Stationary Ordinal Utility and Impatience

Tjalling C. Koopmans

Econometrica, Volume 28, Issue 2 (Apr., 1960), 287-309.

Stable URL:
http://links.jstor.org/sici?sici=0012-9682%28196004%2928%3A2%3C287%3ASOUAI%3E2.0.CO%3B2-N

Econometrica, Vol 28, 2 (April 1960)

STATIONARY ORDINAL UTILITY AND IMPATIENCE[1]

BY TJALLING C. KOOPMANS[2]

This paper investigates Böhm-Bawerk's idea of a preference for advancing the timing of future satisfactions from a somewhat different point of view. It is shown that simple postulates about the utility function of a consumption program for an *infinite* future logically imply impatience at least for certain broad classes of programs. The postulates assert continuity, sensitivity, stationarity of the utility function, the absence of intertemporal complementarity, and the existence of a best and a worst program. The more technical parts of the proof are set off in starred sections.

1. INTRODUCTION

EVER SINCE the appearance of Böhm-Bawerk's *Positive Theorie des Kapitals*, the idea of a preference for advancing the timing of future satisfaction has been widely used in economic theory. The question of how to define this idea precisely has, however, been given insufficient attention. If the idea of preference for early timing is to be applicable also to a world of changing prices, money expenditure on consumption is not a suitable measure of "satisfaction level," and money expenditure divided by a consumers' goods price index is at best an approximate measure, useful for econometric work but not providing the sharp distinctions that theory requires. It seems better, therefore, to try to define preference for advanced timing entirely in terms of a *utility function*. Moreover, if the idea of preference for early timing is to be expressed independently of assumptions that have made the construction of cardinal utility possible[3] (such as choice between uncertain prospects, or stochastic choice, or independence of commodity groups in the preference structure) it will be necessary to express it in terms of an *ordinal* utility function, that is, a function that retains its meaning under a monotonic (increasing) transformation. It would seem that this can be done only if one postulates a certain persistency over time in the structure of preference.

This study started out as an attempt to formulate postulates permitting a sharp definition of *impatience*, the short term Irving Fisher has introduced for preference for advanced timing of satisfaction. To avoid complications connected with the advancing age and finite life span of the individual consumer, these postulates were set up for a (continuous) utility function of a consumption program extending over an infinite future period. The

[1] This study was carried out in part under a grant from the National Science Foundation.

[2] I am indebted to Gerard Debreu and Herbert Scarf for extremely valuable comments and suggestions on the subject and methods of this paper.

[3] For a recent discussion, see Debreu [2].

surprising result was that only a slight strengthening of the continui postulate (incorporated in Postulate 1 below) permits one to conclude fro the existence of a utility function satisfying the postulates, that impatien prevails at least in certain areas of the program space. In other wor conditions hardly stronger than those that appear needed to *define* impatien in a meaningful way are sufficient to *prove* that there are zones of ir patience. Intuitively, the reason is that if there is in all circumstances preference for postponing satisfaction—or even neutrality toward timing then there is not enough room in the set of real numbers to accommoda and label numerically all the different satisfaction levels that may occur relation to consumption programs for an infinite future.

This paper thus has become a study of some implications of a continuo and stationary (see Postulate 3) ordering of infinite programs. Flexibili of interpretation remains as to whether this ordering may serve as a fi approximation to the preferences of an individual consumer, or may perha be an "impersonal" result of the aggregation of somewhat similar individ preferences (interpreting "consumption" as "consumption per head" in t case of a growing population), or finally may guide choices in a centra planned economy. In each of these interpretations further modificatic and refinements may be called for.

The first paper in the literature basing the study of utility on a set behavior axioms (or postulates), known to this author, was by Profes Frisch [**5**]. Since then this method has been widely applied to establi utility concepts appropriate to a variety of choice problems. In most cases t postulates have been in terms of preferences rather than of a utility fu tion. To limit the mathematical difficulties, the postulates of the pres study are in terms of a utility function, with the understanding that alternative with higher utility is always preferred over one with lo utility, and indifference exists between alternatives of equal utility. Stud deriving the existence of an ordinal utility function from postulates ab preferences have been made by Wold [**10**] and by Debreu [**3**].

Two levels of discussion are separated in what follows. The conte and findings of each section are first stated in general terms. Then, wh needed, the more technical stipulations, proofs and discussions are gi in a starred section bearing the same number. The starred sections can passed up by readers interested primarily in the results and in the l technical phases of the reasoning.

2. THE PROGRAM SPACE — NOTATION

A program for an infinite future will be denoted

$$(1) \qquad {}_1x = (x_1, x_2, x_3, \ldots, x_t, \ldots) = (x_1, {}_2x) = \text{etc.}$$

Each symbol x_t, $t = 1, 2, \ldots$, represents a vector (bundle)

$$x_t = (x_{t1}, x_{t2}, \ldots, x_{tn}) \tag{2}$$

of the nonnegative amounts of n listed commodities to be consumed in the period t. Subvectors of (1) consisting of several consecutive vectors (2) will be denoted

$$_tx_{t'} = (x_t, x_{t+1}, \ldots, x_{t'}) , \tag{3}$$

where omission of the right subscript t' of $_tx_{t'}$ indicates that $t' = \infty$. The subscript t of x_t is called the *timing* of the consumption vector x_t, the subscript s of $_sx = (x_s, x_{s+1}, \ldots)$ the *time* of choice between $_sx$ and its alternatives $_sx'$, $_sx''$, ... A constant program is denoted

$$_{\text{con}}x = (x, x, x, \ldots) . \tag{4}$$

Finally, $\equiv$ denotes equality by definition.

2*. Each consumption vector x_t is to be selected from a connected subset X of the n-dimensional commodity space, which we take to be the same for all t. Hence $_tx = (x_t, x_{t+1}, \ldots)$ belongs to the cartesian product $_1X$ of an infinite sequence of identical sets X. Expressions such as "for some x_t," "for all $_tx$," etc., will in what follows always mean "for some $x_t \in X$," "for all $_tx \in {}_1X$," etc., and all functions of x_t or $_tx$ are to be thought of as defined on X or on $_1X$, respectively.

3. EXISTENCE OF A CONTINUOUS UTILITY FUNCTION

Before stating the basic postulate asserting this existence, the meaning of continuity needs to be clarified. Continuity of a function $f(y)$ of a vector y means that, for every y, one can make the absolute difference $|f(y') - f(y)|$ as small as desired by making the distance $d(y', y)$ between y' and y sufficiently small, regardless of the direction of approach of y' to y. For vectors $y = (y_1, \ldots, y_n)$ with a finite number n of components there is a wide choice of definitions of the distance function $d(y', y)$, all of which establish the same continuity concept, and the maximum absolute difference for any component,

$$d(y', y) = |y' - y| \equiv \max_k |y_k' - y_k| \tag{5}$$

is as suitable as any of a large class of alternatives. But in an infinite-dimensional space the continuity concept is sensitive to the choice of the distance function used. In what follows we shall employ as a "distance" between two programs $_1x'$, $_1x$, the function

$$d(_1x', {}_1x) \equiv \sup_t |x_t' - x_t| . \tag{6}$$

This is the maximum distance in the sense of (5) between any two corresponding one-period consumption vectors x_t', x_t, whenever such a maximum

exists.[4] This definition treats all future periods alike, and, if anything, has a bias toward neutrality with regard to the timing of satisfaction.

POSTULATE 1. *There exists a utility function* $U({}_1x)$, *which is defined for all* ${}_1x = (x_1, x_2, \ldots)$ *such that, for all* t, x_t *is a point of a connected subset* X *of the* n*-dimensional commodity space. The function* $U({}_1x)$ *has the continuity property that, if* U *is any of the values assumed by that function, and if* U' *and* U'' *are numbers such that* $U' < U < U''$, *then there exists a positive number* δ *such that the utility* $U({}_1x')$ *of every program* ${}_1x'$ *having a distance* $d({}_1x', {}_1x) \leqq \delta$ *from some program* ${}_1x$ *with utility* $U({}_1x) = U$ *satisfies* $U' \leqq U({}_1x') \leqq U''$.

