The Optimal Linear Income-tax $^{1,2}$

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1. INTRODUCTION

The conflict between equity and efficiency considerations in income taxation is a familiar problem, but no general rules that take both of these considerations into account have yet been established. On the basis of a concrete model, a variety of examples has been recently analyzed and calculated by Mirrlees [2]. In these examples, the optimal income tax schedule appears to be progressive with a negative tax at low incomes. The optimal marginal tax rate is approximately constant in the vicinity of 60 per cent. It is not known, however, how sensitive these results are with respect to various assumptions of the model.

In Mirrlees' model, individuals are assumed to maximize identical utility functions that depend on consumption and labour. The return to labour is assumed to depend partly on an innate skill factor associated with each individual. Through individual decisions, the distribution of skill in the population generates distributions of labour, consumption, and utility. Income taxation is then introduced in order to improve income distribution and social welfare, defined as the sum of individual utilities.$^3$

In this paper we use the above model to prove the following result: if the supply of labour in the economy is a non-decreasing function of the net wage rate then, among linear tax functions, the optimal tax schedule provides a positive lump-sum at zero income, and the optimal marginal tax rate is bounded above by a fraction that decreases with the minimum elasticity of the labour supply.$^4$

It may be noted that this result does not depend on any assumption about the form of the individual utility functions or on the underlying distribution of ability in the population.

2. INDIVIDUAL BEHAVIOUR

Individuals are assumed to have an identical utility function, $u$, that depends on consumption, $c$, and labour, $\ell$:

$$ u = u(c, \ell). \quad \ldots \quad (1) $$

It is assumed that $u$ is continuously differentiable, strictly concave,$^5$ with a positive marginal utility for consumption and a negative marginal utility for labour

$$ u_1 > 0, \quad u_2 < 0, \quad u_{11} > 0, \quad u_{22} > 0, \quad u_{11}u_{22} - u_{12}^2 > 0. \quad \ldots \quad (2) $$

Let $y$ be before-tax income and let $t(y)$ be a linear income-tax function defined on $y$

$$ t(y) = -\alpha + (1 - \beta)y \quad \ldots \quad (3) $$

$^1$ First version received March 1971; final version received August 1971 (Eds.).

$^2$ This paper was written while I was visiting at the Department of Economics, Harvard University. I am indebted to a referee and to the editor for useful comments.

$^3$ The ethical postulates required to make social welfare a sum of measurable and comparable individual utilities are known, e.g. Harsanyi [1] and Sen [3]. It has been argued that if one accepts individualistic ethics and sets public policy the task of satisfying the preferences of the individual members of society, the social welfare function will always tend to have the form of a sum of individual utilities.

$^4$ A similar result has also been obtained for a somewhat different model [2].

$^5$ For the analysis of individual behaviour it is only required that $u$ be strictly quasi-concave, i.e.

$$ u_1u_2^2 - 2u_1u_2u_2 + u_2^2 < 0 $$

but for the social welfare analysis, concavity of $u$ is required in order to make complete equality the first-best optimum.
where $\alpha$ and $\beta$ are the tax-schedule parameters. $\alpha$ is a lump-sum tax ($\alpha<0$) or subsidy ($\alpha>0$) given to an individual with no income. $1-\beta$ is the marginal tax rate, i.e. $0 \leq \beta \leq 1$ implies a non-negative marginal tax rate, not exceeding unity.

After-tax income is equal to consumption:

$$c = y - t(y) = \alpha + \beta y.$$  \hfill (4)

Income is equal to the amount of labour times the wage rate. The latter is assumed to depend positively on the innate ability of the individual, denoted by an index-number $n$ ($0 \leq n \leq \infty$). For simplicity, it is assumed that this relation is linear. Hence

$$y = y(n, \ell) = n \cdot \ell.$$  \hfill (5)

Each individual decides about the optimal amount of consumption and labour so as to maximize his utility (1) subject to the budget constraint (4). By (5), the first order condition for this maximum is

$$-\frac{u_2}{u_1} = \beta n \quad \text{or} \quad -\beta n u_1 = u_2.$$  \hfill (6)

for all $n \geq 0$.

