1991) in numerical results: consumer wealth positions follow random walks, with inequality growing over time without bound. As found by Thomas and Worrall (1990), ever-increasing dispersion of utility combined with concave utility implies that average utility continually decreases. The efficient allocation delivers an ever-increasing fraction of resources to an ever-diminishing fraction of society's population.

Our applications to specific cases rest very much on the nature of the parametric preference families studied. Roughly speaking, we restrict attention to preferences that have the property that a specific moment of the utility distribution suffices to determine resource allocation. In these cases, if we know how best to allocate risk among consumers at any one wealth level, we can scale this allocation to suit any wealth level, and hence any wealth distribution. Thus we obtain a theory of distribution applicable to situations where distribution does not matter allocatively. But the Bellman equation we propose is much more general (though our derivation rests on the multiplicative character of the privately observed shocks). It will be interesting to see whether it can be applied to situations in which changes in the wealth distribution feed back on the nature of risk-pooling opportunities.

When equilibrium in our setting is calculated under some incomplete market structures—such as the cash-in-advance monetary system used in Lucas (1978)—in invariant distribution of wealth is found, as contrasted to the growing inequality found here. Similarly, Taub (1990) finds an invariant distribution when a lower bound is imposed on utility. These results are not to be interpreted as an advantage of monetary or credit-constrained equilibria, but rather as evidence of their inefficiency. The growing inequality we find is not a pathology, but a normative prescription of the model.

If we think of the infinite-lived agents in this economy as a dynastic family with successive generations, however, the prescription of growing inequality highlights a feature of our notion of efficiency that surely merits further examination. In our formulation, the welfare of any member of future generation is the *sole* responsibility of his currently living progenitor, and the latter is granted an unlimited right to borrow against the entitlements of his heirs to satisfy his current consumption needs. If one were to re-examine the efficiency question in an overlapping generations framework with some minimum placed on the welfare of future generations, then it seems likely that efficient wealth distributions would converge to some distribution with finite dispersion, rather than exhibit the ever-growing dispersion implied by our notion of efficiency. In such a setting, the prescription of growing inequality for society as a whole would transpose into a prescription of growing inequality within each cohort, and the variance of the function $h(\theta)$, shown in Section 8 to govern the rate of inequality growth in our model, would be a main determinant of the (finite) variance of wealth across all individuals.

APPENDIX

Part I: Proofs of Lemmas 3.1 *and* 3.2

In order to establish the two lemmas, we need to relate incentive compatibility in the sense of (3.3) to temporary incentive compatibility as defined in (3.7). This requires some additional notation.

Under a given plan u, a consumer has the opportunity to re-think his reporting strategy at any date $r \geqq 1$, after he has already submitted reports $\hat{z}_r = (z_0, \ldots, z_{r-1})$ in the preceding r periods. To consider his incentives for reporting truthfully, we need a notation for describing his options at any date r. Let $z = \{z_t(\theta^t)\}_{t=0}^{\infty} \in Z$ denote his reporting strategy from date r onwards. Then the utility he will be assigned by the plan u at date $r+t$ will be

$$u_{r+t}[w_0, (z_0, \ldots, \dot{z}_{r-1}, z_0(\theta_r), z_1(\theta_r, \theta_{t+1}), \ldots, z_t(\theta_r, \ldots, \theta_{r+t}))],$$

where the first r co-ordinates of his reports through period $r+t$ are arbitrary numbers in Θ and the last $t+1$ are functions of the shocks he actually receives from period r through period $r+t$. We use $(\hat{z}_r, z)$ to refer to an r-vector of given past reports followed by the reporting strategy z, and write the agent's utility assignment at $r+t$ as $u_{r+t}[w_0, (\hat{z}_r, z^t(\theta^t))]$, where $z^t = (z_0, \ldots, z_t)$. Define the expected, discounted utility from period r on by:

$$U_r[w_0, u, (\hat{z}_r, z)] = (1-\beta)\sum_{t=0}^{\infty} \beta^t \int_{\Theta^{t+1}} u_{r+t}[w_0, (\hat{z}_r, z^t(\theta^t))]\theta_t d\mu^{t+1}.$$

These functions $U_r : D \times S \times \Theta^r \times Z \to D$ satisfy:

$$U_r[w_0, u, (\hat{z}_r, z)] = \int_{\Theta} \{(1-\beta)u_r[w_0, (\hat{z}_r, z_0(\theta))]\theta + \beta U_{r+1}[w_0, u, (\hat{z}_r, z_0(\theta), z'(\theta))]\} d\mu, \tag{A.1}$$

for $r \geqq 0$, where $z'(\theta)$ denotes the continuation $\{z_t(\theta^t)\}_{t=1}^{\infty}$ of the reporting plan $z = \{z_t(\theta^t)\}_{t=0}^{\infty}$, with the first coordinate of each history θ^t equal to θ. We now provide a preliminary result that will help in linking the two incentive compatibility requirements (3.3) and (3.7).

