Harald Lang

Comments on difference equations

1. Backward and forward solutions to differences

Let A_t be an unknown sequence satisfying

$$A_t - A_{t-1} = q_t \tag{1.1}$$

where q_t is a known sequence. Replace t by k in (1.1) and add from 1 to t :

$$\sum_1^t (A_k - A_{k-1}) = \sum_1^t q_k$$

The first sum is "telescoping"—all A_k :s except the first and the last cancel—hence

$$A_t - A_0 = \sum_1^t q_k$$

i.e.,

$$A_t = A + \sum_1^t q_k \tag{1.2}$$

where $A = A_0$ is an arbitrary constant. This is true also for $t \leq 0$ with the summation convention

$$\sum_1^t q_k = \begin{cases} 0 & \text{if } t = 0 \\ -q_0 - q_{-1} - \cdots - q_{-t+1} & \text{if } t < 0 \end{cases}$$

In many applications in economics one prefers another representation of the solution. *If* the sum $\sum_{-\infty}^0 q_k$ converges, then we can replace A by the new constant $A + \sum_{-\infty}^0 q_k$ in (1.2) to get

$$A_t = A + \sum_{-\infty}^t q_k \tag{1.3}$$

The solution (1.3) is the *backward* solution to (1.1). In a similar manner, we can define the *forward* solution to (1.1) as

$$A_t = A - \sum_{t+1}^{\infty} q_k \tag{1.4}$$

which is well defined if the sum $\sum_0^\infty q_k$ converges; again A is an arbitrary constant. Indeed, replacing A by $A - \sum_0^\infty q_k$ in (1.1) yields (1.4). In summary:

The following expressions are all solutions to equation (1.1), conditional on the infinite sums be convergent:

$$A_t = A + \sum_{1}^{t} q_k$$

$$A_t = A + \sum_{-\infty}^{t} q_k \quad \text{“backward solution”}$$

$$A_t = A - \sum_{t+1}^{\infty} q_k \quad \text{“forward solution”}$$

2. First order difference equations

Consider the first order difference equation

$$x_t = a x_{t-1} + h_t \tag{2.1}$$

where x_t is an unknown sequence, $a \neq 0$ a constant and h_t a known sequence. Define a new sequence A_t by

$$x_t = a^t A_t$$

Inserting into (2.1) gives, after some simplification,

$$A_t - A_{t-1} = a^{-t} h_t \tag{2.2}$$

which is an equation of the type studied in the previous section.

In particular, assume that $h_t = h$ for all t, i.e., h_t is a constant independent of t. If $a \neq 1$, we can solve (2.2) by summing a geometric series:

$$A_t = A + \sum_{1}^{t} a^{-k} h = A + \frac{a^{-1} - a^{-t-1}}{1 - a^{-1}} h = A + \frac{1 - a^{-t}}{a - 1} h$$

i.e.,

$$x_t = A_t a^t = \left(A + \frac{h}{a-1} \right) a^t + \frac{h}{1-a} = B a^t + \frac{h}{1-a}$$

where B is an arbitrary constant.

3. An example: Dynamic and intertemporal budget constraints

Let w_t be an individual's wage income in period t, $t = 0, 1, \ldots$, c_t her consumption and a_t her net assets (savings minus debt). Her *dynamic* budget constraint is then

$$a_t = a_{t-1} + ra_{t-1} + w_t - c_t$$

where r is the interest rate. We write this as

$$a_t = Ra_t + w_t - c_t$$

where $R \equiv 1 + r > 1$. The solution is $a_t = A_t R^t$, where

$$A_t - A_{t-1} = R^{-t}(w_t - c_t)$$

We solve this forward:

$$A_t = A - \sum_{k=t+1}^{\infty} R^{-k}(w_k - c_k)$$

which implies

$$a_t R^{-t} = A - \sum_{k=t+1}^{\infty} R^{-k}(w_k - c_k) \tag{3.1}$$

We assume that the individual is not permitted to let her debt grow indefinitely at a rate R or more: $\lim_{t\to\infty} a_t R^{-t} \geq 0$; this is called a *no Ponzi-game condition.* Nor, do we assume, does the individual want to for ever build up a positive wealth, at a rate R or more, which is never consumed: $\lim_{t\to\infty} a_t R^{-t} \leq 0$; hence $\lim_{t\to\infty} a_t R^{-t} = 0$. Letting $t \to \infty$ in (3.1) thus gives $A = 0$, i.e.,

$$a_t = -\sum_{k=t+1}^{\infty} R^{t-k}(w_k - c_k)$$

or

$$\sum_{k=t+1}^{\infty} R^{t-k} c_k = a_t + \sum_{k=t+1}^{\infty} R^{t-k} w_k \tag{3.2}$$

or in words: at any point in time, the present value of consumption equals the present value of future wage incomes plus the current net wealth. The constraint (3.2) is the individual's *intertemporal* budget constraint.

4. CAPM; another example

You will eventually read more about this in the macro course; Blanchard and Fischer's book chapter 6.