This equation, together with the budget constraint (4), define implicitly the optimal labour supply and consumption as functions of $\beta n$ and $\alpha$

$$\ell = \ell(\beta n, \alpha), \quad \text{and} \quad c = c(\alpha, n) = \alpha + \beta n \ell(\beta n, \alpha).$$  \hfill (7)

We assume that leisure is a normal good; that is,

$$\frac{\partial \ell}{\partial \alpha} \leq 0.$$  \hfill (8)

Differentiating the first order condition, we can obtain the derivative of the labour supply function with respect to $\alpha$ in terms of the derivatives of the utility function

$$\frac{\partial \ell}{\partial \alpha} = \frac{u_1 u_2 u_{11} - u_1^2 u_{12}}{u_{11} u_2 - 2 u_1 u_{12} + u_{22} u_1^2}.$$  \hfill (9)

Hence, in view of (2), normality requires

$$u_{11} - \frac{u_1}{u_2} u_{12} \leq 0.$$  \hfill (10)

It is also assumed that the supply of labour is a non-decreasing function of the net wage rate; that is $^2$

$$\frac{\partial \ell}{\partial \beta} \geq 0.$$  \hfill (11)

Let $\lambda$ be the lowest elasticity of the labour supply function,$^3$ i.e.

$$\frac{\beta}{\ell} \frac{\partial \ell}{\partial \beta} \geq \lambda.$$  \hfill (12)

for all $n$. Given the tax parameters $\alpha$ and $\beta$, the amounts of labour, income and utility

$^1$ Assuming the optimal solutions are strictly positive. In view of (2), the second-order condition is also satisfied.

$^2$ It is possible, of course, that $\ell = 0$ for small $\beta$ (when the more general first-order condition $u_2/u_1 \leq \beta n$ yields a corner solution). With a "backward-bending" supply function of labour, income could be decreasing as $n$ increases. It would then be impossible to redistribute income by means of a progressive income-tax!

$^3$ More precisely, $\lambda = \lim \inf_{n} \left[ \frac{\beta}{\ell} \frac{\partial \ell}{\partial \beta} \right]$. 


can be regarded as functions of \( n \). Since \( \hat{\ell} \) depends on the product \( \beta \cdot n \), the signs of \( \frac{\partial \hat{\ell}}{\partial \beta} \) and \( \frac{\partial \hat{\ell}}{\partial n} \) are clearly the same. Hence, in view of (11), \( \hat{\ell} \) is a non-decreasing function of \( n \). It follows that \( y \) is strictly increasing in \( n \)

\[
\frac{\partial y}{\partial n} = \frac{\partial (n \cdot \hat{\ell})}{\partial n} = \hat{\ell} + n \frac{\partial \hat{\ell}}{\partial n} > 0. \quad \text{(13)}
\]

Utility is also seen to be strictly increasing in \( n \). By (4) and (6)

\[
\frac{\partial u}{\partial n} = u_1 \frac{\partial c}{\partial n} + u_2 \frac{\partial \hat{\ell}}{\partial n} = u_1 \left[ \frac{\partial c}{\partial n} - \beta n \frac{\partial \hat{\ell}}{\partial n} \right] = u_1 \cdot \beta \cdot \hat{\ell} > 0. \quad \text{(14)}
\]

3. THE OPTIMAL LINEAR INCOME-TAX

Let \( f(n) \) be the density function of ability, i.e. the ratio of the number of individuals with ability \( n \) to the total number of individuals, \( \int_0^\infty f(n)dn = 1 \).

The social welfare function, \( V \), is assumed to be the sum of individual utilities. Normalizing for size,

\[
V = \int_0^\infty u(c, \ell)f(n)dn. \quad \text{(15)}
\]

The optimal linear income-tax problem is to find parameters \( \alpha \) and \( \beta \) that maximize (15), subject to the constraint that total tax proceeds be equal to zero, i.e.

\[
\int_0^\infty t(y)f(n)dn = \int_0^\infty \left( y - \alpha - \beta y \right)f(n)dn = 0
\]

or

\[
\alpha = (1 - \beta) \int_0^\infty yf(n)dn. \quad \text{(16)}
\]