Lemma A.1. *A plan u satisfies the constraint* (3.3) [*for all $w \in D$ and all $z \in Z$, $U(w, u, z^*) \geqq U(w, u, z)$*] *if and only if it satisfies*:

$$(1-\beta)u_r[w, [w, (\hat{z}_r, \theta)]\theta + \beta U_{r+1}[w, u, (\hat{z}_r, \theta, z^*)] \geqq (1-\beta)u_r[w, (\hat{z}_r, \hat{\theta})]\theta + \beta U_{r+1}[w, u, (\hat{z}_r, \hat{\theta}, z)] \tag{A.2}$$

for all $w \in D$, all $r \geqq 0$, all $\hat{z}_r \in \Theta^r$, all reporting strategies z, and all $\theta, \hat{\theta} \in \Theta$.

Condition (A.2) requires that for all past reports $\hat{z}_r$ and for all reporting strategies that might be adopted from next period on, the agent is induced to report his shock θ truthfully in the current period.

Proof. Condition (3.3) is a special case of (A.2), so the sufficiency of (A.2) is immediate.

The proof of necessity is by contradiction. Assume that (3.3) holds but that for some date $r \geqq 1$, some $w \in D$, and some sequence of taste shock reports $\hat{z}_r$, the inequality (A.2) fails to hold. Use $\tilde{z}$ to denote the taste-shock reporting strategy from date r onwards that dominates truth telling z^* in these circumstances. Then for some $\bar{\theta}$,

$$(1-\beta)u_r[w, (\hat{z}_r, \bar{\theta})]\bar{\theta} + \beta U_{r+1}[w, u, (\hat{z}_r, \bar{\theta}, z^*)]$$
$$< (1-\beta)u_r[w, (\hat{z}_r, \tilde{z}_0(\bar{\theta})]\bar{\theta} + \beta U_{r+1}[w, u, (\hat{z}_r, \tilde{z}_0(\bar{\theta}), \tilde{z}')]$$

where $\tilde{z}'$ is the continuation of the strategy $\tilde{z}$ after $\bar{\theta}$ has been realized. We will use $\tilde{z}$ to construct a reporting strategy $\bar{z}$ which begins at date 0 and satisfies $U(w, u, \bar{z}) > U(w, u, z^*)$, thus contradicting (3.3).

Define the reporting strategy $\bar{z}$ as follows: (1) for $t < r$, let $\bar{z}_t(\theta^t) = z_t^*(\theta_t) = \theta_t$ for all $\theta^t \in \Theta^{t+1}$; (2) for $t \geqq r$, continue with truth-telling unless the shock history $(\hat{z}_r, \bar{\theta})$ has been realized; (3) if the history $(\hat{z}_r, \bar{\theta})$ has occurred, adopt the reporting strategy $\tilde{z}$ from date r onwards. The strategy $\bar{z}$ yields the same utilities as does z^* in the first $r-1$ periods, and the same utilities from t on provided $\theta^{r+1} \neq (\hat{z}_r, \bar{\theta})$. If $\theta^{r+1} = (\hat{z}_r, \bar{\theta})$, $\bar{z}$ yields a strictly higher expected utility from date r on, conditional on this event. Since all shocks have positive probability, this event occurs with positive probability. Thus $U(w, u, \bar{z}) > U(w, u, z^*)$, which is the desired contradiction. ||