Comparison with the above definition of continuity of a function $f(y)$ will show that we are here making a slightly stronger requirement (which obviously implies ordinary continuity). For any U' and U'' bracketing the given U, we want *the same* maximum distance δ between ${}_1x'$ and ${}_1x$ to guarantee that $U' \leqq U({}_1x') \leqq U''$ regardless of which is the member ${}_1x$ of the class of all programs with utility equal to U, to which the program ${}_1x'$ has a distance $\leqq \delta$.

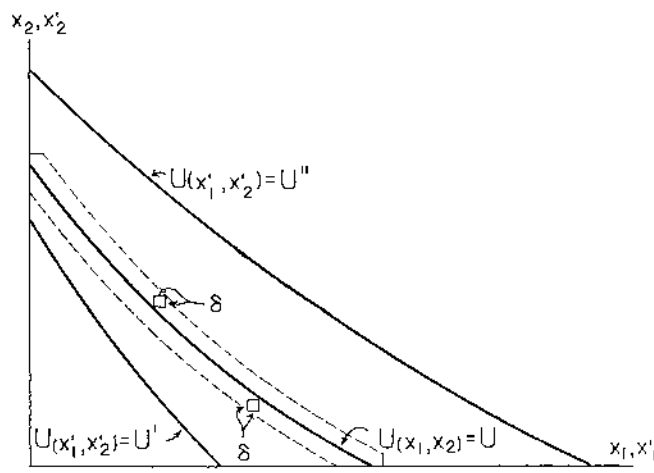

FIGURE 1

Figure 1 shows a simplified case where ${}_1x$ has only two scalar components x_1 and x_2. We then require that there be a band consisting of all points no further than δ away from some point of the indifference curve $U(x_1, x_2) = U$,

[4] If no largest $|x' - x_t|$ exists, but if there is a number exceeding $|x'_t - x_t|$ for all t, then there exists a smallest number with that property, and $\sup_t |x'_t - x_t|$ denotes that number. If no number exceeding $|x'_t - x_t|$ for all t exists, $\sup_t |x'_t - x_t| = \infty$.

which band is to fall entirely within the zone $U' \leqq U(x_1', x_2') \leqq U''$. Essentially, then, we are requiring that the utility function not be infinitely more sensitive to changes in the quantities of one program than it is to any such changes in another *equivalent* program.

3*. If we call the set $\{{}_1x \in {}_1X \mid U({}_1x) = U\}$ the *equivalence class* defined by U, then the continuity property defined by Postulate 1 may be called *uniform continuity on each equivalence class.*[5]

Since $U({}_1x)$ is continuous on a connected set ${}_1X$, the set of values assumed by $U({}_1x)$ is an interval I_U.

4. SENSITIVITY

There would not be much interest in a utility function that assumes the same value for all programs. Such a utility function would not discriminate among any alternatives. In fact, we shall need a somewhat stronger sensitivity postulate than just a statement that the utility function is not a constant. We shall require that utility can be changed by changing the consumption vector in some designated period. The use of the first period for this purpose in the following postulate is a matter of convenience, not of necessity.

POSTULATE 2. *There exist first-period consumption vectors* x_1, x_1' *and a program* ${}_2x$ *from-the-second-period-on, such that*

$$U(x_1, {}_2x) > U(x_1', {}_2x).$$

4*. The need for placing the program change for which sensitivity is postulated in a designated period can be illustrated by an example suggested by Scarf. Let there be only one commodity (hence x_t is a scalar, amount of bread, say) and consider

$$U({}_1x) \equiv \lim_{\tau \to \infty} \sup_{t \geqq \tau} x_t .$$

This function satisfies all the postulates except Postulate 2. A decision-maker guided by it has a heroic unconcern for any (upward or downward) changes in the program that affect only a finite number of periods, no matter how many. His eyes are only on the highest consumption level that is repeated or approximat-

[5] It has been pointed out to me by Debreu that the postulates of this paper do not precisely fit those of his study [3] of the existence of a utility function cited above. Since in the topology generated by the distance function (6) the space ${}_1X$ is not separable, Debreu's theorems do not apply to the present case. Neither can we say, in the topology generated by (6), that, if we specify that X is a compact set, mere continuity of $U({}_1x)$ implies the stronger continuity of Postulate 1. Both statements would become valid if the so-called *product topology* were substituted for that used here. For a definition of the product topology, see, for instance, Taylor [9, § 2.5, p. 79].

ed infinitely often, no matter how long the wait for the first occurrence of a level close to that top, or the waits between successive occurrences. Postulate 2 excludes him.

5. AGGREGATION BY PERIODS

Having rejected expenditure on consumption as a measure for the satisfaction levels reached in particular periods, we must find another means of labeling such levels. This can be done if we are willing to postulate that the particular bundle of commodities to be consumed in the first period has no effect on the preference between alternative sequences of bundles in the remaining future, and conversely. One cannot claim a high degree of realism for such a postulate, because there is no clear reason why complementarity of goods could not extend over more than one time period. It may be surmised, however, that weaker forms of this postulate would still allow similar results to be reached. The purpose of the present form is to set the simplest possible stage for a study of the effect of timing alone on preference.

POSTULATE 3 (3a *and* 3b). *For all* x_1, x_1', ${}_2x$, ${}_2x'$,

(3a) $\quad U(x_1, {}_2x) \geqq U(x_1', {}_2x)$ *implies* $U(x_1, {}_2x') \geqq U(x_1', {}_2x')$,

(3b) $\quad U(x_1, {}_2x) \geqq U(x_1, {}_2x')$ *implies* $U(x_1', {}_2x) \geqq U(x_1', {}_2x')$.

We shall show that, as a consequence of Postulate 3, the utility function can be written in the form

$$U({}_1x) = V(u_1(x_1), U_2({}_2x)) , \tag{7}$$

where $V(u_1, U_2)$ is a continuous and increasing function of its two variables u_1, U_2, and where both $u_1(x_1)$ and $U_2({}_2x)$ have the stronger continuity property attributed to $U({}_1x)$ in Postulate 1. We shall call $u_1(x_1)$ *immediate utility* or *one-period utility* (at time $t = 1$), interpreting it as a numerical indicator of the satisfaction level associated with the consumption vector x_1 in period 1. $U_2({}_2x)$ will be called *prospective utility* (as from time $t = 2$), with a similar interpretation with regard to the remaining future. Whereas this suggests calling $U({}_1x)$ prospective utility as from time 1, we shall for contrast call it *aggregate utility* (aggregated, that is, over *all* future time periods). Finally, the function $V(u_1, U_2)$, to be called the *aggregator*, indicates how any given pair of utility levels, immediate (u_1) and prospective (U_2) stacks up against any other pair in making choices for the entire future.

5*. Since x_1 and x_1' as well as ${}_2x$ and ${}_2x'$ can be interchanged in Postulate 3a, and since "$>$" means "$\geqq$ and not $\leqq$" and "$=$" means "$\geqq$ and $\leqq$," Postulate 3a implies that, for all x_1, x_1', ${}_2x$, ${}_2x'$,

$$(8>) \qquad U(x_1, {}_2x) > U(x_1', {}_2x) \quad \text{implies} \quad U(x_1, {}_2x') > U(x_1', {}_2x') ,$$

$$(8=) \qquad U(x_1, {}_2x) = U(x_1', {}_2x) \quad \text{implies} \quad U(x_1, {}_2x') = U(x_1', {}_2x') .$$

We assign to ${}_2x$ a particular value ${}_2x^0$ for which the statement made in Postulate 2 is valid, and define

$$(9) \qquad u_1(x_1) \equiv U(x_1, {}_2x^0) .$$

We then read from (8=) that

$$u_1(x_1) = u_1(x_1') \quad \text{implies} \quad U(x_1, {}_2x') = U(x_1', {}_2x') \quad \text{for all} \quad {}_2x'.$$

Again writing ${}_2x$ for ${}_2x'$, this means that

$$U(x_1, {}_2x) = F(u_1(x_1), {}_2x) .$$

Applying a similar argument to Postulate 3b and defining

$$(10) \qquad U_2({}_2x) \equiv U(x_1^0, {}_2x) ,$$

we obtain for $U({}_1x)$ the form (7). It follows from the definitions (9) and (10) that $u_1(x_1)$ and $U_2({}_2x)$ have the same continuity property as $U({}_1x)$.