To solve the problem, we form the function

\[
W = \int_0^\infty \left[ u(c, \ell) - q(\alpha - (1 - \beta)y) \right]f(n)dn \quad \text{(17)}
\]

where \( q \) is the shadow-price of constraint (16). The first-order conditions for a maximum of \( W \) with respect to \( \alpha \) and \( \beta \), using (4)-(6), are

\[
\frac{\partial W}{\partial \alpha} = \int_0^\infty (u_1 - q)f(n)dn + q(1 - \beta) \int_0^\infty \frac{\partial \hat{\ell}}{\partial \alpha} f(n)dn = 0 \quad \text{(18)}
\]

\[
\frac{\partial W}{\partial \beta} = \int_0^\infty (u_1 - q)n\hat{\ell}f(n)dn + q(1 - \beta) \int_0^\infty n \frac{\partial \hat{\ell}}{\partial \beta} f(n)dn = 0. \quad \text{(19)}
\]

Denote the optimal parameters by \( \alpha^* \) and \( \beta^* \). We now prove the following:

**Theorem.** Under assumptions (2), (8) and (11), \( \alpha^* > 0 \) and \( \frac{\beta^*}{1 + \beta} < 1 \).

**Proof.** We first observe that \( q > 0 \). From (18) and (19) it is immediately seen that \( q = 0 \) is impossible. Suppose \( q < 0 \). Then the first term in (18) is positive, and, in view of (8), (18) implies that \( \beta^* > 1 \). But, by (11), (19) cannot hold when \( q < 0 \) and \( \beta^* > 1 \). Hence, \( q > 0 \).
Obviously \( \beta^* > 0 \) (or else \( c = \lambda = 0 \) for all \( n \)). Suppose now that \( \beta^* \geq 1 \). Then, in view of (8) and (18)

\[
\int_0^\infty (u_1 - q)f(n)dn \leq 0.
\] ...

(20)

We now wish to prove that (20) implies that

\[
\int_0^\infty (u_1 - q)n^\lambda f(n)dn < 0.
\] ...

(21)

From (2), (4)-(6), (10) and (11), \( u_1 \) is seen to be strictly decreasing in \( n \)

\[
\frac{\partial u_1}{\partial n} = u_{11} \frac{\partial c}{\partial n} + u_{12} \frac{\partial \lambda}{\partial n} = u_{11} \beta^* + \left( u_{12} - \frac{u_2}{u_1} u_{11} \right) \frac{\partial \lambda}{\partial n} < 0.
\] ...

(22)

It has been shown that \( y = n^\lambda \) is strictly increasing in \( n \). Thus, (21) is obvious when \( u_1 - q \leq 0 \) for all \( n \). More generally, there exists a number \( n_0 \geq 0 \) such that \( u_1 - q > 0 \) for \( 0 \leq n < n_0 \) and \( u_1 - q < 0 \) for \( n > n_0 \). Let \( \lambda(n_0) = \lambda_0 \). In view of (14) and (23)

\[
(u_1 - q)n^\lambda < n_0^\lambda \lambda_0(u_1 - q)
\]

for all \( n \). Multiplying both sides of this inequality by \( f(n) \) and integrating, we find, in view of (21)

\[
\int_0^\infty (u_1 - q)n^\lambda f(n)dn < n_0 \lambda_0 \int_0^\infty (u_1 - q)f(n)dn \leq 0.
\]

This establishes (21). From (19), however, it is seen that (21) is impossible when \( \frac{\partial \lambda}{\partial \beta^*} = 0 \), and when \( \frac{\partial \lambda}{\partial \beta^*} > 0 \), (21) implies that \( \beta^* < 1 \), contrary to assumption. Thus, \( \beta^* < 1 \). From (16) we then find that \( \alpha^* > 0 \).

Now let us rewrite (19)

\[
\int_0^\infty u_1 n^\lambda f(n)dn + q \int_0^\infty \left[ \left( \frac{1 - \beta^*}{\beta^*} \right) \left( \frac{\beta}{\lambda} \frac{\partial \lambda}{\partial \beta^*} \right) \right] n^\lambda f(n)dn = 0.
\] ...