Under a given allocation u, two agents may arrive at date t with different initial entitlements w_0 and $\hat{w}_0$ and different shock histories θ^{t-1} and $\hat{\theta}^{t-1}$, yet be entitled to the same expected utility from date t on, as defined by the function U_t. In general, these two agents need not receive the same current utility u_t, even if they receive the same shock θ_t. Obviously, such an allocation u cannot be reproduced for all agents by an allocation rule. It would be useful, therefore, to be able to restrict attention to allocations u that have the following identical treatment property:

for all $t \geqq 0$, all $w_0, \hat{w}_0 \in D$, and all $\theta^{t-1}, \hat{\theta}^{t-1} \in \Theta^{t-1}$,

$$U_t[w_0, u, (\theta^{t-1}, z^*)] = U_t[\hat{w}_0, u, (\hat{\theta}^{t-1}, z^*)] \tag{A.3}$$

implies

$$u_t[w_0, (\theta^{t-1}, \theta)] = u_t[\hat{w}_0, (\hat{\theta}^{t-1}, \theta)]$$

and

$$U_{t+1}[w_0, u, (\theta^{t-1}, \theta, z^*)] = U_{t+1}[\hat{w}_0, u, (\hat{\theta}^{t-1}, \theta, z^*)]$$

for all $\theta \in \Theta$. The next result justifies restricting attention to allocations with this property.

Lemma A.2. *Let $\psi \in M$ and suppose the allocation u attains ψ with resources y. Then there is an allocation u' that satisfies* (A.3) *and also attains ψ with resources y.*

Proof. Let $\psi \in M$ be given and suppose that u attains ψ with resources y. If u does not satisfy (A.3), let t be the first date at which it fails to do so. Define the sets $H_t(w)$, $w \in D$, by:

$$H_t(w) = \{(w_0, \theta^{t-1}) \in D \times \Theta^t \colon U_t[w_0, u, (\theta^{t-1}, z^*)] = w\},$$

so that H_t defines a partition of $D \times \Theta^t$. Let P^t_w denote the conditional distribution of (w_0, θ^{t-1}) given $(w_0, \theta^{t-1}) \in H_t(w)$. For all $(\hat{w}_0, \hat{\theta}^{t-1}) \in H_t(w)$, all $s \geqq 0$, and all sequences of reports z^s for periods t through $t+s$, define u'_{t+s} as the average:

$$u'_{t+s}[\hat{w}_0, (\hat{\theta}_{t-1}, z^s))] = \int_{H_t(w)} u_{t+s}[w_0, (\theta^{t-1}, z^s))] dP^t_w.$$

Define the coordinates of the utility plan u' as equal to those of u for dates $t-1$ and earlier, and equal to u'_{t+s} for $s \geqq 0$. Then u' satisfies (3.2) and (3.3), so u' is an allocation. Since the function C is convex, the allocation u' also satisfies (3.4). Clearly the allocation u' satisfies (A.3) for all $s \leqq t$. Continuing the construction through all dates t completes the proof of the Lemma. ||

We remark that this proof uses the convexity of the set of plans satisfying (3.2) and (3.3) to ensure that the constructed plan u' is an allocation. This is the only point in the argument of Section 3 at which we use the assumption that the taste shocks θ enter multiplicatively. If the incentive constraints did not define a convex set, one would need to define allocations and allocation rules as measures, as do Phelan and Townsend (1991). We have not yet pursued this line of generalization.

We now proceed with the proof of Lemmas 3.1 and 3.2.

Proof of Lemma 3.1. Let $\psi \in M$ be given and suppose that u attains ψ with resources y. In view of Lemma A.2, we may take u to have the property (A.3). If a set $H_t(w)$ is empty, let $f_t(w, \theta) = g_t(w, \theta) = w$. If not, for $U_t[w_0, u, (\theta^{t-1}, z^*)] = w$, define (f_t, g_t) by:

$$f_t(w, \theta) = u_t[w_0, (\theta^{t-1}, \theta)]$$

$$g_t(w, \theta) = U_{t+1}[w_0, u, (\theta^{t-1}, \theta, z^*)].$$

That these functions satisfy (3.6) and (3.7) follows from (3.3) and (3.4), applying Lemma A.1 and (A.2). The boundedness property (3.5) follows from (3.1). Hence $\sigma = \{(f_t, g_t)\}$ is an allocation rule.

It remains to verify that (3.8) holds for all t. Let $\{\psi_t\}$ be the sequence of distributions on D defined recursively from ψ_0 by $\psi_{t+1} = S_{g_t}\psi_t$, with the functions g_t defined as above. We next use an induction to show that for any integrable function $F \colon D \to R$,

$$\int_D F(w) d\psi_t = \int_{D\times\Theta^t} F[U_t(w_0, u, (\theta^{t-1}, z^*))] d\psi_0 d\mu^t. \tag{A.4}$$