Since $u_1(x_1)$ is defined on a connected set X, its continuity implies that the set of values assumed by $u_1(x_1)$ on X is an interval I_{u_1}. By Postulate 2, I_{u_1} has more than one point. By (8>) and (9) we see that $V(u_1, U_2)$ is increasing in u_1 on I_{u_1}, for all U_2. Moreover, since for any ${}_2x \in {}_1X$ the function $U(x_1, {}_2x)$ is continuous with regard to x_1 on X, the set of values assumed by $V(u_1, U_2)$ for all u_1 in I_{u_1} and any given U_2 is also an interval. Since an increasing function that assumes all values in an interval must be continuous, it follows that $V(u_1, U_2)$ is continuous with regard to u_1, for all U_2.

By similar reasoning, the set of values assumed by $U_2({}_2x)$ on ${}_1X$ is an interval I_{U_2}, and *if* I_{U_2} contains more than one point, $V(u_1, U_2)$ is increasing and continuous with regard to U_2 on I_{U_2}, for all u_1. It is easily seen that, in this case, $V(u_1, U_2)$ is continuous in (u_1, U_2) jointly on $I_{u_1} \times I_{U_2}$.

It may be anticipated here that Postulate 4 of the next section will ensure that I_{U_2} contains more than one point. To see this, let x_2, x_2', ${}_3x$ be vectors satisfying Postulate 2, hence

$$U(x_2, {}_3x) > U(x_2', {}_3x) .$$

We insert ${}_2x \equiv (x_2, {}_3x)$, ${}_2x' \equiv (x_2', {}_3x)$ in the implication,

$$U({}_2x) > U({}_2x') \text{ implies } U(x_1, {}_2x) > U(x_1, {}_2x') ,$$

of Postulate 4, and find that

$$V(u_1(x_1), U_2({}_2x)) > V(u_1(x_1), U_2({}_2x')) ,$$

which is possible only if $U_2({}_2x)$ assumes more than one value.

6. STATIONARITY

Postulate 3b says that the preference ordering within a class of programs ${}_1x$ with a common first-period consumption vector x_1 does not depend on what that vector x_1 is. We now go a step further and require that that preference

ordering be the same as the ordering of corresponding programs obtained by advancing the timing of each future consumption vector by one period (and, of course, forgetting about the common first-period vector originally stipulated). This expresses the idea that the passage of time does not have an effect on preferences.

POSTULATE 4. *For some* x_1 *and all* ${}_2x$, ${}_2x'$,

$$U(x_1, {}_2x) \geqq U(x_1, {}_2x') \text{ if and only if } U({}_2x) \geqq U({}_2x') .$$

In the light of (7) and the fact that $V(u_1, U_2)$ increases with U_2, this is equivalent to

$$U_2({}_2x) \geqq U_2({}_2x') \text{ if and only if } U({}_2x) \geqq U({}_2x') .$$

By reasoning similar to that in Section 5*, it follows that

$$U_2({}_2x) = G(U({}_2x)) ,$$

where $G(U)$ is a continuous increasing function of U. If $U = G^{-1}(U_2)$ denotes its inverse,[6] the monotonic transformation

$$U^*({}_1x) \equiv U({}_1x) , \quad u_1^*(x_1) \equiv u_1(x_1) ,$$

$$U_2^*({}_2x) \equiv G^{-1}(U_2({}_2x)) , \quad V^*(u_1^*, U_2^*) \equiv V(u_1^*, G(U_2^*({}_2x)))$$

preserves the preference ordering defined by $U({}_1x)$, and makes the functions $U_2^*({}_2x)$ and $U^*({}_2x)$ identical. We can therefore hereafter drop the time subscripts from the symbols u_1^*, $u_1^*(\ )$, U_2^*, $U_2^*(\ )$. If, now that the reasoning has been completed, we also drop all the asterisks, we have, instead of (7), the simpler relation

$$U({}_1x) = V(u(x_1), U({}_2x)) . \tag{11}$$

This relation will be the point of departure for all further reasoning. It says that the ordering of pairs of utility levels—immediate, $u(x_1)$, and prospective, $U({}_2x)$—defined by the aggregator $V(u, U)$ is such as to produce an ordering of programs for all future time, identical but for a shift in time with the ordering of programs that start with the second period. Of course, ${}_2x$ can again be substituted for ${}_1x$ in (11), giving $U({}_2x) = V(u(x_2), U({}_3x))$, and so on. The function $V(u, U)$ is again continuous and increasing in its arguments u, U.

Since both $u(x_1)$ and $U({}_2x)$ are continuous, the arguments u, U of $V(u, U)$ can take any value in an interval I_u, I_U, respectively, and the values attained by $V(u, U)$ fill the interval I_U. Since we are dealing with ordinal utility, there is still freedom to apply separate increasing transformations to $u(x_1)$ and to $U({}_2x)$, with corresponding transformations of $V(u, U)$, so as to make both I_u and I_U coincide with the unit interval extending from

[6] That is, a function such that $G(G^{-1}(U_2)) = U_2$ for all U_2.

0 to 1. The aggregator $V(u, U)$ can then be represented, though incompletely, by its niveau lines in the unit square, which are descending to the right, as shown in Figure 2.

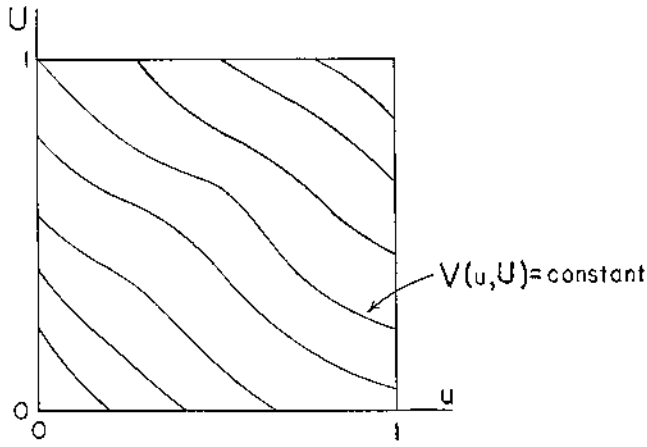

FIGURE 2

The representation is incomplete in that one still has to associate with each niveau line a numerical value of the function, which is to be referred to the vertical scale. It is also somewhat arbitrary in that separate increasing transformations of u and U that preserve the common end points 0, 1 of I_u and I_U are still permitted. The information conveyed by $V(u, U)$ is therefore as yet somewhat hidden in those interrelations between the niveau lines, the verticals, the horizontals, and the numerical niveaus themselves, which are invariant under such transformations.

6*. The question whether I_u or I_U or both include one or both end points, 0 and 1, of the unit interval, still left open by the preceding postulates, will be answered by the next postulate.

7. EXTREME PROGRAMS

In order to sidestep a mathematical complication, we shall only consider the case in which there exist a best program ${}_1\bar{x}$ and a worst program ${}_1\underline{x}$.