Let V_t be the value of a firm as of time t, and let π_t be its cash flow. $u(c_t)$ is the utility of consuming c_t for a representative consumer, and θ is the subjective discount rate, i.e., the total utility of as of now of consuming c_t this period and c_{t+1} next period is $u(c_t) + (1+\theta)^{-1}u(c_{t+1})$. The value of the firm then satisfies

$$(V_t + \pi_t)u'(c_t)(1+\theta)^{-1} = V_{t-1}u'(c_{t-1}) \tag{4.1}$$

To prove this, consider a consumer who has optimized her consumption path, and who considers buying a small portion δ of the firm this period $t-1$, refraining from consumption of equal value, and then selling it off in the next period t in order to increase her consumption then. The utility gain would be

$$-\delta V_{t-1}u'(c_{t-1}) + (1+\theta)^{-1}\delta(V_t + \pi_t)u'(c_t)$$

Obviously this must be ≤ 0; but this must be true also if δ is negative—she can sell of some today, etc.—, hence the expression above must be $= 0$, which proves (4.1).

Introduce the notation $z_t = u'(c_t)V_t$. We get

$$z_t = (1+\theta)z_{t-1} - \pi_t u'(c_t)$$

The general solution is $z_t = A_t(1+\theta)^t$ where A_t is a solution to

$$A_t - A_{t-1} = -(1+\theta)^{-t}\pi_t u'(c_t)$$

We solve this equation forward:

$$A_t = A + \sum_{k=t+1}^{\infty}(1+\theta)^{-k}\pi_k u'(c_k)$$

i.e.,

$$z_t = A_t(1+\theta)^t = A(1+\theta)^t + \sum_{k=t+1}^{\infty}(1+\theta)^{t-k}\pi_k u'(c_k)$$

and finally

$$V_t = \frac{z_t}{u'(c_t)} = \frac{A}{c'(c_t)}(1+\theta)^t + \sum_{k=t+1}^{\infty}(1+\theta)^{t-k}\frac{u'(c_k)}{c'(c_t)}\pi_k$$

The first term in the RHS is considered a *"bubble"* and is (rightly?) assumed zero (i.e., the constant $A = 0$,) furthermore the future is usually uncertain, so the final formula contains expectations:

$$V_t = \sum_{k=t+1}^{\infty}(1+\theta)^{t-k}E_t\left[\frac{u'(c_k)}{c'(c_t)}\pi_k\right]$$

5. Particular solution to second order difference equations

In Sydsæter and Hammond's book, it is shown how to find the general solution to the homogeneous difference equation

$$x_{t+2} + ax_{t+1} + bx_t = 0 \tag{5.1}$$

i.e., we can construct a solution x_t with arbitrary initial conditions $x_0 = d_0$, $x_1 = d_1$.

Let z_t be the solution to (5.1) which satisfies $z_0 = 0$, $z_1 = 1$. We can now construct a particular solution u_t^* to the *inhomogeneous* equation

$$x_{t+2} + ax_{t+1} + bx_t = c_t \tag{5.2}$$

A particular solution u_t^* to (5.2) is given by

$$u_t^* = \sum_{k=1}^{t} z_{t-k} c_{k-1}$$

This solution satisfies $u_0^* = u_1^* = 0$.

Proof:

$$\begin{aligned} u_{t+2}^* + au_{t+1}^* + bu_t^* &= \sum_{k=1}^{t+2} z_{t+2-k} c_{k-1} + a \sum_{k=1}^{t+1} z_{t+1-k} c_{k-1} + b \sum_{k=1}^{t} z_{t-k} c_{k-1} \\ &= \sum_{k=1}^{t} (z_{t+2-k} + az_{t+1-k} + bz_{t-k}) c_{k-1} + z_0 c_{t+1} + z_1 c_t + az_0 c_t \\ &= \ldots \end{aligned}$$

here the first sum vanishes, since $z_{t+2-k} + az_{t+1-k} + bz_{t-k} = 0$, since z_t is a solution to the homogeneous equation (5.1). hence we can continue:

$$\begin{aligned} \ldots &= 0c_{t+1} + 1c_t + a\,0\,c_t \\ &= c_t \end{aligned}$$

which proves that u_t^* is indeed a particular solution to (5.2). *Q.E.D.*

Also in this case we can use backward and forward solutions if appropriate for the application.

Example: solve the equation $x_{t+2} - 3.5x_{t+1} + 1.5x_t = c_t$, where c_t is some bounded sequence.