For $t = 0$, (A.4) follows from (3.2). Suppose (A.4) holds for t. Then

$$\begin{aligned}\int_D F(w) d\psi_{t+1} &= \int_{D\times\Theta} F[g_t(w, \theta)] d\mu d\psi_t \\ &= \int_{D\times\Theta^t} \left\{ \int_\Theta F[g_t(U_t(w_0, u, (\theta^{t-1}, z^*)), \theta_t)] d\mu \right\} d\psi_0 d\mu^t \\ &= \int_{D\times\Theta^{t+1}} F[U_{t+1}(w_0, u, (\theta^t, z^*)), \theta_t)] d\psi_0 d\mu^{t+1},\end{aligned}$$

where the first equality is implied by the definition of the sequence $\{\psi_t\}$, the second by the induction hypothesis, and the third by the definition of the function g_t. This proves (A.4).

To verify (3.8), then, we apply (A.4) to the function $F(w)=\int_\Theta C[f_t(w,\theta)]d\mu$ and use the definition of f_t:

$$\int_{D\times\Theta} C[f_t(w,\theta)]d\mu d\psi_t = \int_{D\times\Theta^{t+1}} C[f_t(U_t(w_0,u,(\theta^{t-1},z^*)),\theta_t)]d\psi_0 d\mu^{t+1}$$

$$= \int_{D\times\Theta^{t+1}} C[u_t(w_0,(\theta^{t-1},\theta_t))]d\psi_0 d\mu^{t+1}$$

This proves Lemma 3.1. ‖

Proof of Lemma 3.2. Suppose the allocation rule $\sigma=\{f_t, g_t\}$ attains ψ with resources y. Let u be generated by σ, so that $u\in S$. We need to show that u satisfies (3.2)-(3.4).

To verify (3.2), we will show that:

for all $t\geqq 0$, $w_0\in D$, and $\theta^{t-1}\in\Theta^t$,

$$w_t(w_0,\theta^{t-1}) = U_t(w_0,u,\theta^{t-1},z^*). \tag{A.5}$$

Then (3.2) will be the statement (A.5) for $t=0$.

To prove (A.5), note first that for all t, w_0, θ^{t-1}, (3.6) implies:

$$w_t(w_0,\theta^{t-1}) = \int_\Theta \{(1-\beta)u_t[w_0,(\theta^{t-1},\theta)]\theta+\beta w_{t+1}[w_0,(\theta^{t-1},\theta_t)]\}d\mu,$$

while (A.1) implies:

$$U_t[w_0,u,\theta^{t-1},z^*] = \int_\Theta \{(1-\beta)u_t[w_0,(\theta^{t-1},\theta)]\theta+\beta U_{t+1}[w_0,u,(\theta^{t-1},\theta,z^*)]\}d\mu,$$

Subtracting gives:

$$|w_t(w_0,\theta^{t-1})-U_t[w_0,u,\theta^{t-1},z^*]|\leqq\beta \sup_{\theta\in\Theta}|w_{t+1}[w_0,(\theta^{t-1},\theta)]-U_{t+1}[w_0,u,(\theta^{t-1},\theta,z^*)]|.$$

Since this inequality holds for all t and (w_0,θ^{t-1}), we have:

$$\sup_{(w_0,\theta^{t-1})}|w_t(w_0,\theta^{t-1})-U_t[w_0,u,\theta^{t-1},z^*]|\leqq\beta^s \sup_{(w_0,\theta^{t+s-1})}|w_{t+s}[w_0,\theta^{t+s-1}]-U_{t+s}[w_0,u,(\theta^{t+s-1},z^*)]| \tag{A.6}$$

for all s, t. Then (3.1) and (3.5) imply that as $s\to\infty$ the right-hand side of the inequality (A.6) goes to zero, which establishes (A.5), and hence (3.2) as well.

Next, we show that the temporary incentive compatibility condition (3.7) implies the incentive compatibility condition (3.3). In fact, we verify that (3.7) implies (A.2), which by Lemma A.1 implies (3.3). We will prove (A.2) first for reporting strategies z that differ from the truth-telling strategy z^* only in the first period, then use an induction to prove (A.2) for strategies z that equal z^* after N periods, and finally verify (A.2) for all strategies z.