POSTULATE 5. *There exist* ${}_1\underline{x}$, ${}_1\bar{x}$ *such that*

$$U({}_1\underline{x}) \leqq U({}_1x) \leqq U({}_1\bar{x}) \text{ for all } {}_1x .$$

As a result of the transformations already applied, we must then have

$$U({}_1\underline{x}) = 0, \quad U({}_1\bar{x}) = 1 . \tag{12}$$

Furthermore, if ${}_1\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots)$, we must also have

$$u(\bar{x}_t) = 1 \text{ for all } t,$$

because, if we had $u(\bar{x}_\tau) < 1$ for some τ, there would exist a program $\bar{x}'$ with $u(\bar{x}'_\tau) > u(\bar{x}_\tau)$ and $\bar{x}'_t = \bar{x}_t$ for all $t \neq \tau$, which would be a better one, in view of (11) and the monotonicity of $V(u, U)$. From this and similar reasoning for the worst program ${}_1\underline{x}$ we have

$$0 = u(\underline{x}_1) \leqq u(x) \leqq u(\bar{x}_1) = 1 \quad \text{for all } x. \tag{13}$$

It follows that in the present case the intervals $I_u = I_U$ contain both end points 0, 1. Finally, if ${}_1\bar{x}$ is a best (${}_1\underline{x}$ a worst) program, it follows from (11) and the monotonicity of $V(u, U)$ that ${}_2\bar{x}$ (or ${}_2\underline{x}$) is likewise a best (worst) program. Hence, by inserting ${}_1\underline{x}$ and ${}_1\bar{x}$ succes sively into (11) and using (12) and (13), we find that

$$V(0, 0) = 0, \quad V(1, 1) = 1\,. \tag{14}$$

8. A DEFINITION OF IMPATIENCE

Now that we have succeeded in associating with each period's consumption vector x_t a utility level $u_t = u(x_t)$ *derived from the same function* $u(\ )$ *for each period*, we are in a position to define impatience as an attribute of a program ${}_1x$.

DEFINITION 1. *A program* ${}_1x$ *with first- and second-period utility levels* $u_1 \equiv u(x_1)$, $u_2 \equiv u(x_2)$ *and prospective utility* $U_3 \equiv U({}_3x)$ *from-the-third-period-on will be said to meet the impatience condition if*

$$V(u_1, V(u_2, U_3)) \left\{\begin{matrix}\geqq\\ =\\ \leqq\end{matrix}\right\} V(u_2, V(u_1, U_3)) \quad \textit{when} \quad u_1 \left\{\begin{matrix}\geqq\\ =\\ \leqq\end{matrix}\right\} u_2\,.$$

Obviously, any program with $u_1 = u_2$ meets this condition. If $u_1 > u_2$, the condition says that interchange of the first-period consumption vector x_1 with the *less desirable* second-period vector x_2 *decreases* aggregate utility. Clearly, if ${}_1x = (x_1, x_2, {}_3x)$ meets this condition with $u_1 > u_2$, then ${}_1x' \equiv (x_2, x_1, {}_3x)$ meets the condition with $u'_1 \equiv u(x_2) < u'_2 \equiv u(x_1)$.

Although impatience is here defined as an attribute of a program ${}_1x$, we shall also say that impatience prevails in the point (u_1, u_2, U_3) in a three-dimensional utility space if the above condition is met.

In Sections 9—12 we shall study some preliminary problems in order to turn in Section 13 to the main problem of finding areas in the program space (or in the utility space of u_1, u_2, U_3) where impatience prevails.

9. CORRESPONDING LEVELS OF IMMEDIATE AND PROSPECTIVE UTILITY

In this section we contrast only the first period with the remaining future. Again omitting time subscripts from the corresponding utility variables

u_1 and U_2, we shall study the question whether, if one of the two utilities, immediate (u) or prospective (U) is given, one can find for the other one a value that equates prospective and aggregate utility,

$$V(u, U) = U . \tag{15}$$

A pair (u, U) that satisfies this condition will be called a pair of *corresponding* (immediate and prospective) utility levels. One interpretation of this correspondence is that the immediate utility level u just compensates for the postponement of a program with aggregate utility U by one period. Another still simpler interpretation will be given in Section 10.

The existence of a prospective utility U corresponding to a given immediate utility u is readily established. Let u be a point of I_u. Then there exists a one-period consumption vector x such that $u(x) = u$. The aggregate utility $U(_{\text{con}}x)$ of the constant program in which x is repeated indefinitely then satisfies, by (11),

$$U(_{\text{con}}x) = V(u(x), U(_{\text{con}}x)) , \tag{16}$$

because a shift in time does not modify the program. Hence $U = U(_{\text{con}}x)$ meets the condition (15) in conjunction with the given u.

We shall now prove that for each u there is only one corresponding U, which represents a continuous increasing function

$$U = W(u), \text{ with } W(0) = 0, \quad W(1) = 1, \tag{17}$$

of u, to be called the *correspondence function*. It follows from this that, conversely, to each U there is one and only one corresponding u. Figure 3 illustrates the connection between $V(u, U)$ and $W(u)$.

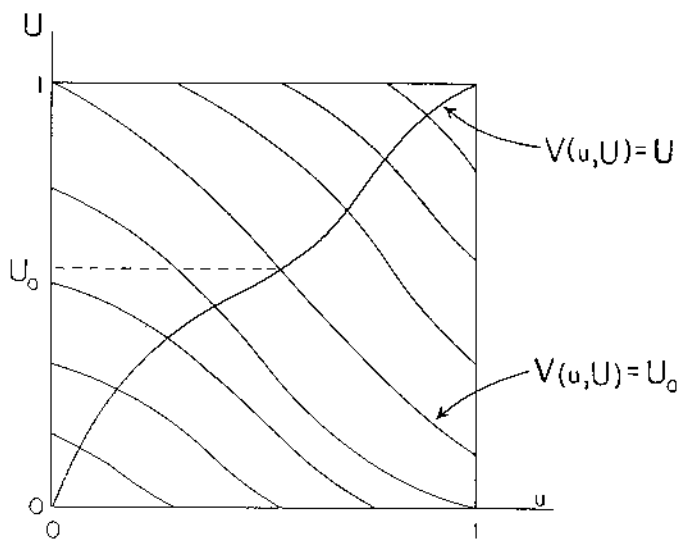

FIGURE 3

9*. We proceed by a sequence of lemmas. With a view to possible later study of the case where no best or worst program exists, Postulate 5 is not assumed in this section 9* (unless otherwise stated).

LEMMA 1a. *Let $u \in I_u$, $U \in I_U$ satisfy* (15) *with $u < 1$. Then there exists no $U' \in I_U$ such that $U' > U$ and*

$$V(u, U'') - U'' \geqq 0 \text{ for all } U'' \text{ such that } U < U'' \leqq U' .$$

PROOF. Suppose there were such a U'. There exist a vector x and a program $_1x$ such that

$$u(x) = u, \quad U(_1x) = U.$$

Since $u < 1$, and since $u(x)$ is continuous on the connected set X, we can in particular choose x in such a way that every neighborhood of x in X contains points x' with $u(x') > u$. Consider the programs

$$(18) \qquad \begin{cases} {}_1x^{(\tau)} \equiv (\overbrace{x, x, \ldots, x}^{\tau \text{ components}}, {}_1x) , \\ {}_1x'^{(\tau)} \equiv (x', x', \ldots, x', {}_1x) . \end{cases}$$

Because of (15),

$$U({}_1x^{(\tau)}) = U({}_1x^{(\tau-1)}) = \ldots = U({}_1x) = U \text{ for all } \tau.$$

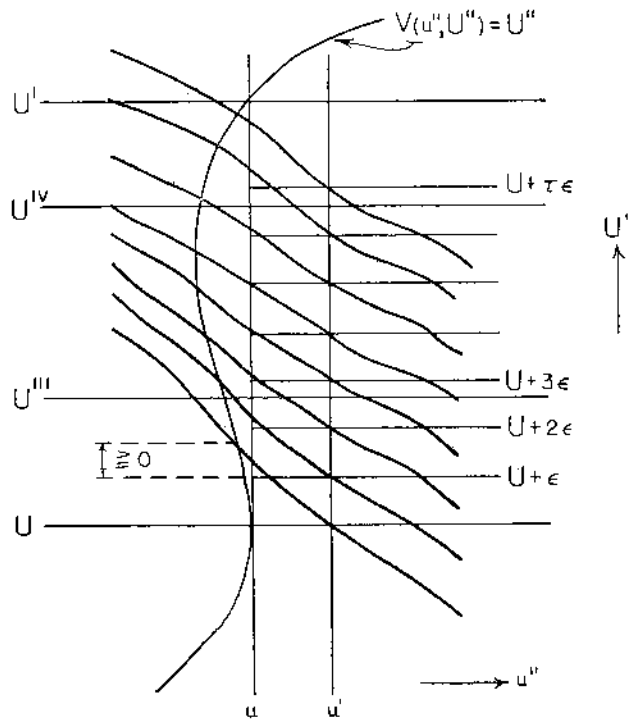

FIGURE 4

Choosing U''', U^{IV} such that $U < U''' < U^{IV} < U'$, we can therefore, because of Postulate 1, choose $\delta > 0$ such that, for all τ,

$$\sup_t |\, x_t' - x_t^{(\tau)} \,| \leqq \delta \quad \text{implies} \quad U({}_1x') \leqq U'''.$$