The roots of the characteristic equation is $r_1 = 3$ and $r_2 = 0.5$. The general solution to the corresponding homogeneous equation is thus $x_t^h = A\,3^t + B\,0.5^t$. We choose constants A and B so as to get a solution z_t satisfying $z_0 = 0$, $z_1 = 1$; we get

$z_t = 0.4\ 3^t - 0.4\ 0.5^t$, and hence the general solution to the non-homogeneous equation is

$$x^t = 3^t A + 0.5^t B + \sum_{k=1}^{t} (0.4\ 3^{t-k} - 0.4\ 0.5^{t-k}) c_{k-1}$$
$$= 3^t A + 0.5^t B + 0.4 \sum_{k=1}^{t} 3^{t-k} c_{k-1} - 0.4 \sum_{k=1}^{t} 0.5^{t-k} c_{k-1}$$

If we want to solve the first sum forward and the second backwards, we replace A by $(A - 0.4 \sum_1^\infty 3^{-k} c_{k-1})$ and B by $(B - 0.4 \sum_{-\infty}^{0} 0.5^{-k} c_{k-1})$:

$$x^t = 3^t (A - 0.4 \sum_1^\infty 3^{-k} c_{k-1}) + 0.5^t (B - 0.4 \sum_{-\infty}^{0} 0.5^{-k} c_{k-1})$$
$$+ 0.4 \sum_{k=1}^{t} 3^{t-k} c_{k-1} - 0.4 \sum_{k=1}^{t} 0.5^{t-k} c_{k-1}$$
$$= 3^t A + 0.5^t B - 0.4 \sum_{k=t+1}^{\infty} 3^{t-k} c_{k-1} - 0.4 \sum_{k=-\infty}^{t} 0.5^{t-k} c_{k-1}$$

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Systems of differential equations

1. Systems of linear equations with constant coefficients

Consider a system of two linear equations with constant coefficients:

$$x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) + b_1$$
$$x_2'(x) = a_{21}x_1(t) + a_{22}x_2(t) + b_2$$

This system may be written more compactly using matrix notation:

$$\underline{x}'(t) = A\underline{x}(t) + \underline{b} \tag{1.1}$$

where

$$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{12} \end{pmatrix} \quad \text{and} \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

We assume that the determinant $\Delta \overset{\text{def}}{=} \det(A) \neq 0$. It is then easy to see that a *steady state* solution—i.e., a solution independent of t—to (1.1) is $\underline{x}^* = -A^{-1}\underline{b}$.

For any $\underline{x}(t)$ we can define $\underline{z}(t)$ by $x(t) = \underline{z}(t) + \underline{x}^*$; substituting into (1.1) yields

$$\underline{z}'(t) = A(\underline{z}(t) + \underline{x}^*) + \underline{b} = A\underline{z}(t) - A(A^{-1}\underline{b}) + \underline{b} = A\underline{z}(t)$$

i.e., $\underline{x}(t)$ is a solution to (1.1) if and only if $\underline{x}(t) = \underline{z}(t) + \underline{x}^*$, where $\underline{x}^* = -A^{-1}\underline{b}$ and $\underline{z}(t)$ is a solution to the *homogeneous* equation

$$\underline{z}(t) = A\underline{z}(t) \tag{1.2}$$

We now proceed to find the general solution.

Theorem: *Let* $\Phi(t) = \begin{pmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{pmatrix}$ *be a solution to*

$$\Phi'(t) = A\Phi(t) \tag{1.3}$$

where

$$\det \Phi(t) = \begin{vmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{vmatrix} \neq 0 \quad \textit{for all } t.$$

Then the general solution to equation (1.2) is $\underline{z}(t) = \Phi(t)\underline{c}$, *where* $\underline{c}$ *is a constant* $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. *In other words, the general solution to equation (1.1) is*

$$\underline{x}(t) = \underline{\phi}_1 c_1 + \underline{\phi}_2 c_2 - A^{-1}\underline{b}$$

where $\underline{\phi}_j$ *is the j:th column of* Φ *and* c_1, c_2 *are arbitrary constants.*

Proof: For any $\underline{z}(t)$, define $\underline{c}(t)$ by $\underline{z}(t) = \Phi(t)\underline{c}(t)$. This is possible, since $\det \Phi(t) \neq 0$. Substituting into (1.1) gives

$$\Phi'(t)\underline{c}(t) + \Phi(t)\underline{c}'(t) = A\Phi(t)\underline{c}(t)$$
$$\Longleftrightarrow$$
$$A\Phi(t)\underline{c}(t) + \Phi(t)\underline{c}'(t) = A\Phi(t)\underline{c}(t)$$
$$\Longleftrightarrow$$
$$\Phi(t)\underline{c}'(t) = 0 \iff \underline{c}'(t) = 0 \iff \underline{c}(t) = \underline{c} = \text{constant}$$

so we have $\underline{z}(t) = \Phi(t)\underline{c}$, *Q.E.D.*

2. Finding $\Phi(t)$

Note that $\Phi(t)$ is a solution to (1.3) if and only if each of the two columns $\underline{\phi}_1(t)$, $\underline{\phi}_2(t)$ are solutions to (1.1). Let us try to find a solution to (1.1) of the form $\underline{\phi}(t) = \underline{v}e^{rt}$ for some constant column matrix $\underline{v}$ and real number r. We get

$$\underline{v}\, re^{rt} = A\underline{v}e^{rt} \iff A\underline{v} = r\underline{v}$$

i.e., we have found a solution iff. $\underline{v}$ is an eigenvector of A and r is the corresponding eigenvalue. The procedure is thus: if the *characteristic equation* $r^2 - \text{Tr}\, A + \det A = 0$ has two distinct real roots r_1, r_2, and $\underline{v}_1$, $\underline{v}_2$ are the corresponding eigenvectors, then we have two solutions $\underline{\phi}_j(t) = \underline{v}_j e^{r_j t}$. Since eigenvectors corresponding to different eigenvalues are linearly independent, it follows that the determinant $\det(\underline{\phi}_1, \underline{\phi}_2) \neq 0$. Thus