From (3.7) and (A.5) we have, for any $w_0\in D$, $r\geqq 0$, , and reporting history $\hat{z}_r$:

$$(1-\beta)u_r[w_0,(\hat{z}_r,\theta)]\theta+\beta U_{r+1}[w_0,u,(\hat{z}_r,\theta,z^*)]\geqq(1-\beta)u_r[w_0,(\hat{z}_r,\hat{\theta})]\theta+\beta U_{r+1}[w_0,u,(\hat{z}_r,\hat{\theta},z^*)], \tag{A.7}$$

for all $\theta,\hat{\theta}\in\Theta$. Now let z^N denote a reporting strategy that is arbitrary for N periods, followed by truth-telling thereafter. As an induction hypothesis, assume that for some value of N and for all w_0, r, and $\hat{z}_r\in\Theta^{r+1}$:

$$(1-\beta)u_r[w_0,(\hat{z}_r,\theta)]\theta+\beta U_{r+1}[w_0,u,(\hat{z}_r,\theta,z^*)]\geqq(1-\beta)u_r[w_0,(\hat{z}_r,\hat{\theta})]\theta+\beta U_{r+1}[w_0,u,(\hat{z}_r,\hat{\theta},z^N)], \tag{A.8}$$

for all z^N and all $\theta,\hat{\theta}\in\Theta$. For $N=0$, this is (A.7). We wish to show that (A.8) holds at $N+1$. From (A.1),

$$U_{r+1}[w_0,u,(\hat{z}_r,\hat{\theta},z^{N+1})] = \int_\Theta \{(1-\beta)u_{r+1}[w_0,(\hat{z}_r,\hat{\theta},z_0^{N+1}(\theta))]\theta+\beta U_{r+2}[w_0,u,(\hat{z}_r,\hat{\theta},z_0^{N+1}(\theta),z^{N\prime})]\}d\mu$$

where $z_0^{N+1}(\theta)$ denotes the first coordinate of the reporting strategy z^{N+1} and $z^{N'}$ denotes its continuation. Thus $z^{N'}$ involves truth-telling after N periods. By the induction hypothesis (A.8), the integrand on the right is maximized by setting $z_0^{N+1}(\theta)=\theta$ and $z^{N'}=z^*$. Hence:

$$U_{r+1}[w_0, u, (\hat{z}_r, \hat{\theta}, z^{N+1})] \leqq \int_\Theta \{(1-\beta)u_{r+1}[w_0, (\hat{z}_r, \hat{\theta}, \theta)]\theta + \beta U_{r+2}[w_0, u, (\hat{z}_r, \hat{\theta}, \theta, z^*)]\}d\mu$$

$$= U_{r+1}[w_0, u, (\hat{z}_r, \hat{\theta}, z^*)]$$

for all $\hat{\theta}$. It follows that:

$$(1-\beta)u_r[w_0, (\hat{z}_r, \hat{\theta})]\theta + \beta U_{r+1}[w_0, u, (\hat{z}_r, \hat{\theta}, z^{N+1})]$$

$$\leqq (1-\beta)u_r[w_0, (\hat{z}_r, \hat{\theta})]\theta + \beta U_{r+1}[w_0, u, (\hat{z}_r, \hat{\theta}, z^*)]$$

$$\leqq (1-\beta)u_r[w_0, (\hat{z}_r, \theta)]\theta + \beta U_{r+1}[w_0, u, (\hat{z}_r, \theta, z^*)]$$

where the last inequality follows from (A.7). This proves (A.8) for all N.

It remains to be shown that there are no reporting strategies that have infinitely many false reports that violate (A.2). But if there were, the boundedness condition (3.1) implies that some strategy z^N with N sufficiently large but finite would also violate (A.2), contradicting (A.8). This completes the proof that (A.2) holds, and hence that (3.3) holds.

Finally, the proof that (3.4) holds is an application of (A.4), used as in the proof of Lemma 3.1. This completes the proof of Lemma 3.2. ‖

Part 2: *Proofs of Lemma* 5.1, 5.2, 5.3 *and* 6.1

Proof of Lemma 5.1. (*Logarithmic Utility*) First, we show that the minimizing choice of r, h must be in a non-empty, compact subset of the constraint set defined by (5.3) and (5.4). The objective in Problem P is continuous, so this step establishes the existence of minimizing choices of r^0, h^0. Then we show that for these choices

$$\int_\theta \exp\{r^0(\theta)\}d\mu = \alpha \int_\theta \exp\{h^0(\theta)\}d\mu. \tag{A.9}$$

This second step establishes that Problem P is equivalent to a problem of minimizing a strictly convex objective, and thus that the minimizing choice of r^0, h^0 is unique.