Choosing next x' such that $|x' - x| \leqq \delta$ and $u' \equiv u(x') > u$, we have in particular

$$U({}_1x'^{(\tau)}) \leqq U''' \text{ for all } \tau. \tag{19}$$

Since $u' > u$ the function $V(u', U'') - V(u, U'')$ is positive. As it is also continuous, we have

$$\varepsilon' \equiv \min_{U \leqq U'' \leqq U'} (V(u', U'') - V(u, U'')) > 0\,,$$

and

$$\varepsilon \equiv \min \{\varepsilon', U' - U^{IV}\} > 0\,.$$

Using, with regard to any program ${}_1x$, the notation

$$\begin{cases} {}_1u \equiv u({}_1x) \equiv (u(x_1), u(x_2), \ldots) \equiv (u_1, u_2, \ldots) \\ V_\tau({}_1u;\, U) \equiv V(u_1, V(u_2, \ldots, V(u_\tau, U) \ldots)), \end{cases} \tag{20}$$

we then have, as long as $\tau\varepsilon \leqq U' - U$, and if ${}_{\text{con}}u' \equiv (u', u', \ldots)$,

$$\begin{aligned} U({}_1x'^{(\tau)}) &= V_\tau({}_{\text{con}}u';\, U) = V_{\tau-1}({}_{\text{con}}u';\, V(u', U)) \geqq V_{\tau-1}({}_{\text{con}}u';\, V(u, U) + \varepsilon) \\ &= V_{\tau-1}({}_{\text{con}}u';\, U + \varepsilon) = V_{\tau-2}({}_{\text{con}}u';\, V(u', U + \varepsilon)) \geqq V_{\tau-2}({}_{\text{con}}u';\, V(u, U + \varepsilon) + \varepsilon) \\ &\geqq V_{\tau-2}({}_{\text{con}}u';\, U + 2\varepsilon) \geqq \ldots \geqq V(u', U + (\tau - 1)\varepsilon) \geqq U + \tau\varepsilon. \end{aligned}$$

But then we can choose τ such that $U + \tau\varepsilon \leqq U'$ but

$$U({}_1x'^{(\tau)}) \geqq U + \tau\varepsilon \geqq U^{IV}\,,$$

a contradiction of (19) which thereby proves Lemma 1. The reasoning is illustrated in Figure 4, where the locus $\{(u'', U'') \mid V(u'', U'') = U''\}$ is drawn in a manner proved impossible in Lemma 1.

Symmetrically, we have

LEMMA 1b. *Let $u \in I_u$, $U \in I_U$ satisfy* (15) *with $u > 0$. Then there exists no $U' \in I_U$ such that $U' < U$ and*

$$V(u, U'') - U'' \leqq 0 \text{ for all } U'' \text{ such that } U' \leqq U'' < U\,.$$

We can now prove, if $\bar{I}_u$ denotes the closure of I_u,

LEMMA 2. *Let $u \in I_u$, $U \in I_U$ satisfy* (15) *with $0 < u < 1$. Then*

(21) $V(u', U') - U' < 0$ *for all $u' \in \bar{I}_u$, $U' \in I_U$ with $u' \leqq u$, $U' \geqq U$, except $(u', U') = (u, U)$.*

(22) $V(u', U') - U' > 0$ *for all $u' \in \bar{I}_u$, $U' \in I_U$ with $u' \geqq u$, $U' \leqq U$, except $(u', U') = (u, U)$.*

PROOF (see Figure 5). We first prove (21) with $u' = u$ by considering its negation. This says that there exists $U'' \in I_U$ with $U'' > U$ such that $V(u, U'') - U'' \geqq 0$. But this implies by Lemma 1a that there exists U''' with $U < U''' < U''$ such that $V(u, U''') - U''' < 0$, and by the continuity of $V(u, U') - U'$ with

respect to U' that there exists a U^{IV} with $U''' < U^{IV} \leqq U''$ such that $V(u, U^{IV}) - U^{IV} = 0$ and $V(u, U') - U' < 0$ for $U''' \leqq U' < U^{IV}$. Inserting U^{IV} for U and U''' for U' in Lemma 1b we find these statements in contradiction with Lemma 1b. This proves (21) with $u' = u$. The remaining cases with $u' < u$, $U' \geqq U$ follow from the increasing property of $V(u', U')$ with respect to u'. The proof of (22) is symmetric to that of (21).

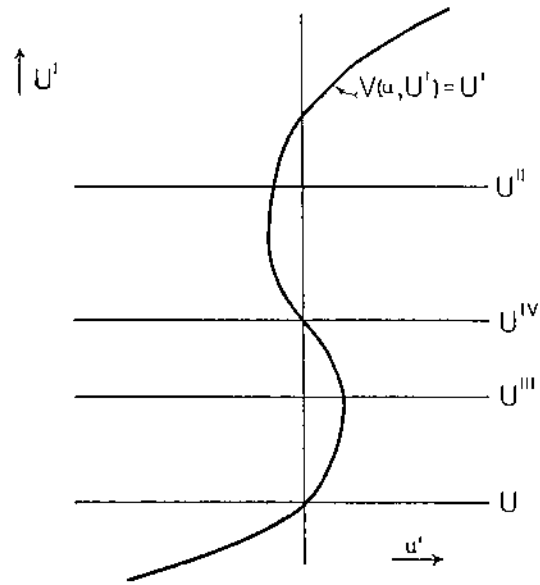

FIGURE 5

Since we know already that there exists for each $u \in I_u$ at least one corresponding U, it follows from Lemma 2 that if $0 < u < 1$ there exists precisely one, to be denoted $W(u)$, and that $W(u)$ increases with u. Moreover, if for $0 < u < 1$ we had

$$W(u) < \lim_{u' \to u+0} W(u') \equiv W(u + 0)$$

the continuity of $V(u, U)$ would entail the existence of two different prospective utility levels, $W(u)$ and $W(u + 0)$, corresponding to the immediate utility level u, contrary to Lemma 2. Hence $W(u)$ is continuous for $0 < u < 1$, and, since $0 \leqq W(u) \leqq 1$, can be extended by

$$W(0) \equiv \lim_{u \to 0} W(u), \quad W(1) \equiv \lim_{u \to 1} W(u)$$

so as to make $W(u)$ continuous and increasing for $0 \leqq u \leqq 1$.

Now if $0 \in I_U$ and hence $0 \in I_u$, we must have $W(0) = 0$, because $W(0) > 0$ would create a contradiction between (14) and Lemma 1a (with 0 substituted for U, and $W(0)$ for U'), since $V(0, U'') - U'' < 0$ for any U'' such that $0 < U'' \leqq W(0)$ is precluded by Lemma 2 and the continuity of $V(u, U'')$ with respect to u. Similar reasoning for the case $1 \in I_u$ completes the proof of (17).

10. EQUIVALENT CONSTANT PROGRAM

Now that the correspondence of utility levels u, U has been shown to be one-to-one and reversible, another interpretation is available. Given an aggregate utility level U, find the corresponding immediate utility u, and a one-period consumption vector x for which it is attained, $u(x) = u$. Then we can reinterpret (16) to mean that the program ${}_{\text{con}}x$ obtained by indefinite repetition of the vector x again has the given aggregate utility $U({}_{\text{con}}x) = U$. The correspondence (17) therefore gives us a means to associate with any program a constant program of the same aggregate utility.

10*. If Postulate 5 is not assumed, the possibility exists of a program ${}_{1}x$ with successive one-period utility levels $u(x_t)$ increasing (or decreasing) with t in such a way that no equivalent constant program and no compensation for a postponement of ${}_{1}x$ by one period exist.