(23)

Since the first term in (23) is positive, clearly

\[
\int_0^\infty \left[ 1 - \left( \frac{1 - \beta^*}{\beta^*} \right) \left( \frac{\beta}{\lambda} \frac{\partial \lambda}{\partial \beta^*} \right) \right] n^\lambda f(n)dn > 0.
\]

From (12), then

\[
\left[ 1 - \left( \frac{1 - \beta^*}{\beta^*} \right) \left( \frac{\beta}{\lambda} \frac{\partial \lambda}{\partial \beta^*} \right) \right] \int_0^\infty n^\lambda f(n)dn \geq \int_0^\infty \left[ 1 - \left( \frac{1 - \beta^*}{\beta^*} \right) \left( \frac{\beta}{\lambda} \frac{\partial \lambda}{\partial \beta^*} \right) \right] n^\lambda f(n)dn > 0
\]

for which it is necessary that

\[
\left( \frac{1 - \beta^*}{\beta^*} \right) \lambda < 1, \text{ i.e. } \beta^* > \frac{\lambda}{1 + \lambda} \quad \text{Q.E.D.}
\]

For empirical applications it is important to notice that in the above, \( \lambda \) can be taken as the lowest elasticity in open intervals over which the density \( f(n) \) is positive.

The upper bound that we found on the optimal marginal tax rate has a natural interpretation, which at the same time brings out its weakness. Suppose that the government, instead of maximizing social welfare, would be interested in maximizing tax revenue
obtained from a proportional income tax (it is assumed that lump-sum taxation is impossible). With \( t(y) = (1-\beta)y \), maximization of tax revenue

\[
\int_0^\infty t(y)f(n)dn = \int_0^\infty (1-\beta)n^\beta f(n)dn
\]
yields the first-order condition

\[
\int_0^\infty \left[ \frac{1-\beta}{\beta} \left( \frac{\beta}{\beta} \frac{\partial \ell}{\partial \beta} \right) - 1 \right] n^\beta f(n)dn = 0. \quad \ldots(24)
\]

Denoting the solution to (24) by \( \bar{\beta} \), it is seen that the theorem holds also for this case: \( \beta > \frac{\lambda}{1+\lambda} \). One expects therefore, that the optimal marginal tax rate will not exceed the revenue-maximizing rate. While this cannot be shown to hold in general, it is true, for example, when the elasticity of labour supply is constant, say \( \lambda \). From (23) and (24) it then follows that, \( \beta^* > \bar{\beta} = \frac{\lambda}{1+\lambda} \).

4. AN EXAMPLE

Suppose that \( u(c, \ell) = c(\ell - \ell) \), where \( \ell > 0 \) is the maximum feasible labour supply, i.e. \( 0 \leq \ell \leq \bar{\ell} \). From the first-order condition (6) and the budget constraint (4) one derives the optimal labour supply

\[
\ell(\beta n, \alpha) = \begin{cases} 
0 & n \leq n_0 \\
\frac{1}{\beta} \left( \frac{\alpha}{\beta n} \right) & n > n_0 
\end{cases} \quad \ldots(25)
\]

where \( n_0 = \frac{\alpha}{\beta \ell} \). We have \( \frac{\partial \ell}{\partial \alpha} < 0 \) and \( \frac{\partial \ell}{\partial \beta} > 0 \) as assumed in (8) and (11).

The elasticity of labour supply is found to be

\[
\frac{\beta}{\ell} \frac{\partial \ell}{\partial \beta} = \frac{\alpha}{\beta \ell - \alpha}, \quad \ldots(26)
\]
a monotonically decreasing function of \( n \). Let \( \bar{n} \) be the highest \( n \) for which \( f(n) > 0 \). In this range

\[
\frac{\beta}{\ell} \frac{\partial \ell}{\partial \beta} \geq \frac{\alpha}{\beta \bar{n} - \alpha} = \lambda.
\]

Thus, the condition \( \beta^* > \frac{\lambda}{1+\lambda} \) is in this case given by

\[
\beta^* > \frac{\alpha^*}{\beta^* \bar{n}^\beta} \quad \text{or} \quad \beta^* > \sqrt[\beta^*]{\frac{\alpha^*}{\bar{n}^\beta}}. \quad \ldots(27)
\]

Now, \( \alpha^* \) is the lowest after-tax income, while \( \bar{n}^\beta \) is approximately the highest before-tax income.\(^1\) For example, if \( \alpha^*/\bar{n}^\beta = 1/9 \) then \( \beta^* > 1/3 \); i.e. the marginal tax rate should not exceed two-thirds.

\(^1\) Since \( n \leq \bar{n} \) (in the relevant range) and \( \ell \leq \bar{\ell} \), it follows that \( y = n\ell \leq \bar{n}\bar{\ell} \).
REFERENCES