The subset of R^{2n} defined by (5.3) and (5.4) is not compact, but the minimization in Problem P can be confined to a compact set as follows. Let (r^*, h^*) be any choice satisfying (5.3)-(5.4), and consider the set $A \subset R$ defined by (5.3), (5.4) and the two inequalities $\int_\Theta \exp\{r(\theta)\}d\mu \leqq \int_\Theta \exp\{r^*(\theta)\}d\mu$ and $\int_\Theta \exp\{h(\theta)\}du \leqq \int_\Theta \exp\{h^*(\theta)\}d\mu$. Clearly a minimum in Problem P must be attained in A. The set A is evidently closed and the co-ordinates of points in A are bounded above. Then (5.3) and the assumption that all θ values have positive probability implies that all co-ordinates are also bounded below. Hence A is compact and there is a point (r^0, h^0) that attains the minimum in Problem P.

To verify the equality (A.9), note that for any constant a, the point $(r^0(\theta)+a,\ h^0(\theta)-[(1-\beta)/\beta]a)$ satisfies (5.3) and (5.4). If equality does not hold, the constant a can be chosen (positive or negative) to yield a lower value to the objective function, contradicting the optimality of (r^0, h^0). The uniqueness of the minimizing value in Problem P now follows from the strict convexity of $\exp\{u\}$. ‖

Proof of Lemma 5.1. (*CRRA case with* $\gamma<0$ *and CARA case*). The proof of Lemma 5.1 in the CRRA utility case with $\gamma<0$ and in the CARA utility case is the same as in the logarithmic case. ‖

In the case of CRRA utility with $\gamma>0$, the proof needs to be modified as follows.

Proof of Lemma 5.1. (*CRRA case* $\gamma>0$). In the CRRA case with $\gamma>0$, since $D=[0,\infty)$ and each θ occurs with positive probability, the constraint set defined by (5.3) and (5.4) is bounded both below and above, and is thus compact. But, since we must admit the possibility that $r(\theta)$ or $h(\theta)$ is equal to zero for some θ, we cannot use the method above to verify the obvious analogue to (A.9). In this case, the following method may be used to verify this equation.

Assume, contrary to (A.9), that (r^0, h^0) satisfy the incentive constraints (5.3) and (5.4) and that

$$\int_\Theta [r^0(\theta)]^{1/\gamma}d\mu > \alpha \int_\Theta [h^0(\theta)]^{1/\gamma}d\mu.$$

Let $\bar{r}(\theta)=0$, $\bar{h}(\theta)=1/\beta$, so that $(\bar{r},\bar{h})$ satisfy the incentive constraints (5.3) and (5.4). Define $r^\lambda=(1-\lambda)r^0+\lambda\bar{r}$, $h^\lambda=(1-\lambda)h^0+\lambda\bar{h}$ for $\lambda\in[0,1]$. Since the incentive constraints (5.3) and (5.4) are convex, these functions also satisfy the incentive constraints. For any value of $\lambda\in(0,1)$, we have

$$\int_\Theta [r^0(\theta)]^{1/\gamma}d\mu > \int_\Theta [r^\lambda(\theta)]^{1/\gamma}d\mu.$$

By continuity, for sufficiently small values of λ, we have

$$\int_\Theta [r^0(\theta)]^{1/\gamma}d\mu > \alpha\int_\Theta [h^\lambda(\theta)]^{1/\gamma}d\mu.$$

Thus, (r^0, h^0) cannot minimize the objective in Problem P. This proves in the CRRA case with $\gamma>0$, that the obvious analogue to (A.9) holds at the choice of (r^0, h^0) that minimize the objective in Problem P. ||

To prove Lemma 5.2, we require the following lemma.

Lemma A.3. (Logarithmic Utility Case). *Let* $C(u)=\exp(u)$, $D=(-\infty,\infty)$, *and* $\varphi(\psi)=\alpha\int_D C(w)d\psi<+\infty$. *If* $f^0(w,\theta)$ *and* $g^0(w,\theta)$ *attain the minimum in* (4.4) *given* α, *then*

$$\int_{D\times\Theta} C[f(w,\theta)]d\mu d\psi=\alpha\int_{D\times\Theta} C[g(w,\theta)]d\mu d\psi. \tag{A.10}$$

Proof: (*Logarithmic Utility Case*). To verify the equality (A.10), note that, for any constant a, the point $(f^0(w,\theta)+a,\ g^0(w,\theta)-[(1-\beta)/\beta]a]$ satisfies (4.2) and (4.3). If equality (A.10) does not hold, the constant a can be chosen (positive or negative) to yield a lower value to the objective function (4.4), contradicting the optimality of (f^0, g^0). ||