11. EQUATING CORRESPONDING UTILITY LEVELS

The correspondence function $W(u)$ can be used to change the scale of one of the two utility types, for instance of u, in such a way as to equate corresponding utility levels. The appropriate increasing transformation is defined by

$$\begin{gathered} u^*(x) \equiv W(u(x)), \quad U^*({}_{1}x) \equiv U({}_{1}x), \\ V^*(u^*, U^*) \equiv V(W^{-1}(u^*), U^*), \end{gathered} \tag{23}$$

where $u = W^{-1}(u^*)$ is the inverse of $u^* = W(u)$. If now u^* and U^* represent corresponding utility levels on the new scales, we have

$$0 = V^*(u^*, U^*) - U^* = V(W^{-1}(u^*), U) - U,$$

and hence, by the definition of $W(u)$,

$$U^* = U = W(W^{-1}(u^*)) = u^*.$$

Hence the new correspondence function $U^* = W^*(u^*)$ is simply the identity $U^* = u^*$, represented in the new form of Figure 3 by the diagonal connecting (0,0) with (1,1). Although this change of scale is not essential for any of the reasoning that follows, we shall make it in order to simplify formulae and diagrams. Dropping asterisks again, the correspondence relation (15) now takes the form

$$V(U, U) = U. \tag{24}$$

12. REPEATING PROGRAMS

A program in which a given sequence ${}_{1}x_\tau$ of τ one-period vectors $x_1, x_2, \ldots, x_\tau$ is repeated indefinitely will be called a *repeating program*, to be denoted

$${}_{\text{rep}}x_\tau \equiv ({}_{1}x_\tau, {}_{1}x_\tau, \ldots).$$

The sequence ${}_1x_\tau$ will be called the *pattern* of the repeating program, τ its *span*, provided no $\tau' < \tau$ exists permitting the same form. We shall use the notation

$$\begin{gathered} {}_{\text{rep}}u_\tau \equiv ({}_1u_\tau, {}_1u_\tau, \ldots), \\ {}_1u_\tau \equiv u_\tau\,({}_1x_\tau) \equiv (u(x_1), \ldots, u(x_\tau)) \equiv (u_1, \ldots, u_\tau) \end{gathered}$$

for the corresponding sequences of one-period utility levels, and call ${}_1u_\tau$ the *utility pattern* corresponding to ${}_1x_\tau$. The function

$$V_\tau({}_1u_\tau;\, U) \equiv V(u_1, V(u_2, \ldots, V(u_\tau, U) \ldots)) \tag{25}$$

then indicates how the utility level U of any program is modified if that program is postponed by τ periods and a pattern with the corresponding utility pattern ${}_1u_\tau$ is inserted to precede it.

Given a utility pattern ${}_1u_\tau = u_\tau({}_1x_\tau)$, we can now ask whether there is a utility level U which is not affected by such a postponement,

$$V_\tau({}_1u_\tau;\, U) = U\,. \tag{26}$$

Obviously, the utility level

$$U = U({}_{\text{rep}}x_\tau) \tag{27}$$

meets this requirement, because the program ${}_{\text{rep}}x_\tau$ itself is not modified by such postponement. By an analysis entirely analogous to that already given for the case $\tau = 1$, one can show that this utility level is unique and hence is a function

$$U = W_\tau({}_1u_\tau) \tag{28}$$

of the utility pattern. This function is a *generalized correspondence function*. One can interpret it either as the aggregate utility of any program, the postponement of which by τ periods can just be compensated by insertion of a sequence ${}_1x_\tau$ with $u_\tau({}_1x_\tau) = {}_1u_\tau$, or as the aggregate utility of the repeating program ${}_{\text{rep}}({}_1x_\tau)$, where again $u_\tau({}_1x_\tau) = {}_1u_\tau$. As before, one can show that $W({}_1u_\tau)$ is continuous and increasing with respect to each of the variables $u_1, \ldots, u_\tau$. Finally, as before in the case $\tau = 1$,

$$U \begin{Bmatrix} \leq \\ = \\ > \end{Bmatrix} V_\tau({}_1u_\tau;\, U) \begin{Bmatrix} \leq \\ = \\ > \end{Bmatrix} W_\tau({}_1u_\tau) \text{ if } U \begin{Bmatrix} \leq \\ = \\ > \end{Bmatrix} W_\tau({}_1u_\tau). \tag{29}$$

12*. The uniqueness of the solution of (26) and the first set of inequalities in (29) are proved by having an arbitrary one of the variables $u_1, \ldots, u_\tau$ play the role performed by u in Section 9*. To prove continuity and monotonicity of $W_\tau({}_1u_\tau)$, that role is assigned successively to each of these variables. The second set of inequalities in (29) then follows from (26), (28) and the fact that $V_\tau(u_\tau;\, U)$ increases with U.

To obtain one further interesting result we revert to the notation (20). By repeated application of (29) we have, for $n = 1, 2, \ldots$,

$$\begin{gathered} U'' < U = W_\tau({}_1u_\tau) < U' \text{ implies} \\ V_{n\tau}({}_{\text{rep}}u_\tau;\, U'') < V_{n\tau}({}_{\text{rep}}u_\tau;\, U) = U < V_{n\tau}({}_{\text{rep}}u_\tau;\, U'), \end{gathered} \tag{30}$$

where $V_{n\tau}(_{\text{rep}}u_\tau,\ U''')$ is increasing with n if $U''' < U$, decreasing if $U''' > U$. It follows that

$$\lim_{n\to\infty} V_{n\tau}(_{\text{rep}}u_\tau;\ U''') \tag{31}$$

exists for all $U''' \in I_U$. But for any such U''' insertion of (31) for U in (26) satisfies that condition, which we know to be satisfied by U only. Hence, by (28),

$$\lim_{n\to\infty} V_{n\tau}(_{\text{rep}}u_\tau;\ U''') \equiv V_\infty(_{\text{rep}}u_\tau) = W_\tau(_1u_\tau) \quad \text{for all} \quad U''' \in I_U. \tag{32}$$

13. ALTERNATING PROGRAMS AND IMPATIENCE

A repeating program with a span $\tau = 2$ will be called an *alternating program*. Its one-period utility sequence alternates between two different levels, u' and u'', say, which we shall always choose such that

$$u' > u''. \tag{33}$$

If we write $w' \equiv (u', u'')$, $w'' \equiv (u'', u')$ for the two possible utility patterns, the two possible alternating programs have the respective utility sequences

$$\begin{cases} (34') & {}_{\text{rep}}w' \equiv (u', u'', u', u'', \ldots), \\ (34'') & {}_{\text{rep}}w'' \equiv (u'', u', u'', u', \ldots). \end{cases} \tag{34}$$

The implications of the preceding analysis for this type of program are illustrated in Figure 6. The aggregate utility level U' corresponding to (34'),

$$U' \equiv W_2(w'), \tag{35}$$

satisfies the condition

$$\Phi'(U') \equiv V(u', V(u'', U')) - U' = 0. \tag{36}$$

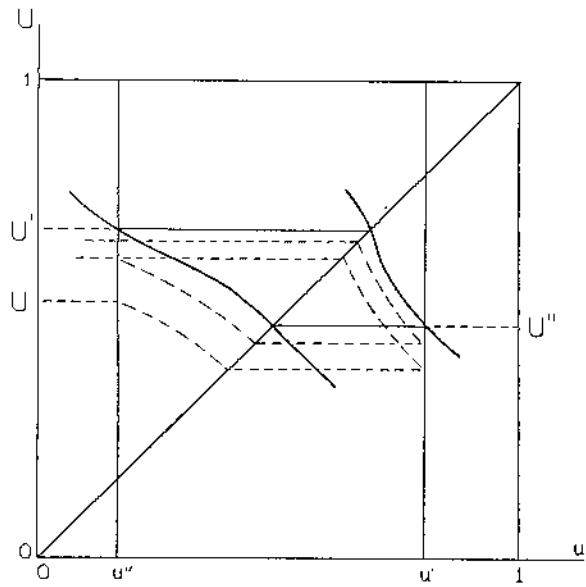

FIGURE 6

Hence U' can be read off, as indicated in Figure 6, from a quadrilateral consisting of two horizontals and two niveau lines (drawn solid), with two vertices on the diagonal of the unit square, the other two vertices on the verticals at $u = u'$ and $u = u''$, respectively. Enlarging on (36), we also have from (29)

$$0 \left\{ \begin{matrix} \leqq \\ > \end{matrix} \right\} \Phi'(U) \equiv V(u', V(u'', U)) - U \left\{ \begin{matrix} \leqq \\ > \end{matrix} \right\} U' - U \text{ if } U \left\{ \begin{matrix} \geqq \\ < \end{matrix} \right\} U' . \tag{37}$$