Theorem: *If the characteristic equation* $r^2 - \text{Tr}\, A + \det A = 0$ *has two distinct real roots* r_1, r_2, *and* $\underline{v}_1$, $\underline{v}_2$ *are the corresponding eigenvectors, then the general solution to equation (1.1) is (assuming* $\det A \neq 0$) :

$$\underline{x}(t) = c_1 e^{r_1 t}\, \underline{v}_1 + c_2 e^{r_2 t}\, \underline{v}_2 - A^{-1}\underline{b}$$

where c_1 *and* c_2 *are arbitrary constants.*

If the characteristic equation has a double root, or complex roots, we state the general solution to equation (1.1) without proof:

Theorem: *If the characteristic equation $r^2 - \operatorname{Tr} A + \det A = 0$ has a double root r, then the general solution of (1.1) is*

$$\underline{x}(t) = \underline{v}_1\, e^{rt} + \underline{v}_2\, te^{rt} - A^{-1}\underline{b}$$

where

$$\underline{v}_1 \text{ is an arbitrary column matrix and } \underline{v}_2 = (A - Ir)\underline{v}_1$$

If the characteristic equation has complex roots $\alpha \pm i\beta$, then the general solution is

$$\underline{x}(t) = \underline{v}_1\, e^{\alpha t}\cos\beta t + \underline{v}_2\, e^{\alpha t}\sin\beta t - A^{-1}\underline{b}$$

where

$$\underline{v}_1 \text{ is an arbitrary column matrix and } \underline{v}_2 = \frac{1}{\beta}(A - I\alpha)\underline{v}_1$$

3. Characterization of solutions.

The solutions can be depicted in a *phase diagram* where the curves $(x_1(t), x_2(t))$ are plotted in an x_1, x_2 diagram. The solution curves are called *trajectories.* We can distinguish between a number of cases. Let $T = \operatorname{Tr} A$ and $\Delta = \det A$. If $4\Delta > T^2$, then the characteristic equation features complex roots, and any trajectory is a *stable spiral* if $\alpha < 0$. In this case the trajectory spiral in towards the stationary solution $\underline{x}^*$. If instead $\alpha > 0$ we have *unstable* spirals; any trajectory will spiral out from $\underline{x}^*$. In either case, the spirals will turn in a positive—i.e., counter clockwise—direction if $a_{21} > 0$ and in negative—clockwise—direction if $a_{21} < 0$. If α happens to be exactly equal to zero, then the trajectories will go around the stationary solution $\underline{x}^*$ in ellipses; we have *periodic* solutions.

If the characteristic equation has distinct real roots, which is the case if $4\Delta < T^2$, then there are three cases depending on the location of the roots. Define $\psi(r) = r^2 - rT + \Delta$. This function is a parabola with it's minimum located at $r = T/2$.

(1.1) $\Delta > 0$ and $T < 0$. In this case $\psi(r)$ has it's minimum point $T/2$ to the left of $r = 0$ where $\psi(T/2) = -T^2/4 + \Delta < 0$ and $\psi(0) = \Delta > 0$. It follows that both roots r_1, r_2 are negative. The phase portrait is a *stable node.* All trajectories will converge towards the stationary solution $\underline{x}^*$ and will become tangent to the eigenvector corresponding to the eigenvalue closest to zero as $t \to \infty$.

(1.2) $\Delta > 0$ and $T > 0$. In this case $\psi(r)$ has it's minimum point $T/2$ to the right of $r = 0$ where $\psi(T/2) = -T^2/4 + \Delta < 0$ and $\psi(0) = \Delta > 0$. It follows that both roots r_1, r_2 are positive. The phase portrait is an *unstable node.* All trajectories will diverge away from the stationary solution $\underline{x}^*$ and will become tangent to the eigenvector corresponding to the eigenvalue closest to zero as $t \to -\infty$.

(1.3) $\Delta < 0$. In this case $\psi(0) = \Delta < 0$, it follows that one root is negative and the other positive; say $r_1 < 0$ and $r_2 > 0$. The phase portrait is a *saddle* *point.* Almost all trajectories will diverge away from the stationary solution $\underline{x}^*$; the only exception are the solutions $\underline{x}(t) = c_1 e^{r_1 t}\,\underline{v}_1 + c_2 e^{r_2 t}\,\underline{v}_2 - A^{-1}\underline{b}$ where $c_2 = 0$. These solutions $\underline{x}(t) = c_1 e^{r_1 t}\,\underline{v}_1 - A^{-1}\underline{b}$ are called *saddle paths* and play an important rule in many macro economic models.