Proof of Lemma A.3 (*CRRA utility case* $\gamma>0$). In this case $D=[0,\infty)$. Since we must admit the possibility that $f^0(w,\theta)$ or $g^0(w,\theta)$ is equal to zero for some (w,θ), we cannot use the method above to verify (A.10). In this case, the same method used to verify (A.9) may be used to verify (A.10).||

Proof of Lemma A.3 (*CRRA utility case* $\gamma>0$) and *CARA utility.* In this case $D=(-\infty,0)$. In the case of CARA utility, since the function C bounded neither above nor below, we must add the assumption that we consider only distributions ψ which satisfy $|\int_D C(w)d\psi|<+\infty$. Since we must admit the possibility that, for some $\theta\in\Theta$ $\sup_w f^0(w,\theta)=0$ or $\sup_w g^0(w,\theta)=0$, we cannot use the methods above to verify (A.10).

In this case, we verify that, in fact, there must exist some set $D_0\subseteq D$ with $\psi(D_0)>0$ such that for all $\theta\in\Theta$, $\sup f^0(w,\theta)<0$ and $\sup g^0(w,\theta)<0$, where the suprema are taken over $w\in D_0$. After having argued this, we then can observe that, for any constant a, the point $\bar{f},\bar{g}$ with $\bar{f}(w,\theta)=f^0(w,\theta)$, $\bar{g}(w,\theta)=g^0(w,\theta)$ for $w\in D-D_0$, and $\bar{f}(w,\theta)=f^0(w,\theta)+a$, $\bar{g}(w,\theta)=g^0(w,\theta)-[(1-\beta)/\beta]a$ for $w\in D_0$ satisfies (4.2) and (4.3). If equality (A.10) does not hold, since $\psi(D_0)>0$, the constant a can be chosen (positive or negative) to yield a lower value to the objective function (4.4), contradicting the optimality of (f^0, g^0).

We verify that there must exist some set $D_0\subseteq D$, $\psi(D_0)>0$, such that, for all $\theta\in\Theta$, $\sup f^0(w,\theta)<0$ and $\sup g^0(w,\theta)<0$, where the suprema are taken over $w\in D_0$ as follows. Suppose the contrary, i.e. that for all $D_0\subseteq D$ with $\psi(D_0)>0$, for some $\theta\in\Theta$, $\sup f^0(w,\theta)=0$ or $\sup g^0(w,\theta)=0$, where the suprema are taken over $w\in D_0$. Define the function $m^0: D\times\Theta\to D\times\Theta$ by

$$m^0(w,\theta)=\max[f^0(w,\theta), g^0(w,\theta)]$$

and $m: D\to D$ by $m(w)=\max_{\theta\in\Theta} m^0(w,\theta)$. Define an increasing set of functions $m_n: D\to D$ by $m_n(w)=m(w)$ if $m(w)\leqq -1/n$ and $m_n(w)=1/n$ otherwise. Let $D_n\subseteq D$ be the set of w for which $m(w)\leqq 1/n$. Our hypothesis implies that $\psi(D_n)=0$. Thus, $\int_D m_n(w)d\psi=1/n$, for all n, and then, in the limit,

$$\int_D m(w)d\psi=0. \tag{A.11}$$

We now proceed to prove our result by contradicting (A.11). For each $\theta\in\Theta$, let $A_\theta\subseteq D$ be that set of $w\in D$ such that $f(w,\theta)\geqq g(w,\theta)$. Since $|\int_D C(w)d\psi|<+\infty$, we have that the two terms in (A.10) are finite.

$$\int_D C|m(w,\theta)]d\psi=\int_A C[f^0(w,\theta)]d\psi+\int_{D-A} C[g^0(w,\theta)]d\psi<+\infty$$

for all $\theta \in \Theta$. Since Θ is finite and each θ occurs with positive probability, we have that $\int_D C[m(w)]d\psi < +\infty$. By Jensen's inequality, $\int_D m(w)\, d\psi < 0$, which contradicts (A.11), and we are done. ||

Proof of Lemma 5.2. (*Logarithmic utility*). As a consequence of Lemma A.3, we may rewrite Problem T when $\varphi(\psi) = \alpha \int_D C(w)d\psi$ as

$$\min_{f,g} \int_{D\times\Theta} C[f(w, \theta)d\mu d\psi$$

subject to the incentive constraints (4.2), (4.3), and the constraint

$$\int_{D\times\Theta} C[f(w, \theta)]d\mu d\psi \geqq \alpha \int_{D\times\Theta} C[g(w, \theta)]d\mu d\psi. \tag{A.12}$$