Hence, for any program with an aggregate utility $U \neq U'$, postponement by two periods with insertion of the utility pattern (u', u'') in the first two periods thereby vacated will bring the aggregate utility closer to U', without overshooting. By (32), indefinite repetition of this operation will make the aggregate utility approach U' as a limit (see dotted lines for a case with $U < U'$). Symmetrically to (37), we have

$$0 \left\{ \begin{matrix} \leqq \\ > \end{matrix} \right\} \Phi''(U) \equiv V(u'', V(u', U)) - U \left\{ \begin{matrix} \leqq \\ > \end{matrix} \right\} U'' - U \text{ if } U \left\{ \begin{matrix} \leqq \\ > \end{matrix} \right\} U'' , \tag{38}$$

with similar interpretations, and where U'' is related to U', u'' and u' by

$$u'' < U'' = V(u'', U') < U' = V(u', U'') < u' , \tag{39}$$

as indicated in Figure 6, and proved in detail below.

We are now ready to draw inferences about the presence of impatience in certain parts of the utility space. The functions $\Phi'(U)$ and $\Phi''(U)$ introduced in (37) and (38) are related to the criterion of impatience by

$$\Phi(U) \equiv \Phi'(U) - \Phi''(U) = V(u', V(u'', U)) - V(u'', V(u', U)) . \tag{40}$$

Since $u' > u''$, impatience is present whenever $\Phi(U) > 0$. Reference to (37) and (38), or to Figure 7 in which the implications of (37) and (38)

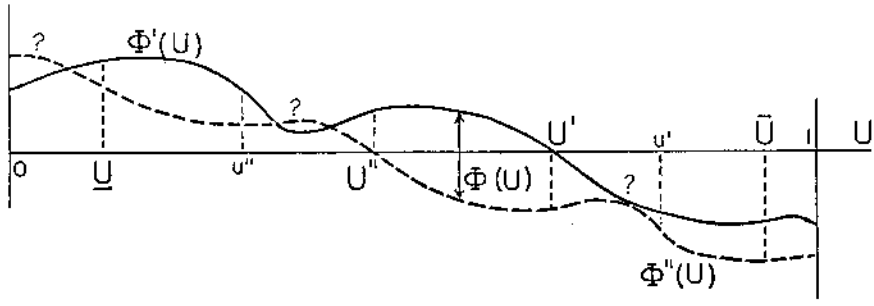

FIGURE 7

are exhibited, shows that, since $\Phi'(U) > 0$ for $0 \leqq U < U'$ and $\Phi''(U) < 0$ for $U'' < U \leqq 1$, we have

$$\Phi(U) > 0 \text{ for } U'' \leqq U \leqq U'. \tag{41}$$

This proves the presence of impatience in a central zone of the space of the

utility triples (u', u'', U), as illustrated in Figure 8. It is to be noted that the result (41) is obtained as long as the two marked points do not fall on the same side of the horizontal at U. This is the case precisely if $U'' \leqq U \leqq U'$.

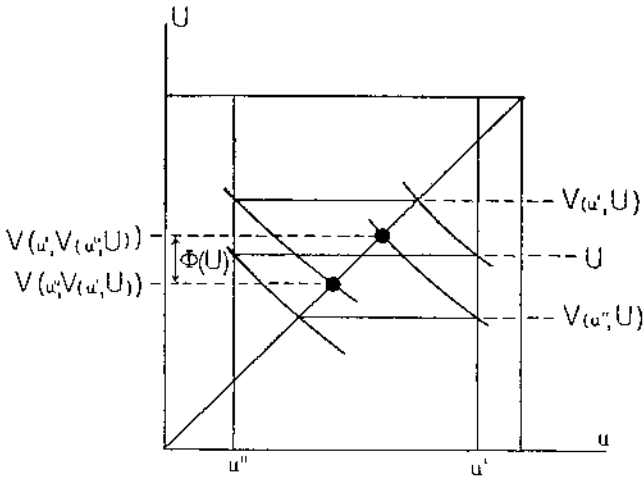

FIGURE 8

Two other zones can be added to this one, on the basis of the monotonicity of $V(u, U)$ with respect to U. If we define $\underline{U}$, $\bar{U}$ by

(42) $$V(u', \underline{U}) = u'', \quad V(u'', \bar{U}) = u',$$

if solutions of these equations exist, and by $\underline{U} = 0$, and/or $\bar{U} = 1$ otherwise, Figure 9 suggests that

(43) $$\Phi(U) > 0 \text{ for } \underline{U} \leqq U \leqq u'' \text{ and for } u' \leqq U \leqq \bar{U}.$$

A detailed proof is given below.

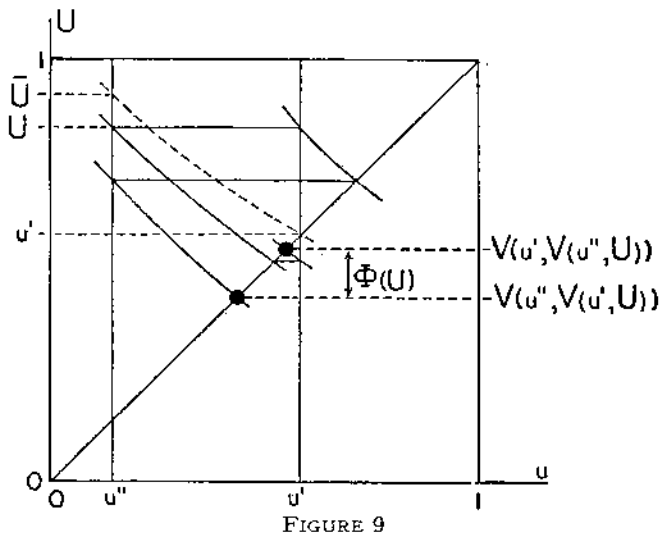

FIGURE 9

There are indications that in the intermediate zones, $u'' < U < U''$ and $U' < U < u'$, impatience is the general rule, neutrality toward timing a conceivable exception. The behavior of $\Phi(U)$ in these zones will not be analyzed further in this paper, in the hope that an argument simpler than that which has furnished these indications may still be found.

For the sake of generality of expression, we shall state the present results in a form that does not presuppose the, convenient but inessential, transformation introduced in Section 11 to equate corresponding utility levels.

THEOREM 1. *If Postulates* 1, 2, 3, 4, *and* 5 *are satisfied, a program* ${}_1x$ *with first- and second-period utilities* $u_1 = u(x_1)$ *and* $u_2 = u(x_2)$ *such that* $u_1 > u_2$ *and with prospective utility as-from-the-third-period* $U_3 = U({}_3x)$ *meets the condition* (40) *of impatience in each of the following three zones:*

(*a*) *If* U_3 *equals or exceeds the utility of a constant program indefinitely repeating the vector* x_1, *provided* U_3 *is not so high* (*if that should be possible*) *that the utility of the program* $(x_2, {}_3x)$ *exceeds that of the constant program* $(x_1, x_1, x_1, \ldots)$;

(*b*) *If* U_3 *equals the utility of either of the alternating programs*

$$(x_1, x_2, x_1, x_2, \ldots)$$
$$(x_2, x_1, x_2, x_1, \ldots)$$

or falls between these two utility levels;

(*c*) *If* U_3 *equals or falls below the utility of the constant program* $(x_2, x_2, x_2, \ldots)$, *provided* U_3 *is not so low* (*if that should be possible*) *that the utility of the program* $(x_1, {}_3x)$ *falls below that of the constant program* $(x_2, x_2, x_2, \ldots)$.