Since $2\alpha = T$ we have in summary:

$T < 0,\ \ 4\Delta > T^2,\ \ a_{21} \begin{cases} > 0 \\ < 0 \end{cases}$	stable spiral $\begin{cases} \text{counter-clockwise} \\ \text{clockwise} \end{cases}$
$T > 0,\ \ 4\Delta > T^2,\ \ a_{21} \begin{cases} > 0 \\ < 0 \end{cases}$	unstable spiral $\begin{cases} \text{counter-clockwise} \\ \text{clockwise} \end{cases}$
$T > 0,\ \ 0 < 4\Delta < T^2$	unstable node
$T < 0,\ \ 0 < 4\Delta < T^2$	stable node
$\Delta < 0$	saddle point

4. Linearization and isoclines.

Consider a non-linear autonomous system of equations

$$\begin{aligned} x_1'(t) &= F(x_1, x_2) \\ x_1'(t) &= G(x_1, x_2) \end{aligned} \tag{4.1}$$

and let (x_1^*, x_2^*) be a stationary solution, i.e., a solution to $F(x_1^*, x_2^*) = G(x_1^*, x_2^*) = 0$. We can then *linearize* the equation:

$$\underline{x}(t) = A\underline{x}(t) - A\underline{x}^* \tag{4.2}$$

where

$$A = \begin{pmatrix} F_{x_1}(x_1^*, x_2^*) & F_{x_2}(x_1^*, x_2^*) \\ G_{x_1}(x_1^*, x_2^*) & G_{x_2}(x_1^*, x_2^*) \end{pmatrix}$$

Theorem: *In any of the cases listed in the summary in the previous section, the the solutions to the linearized system (4.2) are good approximations to the solutions to the nonlinear system (4.1) as long as they stay close to the equilibrium* (x_1^*, x_2^*).

Another help in studying nonlinear systems is the plotting of *isoclines.* The trajectories will traverse the curve $G(x_1, x_2)$—the zero isocline—horizontally, and the curve $F(x_1, x_2)$—the infinity isocline—vertically. These two isoclines divides the x_1, x_2 plane into regions, and in each of these the derivatives of $x_1(t)$ and $x_2(t)$ are constant. These observations are very helpful in the study of the qualitative behavior of the solutions of (4.1).

Optimal Control

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Continuous case

Consider the problem of finding functions $x(t) = (x_1(t), \ldots, x_n(t))$ and $u(t) = (u_1(t), \ldots, u_m(t))$ so as to maximize the integral

$$\int_a^b f(x(t), u(t), t)\, dt$$

subject to

$$\dot{x}_j(t) = g_j(x(t), u(t), t) \quad j = 1, \ldots, n$$

$$\int_a^b k_j(x(t), u(t), t)\, dt = Q_j \quad j = 1, \ldots, J_1$$

$$h_j(x(t), u(t), t) = 0 \quad j = 1, \ldots, J_2$$

$$u(t) \in \Gamma_t \quad \Gamma_t \text{ some convex subset of } \mathbf{R}^m$$

Of course, $J_1 = 0$ means that there are no integral restrictions, and similarly if $J_2 = 0$. Define the *Hamiltonian* H as

$$\begin{aligned} H(x(t), u(t), \lambda(t), \mu, \nu(t), t) =& f(x(t), u(t), t) + \lambda(t) \cdot g(x(t), u(t), t) \\ &+ \mu \cdot k(x(t), u(t), t) + \nu(t) \cdot h(x(t), u(t), t) \end{aligned}$$

where we of course have used vector notation: $g(\cdot) = (g_1(\cdot), \ldots, g_n(\cdot))$; $\lambda(t) = (\lambda_1(t), \ldots, \lambda_n(t))$, etc. Note that the *Lagrangean multipliers* λ_j and ν_j are functions of t, whereas the μ_j's are constants.

Assumption: *The Hamiltonian H is concave in the variables (x, u) and continuously differentiable wrt x.*

Lemma 1. *Let $u \in \operatorname*{argmax}_{u \in \Gamma_t} H(x, u, \lambda, \mu, \nu, t)$. Then, for any $\tilde{u} \in \Gamma_t$,*

$$H(x, u, \lambda, \mu, \nu, t) - H(\tilde{x}, \tilde{u}, \lambda, \mu, \nu, t) \geq H_x(x, u, \lambda, \mu, \nu, t) \cdot [x - \tilde{x}]$$

Proof. Let $\Delta x \equiv \tilde{x} - x, \Delta u \equiv \tilde{u} - u$. For any positive integer m,

$$\begin{aligned} &H(x + \frac{\Delta x}{m}, u + \frac{\Delta u}{m}, \lambda, \mu, \nu, t) \\ &\quad \geq H(x, u, \lambda, \mu, \nu, t) + \frac{1}{m}[H(\tilde{x}, \tilde{u}, \lambda, \mu, \nu, t) - H(x, u, \lambda, \mu, \nu, t)] \\ &\quad \geq H(x, u + \frac{\Delta u}{m}, \lambda, \mu, \nu, t) + \frac{1}{m}[H(\tilde{x}, \tilde{u}, \lambda, \mu, \nu, t) - H(x, u, \lambda, \mu, \nu, t)] \end{aligned}$$