Sufficient conditions for (f, g) to solve this Problem T are that there exist a number $\lambda > 0$ such that, for each $w \in D, f, g$ minimizes

$$\int_\Theta \{(1-\lambda)C[f(w, \theta)] + \lambda\alpha C[g(w, \theta)]\}d\mu$$

subject to the incentive constraints (4.2), and (4.3), and such that

$$\lambda\left\{\int_{D\times\Theta} C[f(w, \theta)]d\mu d\psi - \alpha\int_{D\times\Theta} C[g(w, \theta)]d\mu d\psi\right\} = 0.$$

By (A.9), the values of (r^0, h^0) that solve Problem P (in the logarithmic utility case) minimize $\int_\Theta \exp(r(\theta))d\mu$ subject to the incentive constraints (5.3), (5.4), and the constraint that

$$\int_\Theta \exp(r(\theta))d\mu \geqq \alpha \int_\Theta \exp(h(\theta))d\mu.$$

Then, by Lemma 5.1 and the Kuhn-Tucker Theorem, there exists a $\lambda \geqq 0$ such that r^0, h^0 minimizes

$$\int_\Theta \{(1-\lambda)\exp(r(\theta)) + \lambda\alpha \exp(h(\theta))\}d\mu$$

subject to the incentive constraints (5.3), and (5.4), and satisfies

$$\lambda\left\{\int_\Theta \{\exp(r(\theta)) - \alpha\exp(h(\theta))\}d\mu = 0.\right.$$

It is clear that $\lambda > 0$. It follows that λ and the functions (f^0, g^0) defined in the hypothesis of the lemma satisfy sufficient conditions to solve Problem T. (The proof is similar in the cases of CRRA and CARA utility.) ||

Proof of Lemma 5.3. The function $\phi(\alpha)$ is continuous by the Theorem of the Maximum. We verify that $\phi(\alpha_c) \geqq \alpha_c$ and $\phi(1) \leqq 1$. This plus the continuity of ϕ will ensure the existence of a fixed point of ϕ in $[\alpha_c, 1]$.

The constant functions $r(\theta) = h(\theta) = 0$ are feasible for Problem P, and for $\alpha = 1$ they yield the value 1. Hence $\phi(1) \leqq 1$.

For any value of α, the solution $\phi(\alpha)$ to Problem P must be greater than or equal to the solution of the same minimum problem with the incentive constraints (5.4) discarded. The solution of this latter problem in the logarithmic utility case is readily calculated to be attained at:

$$\exp(r(\theta)) = \lambda\theta \quad \text{and} \quad \exp(h(\theta)) = \frac{\lambda}{\alpha},$$

where $\lambda = \alpha^\beta \exp\{-(1-\beta)E[\theta \log(\theta)]\}$. Inserting either solution into the objective function yields $\phi(\alpha) \geqq \lambda$. At $\alpha = \alpha_c$, this inequality is:

$$\phi(\alpha_c) \geqq \exp\{-\beta E[\theta \log(\theta)]\} \exp\{-(1-\beta)E[\theta \log(\theta)]\} = \alpha_c.$$

The proof of this point is similar in the cases of CRRA and CARA utility.

We can verify that the fixed point of ϕ in the interval $[\alpha_c, \alpha_a]$ is unique by the following argument. Let $\alpha' > \alpha$ and apply the result from Lemma (6.1) that at the choice of (r^0, h^0) which solve Problem P,

$$\phi(\alpha) = \int_\Theta \exp[r(\theta; \alpha)]d\mu = \alpha \int_\Theta \exp[h(\theta; \alpha)]d\mu$$

(in the logarithmic utility case) to obtain:

$$\phi(\alpha') - \phi(\alpha) = \alpha' \int_{\Theta} \exp[h(\theta; \alpha')] d\mu - \alpha \int_{\Theta} \exp[h(\theta; \alpha)] d\mu$$

$$\leqq (\alpha' - \alpha) \frac{\phi(\alpha)}{\alpha},$$

since $(r^0(\theta; \alpha), h^0(\theta; \alpha))$ is feasible for Problem P at α'. Moreover, the inequality must hold strictly since the minimum at α' is unique. Hence ϕ has at most one fixed point. ||

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