This is, in a way, a surprising result. The phenomenon of impatience was introduced by Böhm-Bawerk as a psychological characteristic of human economic preference in decisions concerning (presumably) a *finite* time horizon. It now appears that impatience, at least in one central and two outlying zones of the space of programs, is also a necessary logical consequence of more elementary properties of a utility function of programs with an *infinite* time horizon: continuity (uniform on each equivalence class), sensitivity, aggregation by periods, independence of calendar time (stationarity), and the existence of extreme programs.

13*. PROOF. In order to prove relations (39) and (43) on which Theorem 1 depends, without reference to a diagram, we lift from the already proved statements (37) and (38) the defining relations

(44'') and (44') $\qquad V(u'', V(u', U'')) = U'', \quad V(u', V(u'', U')) = U',$

of U'' and U', respectively. From (44') we read that $V(u'', V(u', V(u'', U'))) = V(u'', U')$, showing that $V(u'', U')$ satisfies the defining relation (44'') of U''.

This, and an argument symmetric to it, establish the equalities in (39). Now assume first that $U'' < U'$. In that case, because $V(u, U)$ increases with U,

$$0 = V(u', U'') - U' < V(u', U') - U' ,$$

whence $U' < u'$ by Lemma 2, since $V(u', u') - u' = 0$. By similar reasoning, $U'' > u''$, establishing the inequalities in (39) for the present case. But the same reasoning applied to the assumption $U'' \geqq U'$ would entail $u'' \geqq U'' \geqq U' \geqq u'$, which is contradicted by the datum that $u' > u''$. This completes the proof of (39).

To prove (43) we note that, given u', u'' with $u' > u''$,

$$\text{if } \begin{Bmatrix} U = u' \\ u' < U < \bar{U} \\ U = \bar{U} \end{Bmatrix} \text{ then } V(u', U) \begin{Bmatrix} = \\ < \\ < \end{Bmatrix} U, \text{ and } V(u'', V(u', U)) \begin{Bmatrix} = \\ < \\ < \end{Bmatrix} V(u'', U) \begin{Bmatrix} < \\ < \\ = \end{Bmatrix} u' ,$$

using in succession (24), Lemma 2, the monotonicity of $V(u, U)$ with respect to U, and (42). But then also

$$V(u', V(u'', U)) \begin{Bmatrix} > \\ > \\ = \end{Bmatrix} V(u'', U) ,$$

using again (24) and Lemma 2. A comparison of these results establishes (43).

The forms here given to the proofs of (39) and (43) have been chosen so that they may carry over by mere reinterpretation to a more general case to be considered in a later paper.

14. PERIOD INDEPENDENCE

It might seem only a small additional step if to Postulate 3 we add[7]

POSTULATE 3′ (3′a *and* 3′b). *For all* $x_1, x_2, {}_3x, x_1', x_2', {}_3x'$,

(3′a) $U(x_1, x_2, {}_3x) \geqq U(x_1', x_2', {}_3x)$ *implies* $U(x_1, x_2, {}_3x') \geqq U(x_1', x_2', {}_3x')$,

(3′b) $U(x_1, x_2, {}_3x) \geqq U(x_1', x_2, {}_3x')$ *implies* $U(x_1, x_2', {}_3x) \geqq U(x_1', x_2', {}_3x')$.

In fact, it follows from a result of Debreu [2], that this would have quite drastic implications. Postulates 1—5 and 3′ together satisfy the premises of a theorem[8] which, translated in our notation and terminology, says that one can find a monotonic transformation of $U({}_1x)$ such that

$$U({}_1x) = u_1(x_1) + u_2(x_2) + U_3({}_3x) . \tag{46}$$

Taken in combination with the stationarity Postulate 4, this would leave only the possibility that

$$U({}_1x) = \sum_{t=1}^{\infty} a^{t-1} u(x_t), \quad 0 < a < 1 , \tag{47}$$

[7] A postulate very similar to Postulate 3′ is contained in an unpublished memorandum, kindly made available to me by Robert Strotz in 1958.

[8] l.c., Section 3.

that is, aggregate utility is a discounted sum of all future one-period utilities, with a constant discount factor α. This form has been used extensively in the literature.[9] Since the form (47) is destroyed by any other transformations than increasing linear ones, one can look on Postulate 3' (as Debreu does) as a basis (in conjunction with the other postulates) for defining a *cardinal* utility function (47). While this in itself is not objectionable, the constant discount rate seems too rigid to describe important aspects of choice over time. If for the sake of argument we assume that the aggregator function $V(u, U)$ is differentiable, it is shown below that the *discount factor*

$$\left(\frac{\partial V(u, U)}{\partial U}\right)_{U=u} \tag{48}$$

is invariant for differentiable monotonic transformations. Obviously, it can take different values for different common values of $U = u$. The main purpose of the system of postulates of this paper therefore is to clarify behavior assumptions that will permit the relative weight given to the future as against the present to vary with the level of all-over satisfaction attained —a consideration which can already be found in the work of Irving Fisher [4].

14*. To prove the invariance of (48), we observe that the increasing transformations of V, u, U that preserve (24) are of the type

$$u^*(x_1) \equiv f(u(x_1)), \quad U^*(x) \equiv f(U(x)), \quad f(0) = 0, \quad f(1) = 1,$$
$$V^*(u^*, U^*) = f(V(f^{-1}(u^*), f^{-1}(U^*))).$$

But then, for so related values of u^*, U^*, u, U,

$$\frac{\partial V^*(u^*, U^*)}{\partial U^*} = \left(\frac{df(U')}{dU'}\right)_{U'=V(u,U)} \cdot \frac{\partial V(u, U)}{\partial U} \cdot \left(\frac{df^{-1}(U^*)}{dU^*}\right)_{U^*=f(U)}.$$

If $u = U$, then, $U' = U$, and the first and third factors of the right hand member are reciprocals, hence cancel.

It should finally be noted that Postulates 3'a and 3'b are not counterparts to each other in the way in which Postulates 3a and 3b are counterparts. The respective counterparts, in that sense, to Postulates 3'a and 3'b are implied in Postulates 1—5, and hence do not need restatement.

Cowles Foundation for Research in Economics at Yale University

[9] See, for instance, Ramsay [6], Samuelson and Solow [7], Strotz [8]. The first two publications find a way to make $\alpha = 1$.

REFERENCES

[1] BÖHM-BAWERK, E. VON: *Positive Theorie des Kapitals*, Dritte Auflage, 1912, especially Buch IV, Abschnitt I, Gegenwart und Zukunft in der Wirtschaft, pp. 426–486, English Translation in *Capital and Interest*, Vol. II, *Positive Theory of Capital*, Book IV, Section I, pp. 257-289, South Holland, Illinois, 1959.

[2] DEBREU, G.: "Topological Methods in Cardinal Utility Theory," Cowles Foundation Discussion Paper No. 76, to appear in *Mathematical Methods in the Social Sciences*, Stanford: Stanford University Press, 1960.

[3] ———: "Representation of a Preference Ordering by a Numerical Function," Chapter XI of Thrall, Coombs and Davis, eds., *Decision Processes*, New York: Wiley, 1954, pp. 159–165.

[4] FISHER, IRVING: *The Theory of Interest*, New York: Macmillan, 1930, Chapter IV, especially §§ 3 and 6.

[5] FRISCH, R.: *Sur un problème d'économie pure*, Norsk Matematisk Forenings Skrifter, Serie I, No. 16, Oslo, 1926, 40 pp.

[6] RAMSAY, F. P.: "A Mathematical Theory of Saving," *Economic Journal*, December, 1928, pp. 543–559.

[7] SAMUELSON, P. A., AND SOLOW, R.: "A Complete Capital Model Involving Heterogeneous Capital Goods," *Quarterly Journal of Economics*, November, 1956, pp. 537–562.

[8] STROTZ, R. H.: "Myopia and Inconsistency in Dynamic Utility Maximization," *Review of Economic Studies*, XXIII, 3 (1957), pp. 165–180.

[9] TAYLOR, A. E.: *Functional Analysis*, New York, 1958.

[10] WOLD, H.: "A Synthesis of Pure Demand Analysis, Part II," *Skandinavisk Aktuaritidskrift*, 26, 1943, pp. 220–263.