The first inequality comes from the concavity, the second from the definition of u; note that $u + \frac{\Delta u}{m} \in \Gamma_t$ since Γ_t is convex. Hence

$$\begin{aligned} &H(x, u, \lambda, \mu, \nu, t) - H(\tilde{x}, \tilde{u}, \lambda, \mu, \nu, t) \\ &\qquad \geq m[H(x, u + \frac{\Delta u}{m}, \lambda, \mu, \nu, t) - H(x + \frac{\Delta x}{m}, u + \frac{\Delta u}{m}, \lambda, \mu, \nu, t)] \\ &\qquad = -H_x(\widehat{x}_m, u + \frac{\Delta u}{m}, \lambda, \mu, \nu, t) \cdot \Delta x \end{aligned}$$

where $\widehat{x}_m$ is some point on the straight line connecting x and $x + \frac{\Delta x}{m}$; the equality comes from the mean value theorem of calculus. Now, letting $m \to \infty$ using the fact that H_x is continuous, we get the result of the lemma; *Q.E.D.*

It is easy to verify that

$$\int_a^b f(x(t), u(t), t)\, dt = \int_a^b [H(\cdot) - \lambda(t) \cdot \dot{x}(t)]\, dt - \mu Q$$

Hence, if $(x(t), u(t))$ is a fixed feasible pair with

$$u(t) \in \operatorname*{argmax}_{u \in \Gamma_t} H(x(t), u(t), \lambda(t), \mu, \nu, t)$$

and $(\tilde{x}(t), \tilde{u}(t))$ is any feasible pair, then

$$\begin{aligned} &\int_a^b f(x(t), u(t), t)\, dt - \int_a^b f(\tilde{x}(t), \tilde{u}(t), t)\, dt \\ &= \int_a^b \{H(x(t), u(t), \lambda(t), \mu, \nu(t), t) - H(\tilde{x}(t), \tilde{u}(t), \lambda(t), \mu, \nu(t), t)\}\, dt \\ &\quad - \int_a^b \lambda(t) \cdot [\dot{x}(t) - \dot{\tilde{x}}(t)]\, dt \\ &\geq \int_a^b H_x(x(t), u(t), \lambda(t), \mu, \nu(t), t) \cdot [x(t) - \tilde{x}(t)]\, dt - \int_a^b \lambda(t) \cdot [\dot{x}(t) - \dot{\tilde{x}}(t)]\, dt \\ &= \int_a^b [H_x(x(t), u(t), \lambda(t), \mu, \nu(t), t) + \dot{\lambda}(t)] \cdot [x(t) - \tilde{x}(t)]\, dt \\ &\quad - \big[\lambda(t) \cdot (x(t) - \tilde{x}(t))\big]_a^b \end{aligned}$$

The following conditions on $(x(t), u(t), \lambda(t), \mu, \nu(t))$ are obviously sufficient for the last expression in the chain of relations above to be 0, and hence *sufficient conditions for $(x(t), u(t))$ to be a solution to the maximization problem:*

(Ex$_j$) $\quad H_{x_j}(x(t), u(t), \lambda(t), \mu, \nu(t), t) + \dot{\lambda}_j(t) = 0 \quad j = 1, \ldots, n$

(Mu$_j$) $\quad u_j \in \underset{u \in \Gamma_t}{\operatorname{argmax}} H(x(t), u(t), \lambda(t), \mu, \nu(t), t) \quad j = 1, \ldots, m$

for any $j = 1, \ldots, n$, either $x_j(b) = x_j^b$ is given, or else

(TUx$_j$) $\quad \lambda_j(b) = 0$

for any $j = 1, \ldots, n$, either $x_j(a) = x_j^a$ is given, or else

(TLx$_j$) $\quad \lambda_j(a) = 0$

If there is no restriction $x_j(b) = x_j^b$, then b is called a *free boundary* for x_j, and the condition *(TUx$_j$)* and *(TLx$_j$)* are called *transversality conditions.* The E-equations are called *Euler equations.*

Infinite horizon

Let in the previous case $b = \infty$. The only impact this has on the analysis above is on the last expression $-[\lambda(t) \cdot (x(t) - \tilde{x}(t))]_a^b$. A sufficient condition for this expression to be ≥ 0 as $b \to \infty$ is, in addition to *(TLx$_j$)* above

(TU$^\infty$x$_j$) $\quad \liminf_{t \to \infty} \lambda(t) \cdot \tilde{x}(t) \geq 0$ *for all feasible $\tilde{x}(t)$ and* $\lim_{t \to \infty} \lambda(t) \cdot x(t) = 0$

The envelope theorem

Let $V = \max \int_a^b f(x(t), u(t), t)\, dt$ and let α be a parameter which does not influence Q, a, b, Γ_t, x^a, if a is not a free boundary, x^b if b is not a free boundary. Assume that the sufficient conditions given above are satisfied and that u is uniquely defined by *(Mu$_j$)* and locally bounded as a function of α. Then

$$\frac{dV}{d\alpha} = \int_a^b H_\alpha(x(t), u(t), \lambda(t), \mu, \nu(t), t)\, dt$$

$$\frac{dV}{da} = -H(x(a), u(a), \lambda(a), \mu, \nu(a), a)$$

$$\frac{dV}{db} = H(x(b), u(b), \lambda(b), \mu, \nu(b), b)$$

$$\frac{dV}{dx_j^a} = \lambda_j(a) \quad \textit{(if a is not a free boundary)}$$

$$\frac{dV}{dx_j^b} = -\lambda_j(b) \quad \textit{(if b is not a free boundary)}$$

$$\frac{dV}{dQ_j} = -\mu_j \quad j = 1, \ldots, J_1$$

Lemma 2. *If β is any parameter (i.e., β may influence a, x_j^a, Q, etc.), then*

$$\int_a^b [H_x \cdot x_\beta + H_\lambda \cdot \lambda_\beta + H_\mu \cdot \mu_\beta + H_\nu \cdot \nu_\beta - \lambda_\beta \cdot \dot{x} - \lambda \cdot \dot{x}_\beta]\, dt - \mu_\beta \cdot Q$$
$$= \lambda(a) \cdot x_\beta(a) - \lambda(b) \cdot x_\beta(b)$$

Proof. The term $H_\mu = k(x, u, t)$ whose integral is Q and $H_\nu = h(x, u, t)$ which is $\equiv 0$. Hence we are left with

$$\int_a^b [H_x \cdot x_\beta + H_\lambda \cdot \lambda_\beta - \lambda_\beta \cdot \dot{x} - \lambda \cdot \dot{x}_\beta]\, dt = \int_a^b [(H_x + \dot{\lambda}) \cdot x_\beta + (H_\lambda - \dot{x}) \cdot \lambda_\beta]\, dt - [\lambda(t) \cdot x_\beta(t)]_a^b$$

The integral is $= 0$, since $H_x + \dot{\lambda} = 0$ by *(Ex)* and $H_\lambda - \dot{x} = g(x, u, t) - \dot{x} = 0$. So we are left with $\lambda(a) \cdot x_\beta(a) - \lambda(b) \cdot x_\beta(b)$ *Q.E.D.*

Proof of the the envelope theorem.

By the "usual" envelope theorem, we can treat u as constant when we differentiate, hence

$$\frac{dV}{d\alpha} = \int_a^b H_\alpha\, dt + \int_a^b [H_x \cdot x_\alpha + H_\lambda \cdot \lambda_\alpha + H_\mu \cdot \mu_\alpha + H_\nu \cdot \nu_\alpha - \lambda_\alpha \cdot \dot{x} - \lambda \cdot \dot{x}_\alpha]\, dt - \mu_\alpha \cdot Q$$

Using lemma 2, we get

$$\frac{dV}{d\alpha} = \int_a^b H_\alpha\, dt + \lambda(a) \cdot x_\alpha(a) - \lambda(b) \cdot x_\alpha(b)$$

If a is a free boundary, then $\lambda(a) = 0$; if $x(a) = x^a$ is given, then $x_\alpha(a) = 0$, since x^a is independent of α by assumption. In any case, the product $\lambda(a) \cdot x_\alpha(a) = 0$, and similarly $\lambda(b) \cdot x_\alpha(b) = 0$, which proves the $dV/d\alpha$ part.

$$\begin{aligned} \frac{dV}{db} &= \frac{d}{db}\left\{ \int_a^b [H(\cdot) - \lambda(t) \cdot \dot{x}(t)]\, dt - \mu \cdot Q \right\} \\ &= H(x(b), u(b), \lambda(b), \mu, \nu, b) - \lambda(b) \cdot \dot{x}(b) \\ &\quad + \int_a^b [H_x \cdot x_b + H_\lambda \cdot \lambda_b + H_\mu \cdot \mu_b + H_\nu \cdot \nu_b - \lambda_b \cdot \dot{x} - \lambda \cdot \dot{x}_b]\, dt - \mu_b \cdot Q \end{aligned}$$

Using lemma 2, we get

$$\frac{dV}{db} = H(x(b), u(b), \lambda(b), \mu, \nu, b) - \lambda(b) \cdot \dot{x}(b) + \lambda(a) \cdot x_b(a) - \lambda(b) \cdot x_b(b)$$

Here the term $\lambda(a) \cdot x_b(a) = 0$, by the same argument as in the previous case. If b is a free boundary, then $\lambda(b) = 0$. In order to analyze the situation when b is not a free boundary, we introduce the temporary notation $x(t; b)$ for the optimal function $x(t)$ given that the upper limit of integration is b. In this case $x(b; b) = \widehat{x}$ where $\widehat{x}$ is a number independent of b. Hence, by differentiation w.r.t. b, $\dot{x}(b,b)+x_b(b;b)=0$. We see that in all cases the sum $-\lambda(b)\cdot\dot{x}(b)-\lambda(b)\cdot x_b(b)=0$, which proves the dV/db part. Of course, the dV/da part is proven similarly.

Using lemma 2, we have

$$\begin{aligned}\frac{dV}{dx_j^a} &= \int_a^b \Big[H_x\cdot\frac{\partial x}{\partial x_j^a}+H_\lambda\cdot\frac{\partial\lambda}{\partial x_j^a}+H_\mu\cdot\frac{\partial\mu}{\partial x_j^a}+H_\nu\cdot\frac{\partial\nu}{\partial x_j^a}-\frac{\partial\lambda}{\partial x_j^a}\cdot\dot{x}-\lambda\cdot\frac{\partial\dot{x}}{\partial x_j^a}\Big]\,dt\\ &\quad -Q\cdot\frac{\partial x}{\partial x_j^a}\\ &=\lambda(a)\cdot\frac{\partial x(a)}{\partial x_j^a}-\lambda(b)\cdot\frac{\partial x(b)}{\partial x_j^a}\end{aligned}$$

Here either $\lambda(b)$ is $=0$ (if b is a free boundary) or $\dfrac{\partial x(b)}{\partial x_j^a}=0$ (if b is not a free boundary) Hence $\dfrac{dV}{dx_j^a}=\lambda(a)\cdot\dfrac{\partial x(a)}{\partial x_j^a}=\lambda_j(a)$. The formula for $\dfrac{dV}{dx_j^b}$ is of course proven similarly.

Finally, employing lemma 2 once again, we have

$$\begin{aligned}\frac{dV}{dQ} &= \frac{d}{dQ}\Big\{\int_a^b [H(\cdot)-\lambda(t)\cdot\dot{x}(t)]\,dt-\mu\cdot Q\Big\}\\ &=\int_a^b H_Q\,dt+\int_a^b [H_x\cdot x_Q+H_\lambda\cdot\lambda_Q+H_\mu\cdot\mu_Q+H_\nu\cdot\nu_Q-\lambda_Q\cdot\dot{x}-\lambda\cdot\dot{x}_Q]\,dt\\ &\quad -Q\cdot\mu_Q-\mu\\ &=\lambda(a)\cdot x_Q(a)-\lambda(b)\cdot x_Q(b)-\mu=-\mu\end{aligned}$$

The argument for the last equality is the same as in previous cases. *Q.E.D.*

Discrete case

Consider the problem of finding sequences $x_t=(x_t^1,\ldots,x_t^n)$ and $u_t=(u_t^1,\ldots,u_t^m)$ so as to maximize the sum

$$\sum_s^T f(x_t,u_t,t)$$

subject to the constraints

$$\begin{aligned}&x_{t+1}^j=g^j(x_t,u_t,t),\quad t=s,\ldots,T\quad j=1,\ldots,n\\ &\sum_s^T k^j(x_t,u_t,t)=Q^j\quad j=1,\ldots,J_1\\ &h^j(x_t,u_t,t)=0\quad t=s,\ldots,T\quad j=1,\ldots,J_2\\ &u_t\in\Gamma_t\quad \Gamma_t \text{ some convex subset of } \mathbf{R}^m\end{aligned}$$

Of course, $J_1=0$ means that there are no summation restrictions, and similarly if $J_2=0$. Define the *Hamiltonian*

$$H(x_t,u_t,\lambda_{t+1},\mu,\nu_t,t)=f(x_t,u_t,t)+\lambda_{t+1}\cdot g(x_t,u_t,t)+\mu\cdot k(x_t,u_t,t)+\nu_t\cdot h(x_t,u_t,t)$$

where we of course have used vector notation: $g(\cdot) = (g^1(\cdot), \dots, g^n(\cdot))$; $\lambda_t = (\lambda_t^1, \dots, \lambda_t^n)$, etc. Note that the *Lagrangean multipliers* λ^j and ν^j are functions of t, whereas the μ^j's are constants. It is easy to verify that

$$\sum_s^T f(x_t, u_t, t) = \sum_s^T [H(\cdot) - \lambda_{t+1} \cdot x_{t+1}] - \mu Q$$

Assumption: *The Hamiltonian H is concave in the variables (x, u) and continuously differentiable wrt x.*

By lemma 1, if $u_t \in \operatorname*{argmax}_{u \in \Gamma_t} H(x_t, u_t, \lambda_{t+1}, \mu, \nu_t, t)$ and $(\tilde{x}_t, \tilde{u}_t)$ is any feasible pair, then

$$H(x, u, \lambda, \mu, \nu, t) - H(\tilde{x}, \tilde{u}, \lambda, \mu, \nu, t) \geq H_x(x, u, \lambda, \mu, \nu, t) \cdot [x - \tilde{x}]$$

Hence, if (x_t, u_t) is a fixed feasible pair such that $u_t \in \operatorname*{argmax}_{u \in \Gamma_t} H(x_t, u_t, \lambda_{t+1}, \mu, \nu_t, t)$ and $(\tilde{x}_t, \tilde{u}_t)$ is any feasible pair, then

$$\begin{aligned}
\sum_s^T f(x_t, u_t, t) - \sum_s^T f(\tilde{x}_t, \tilde{u}_t, t) & \\
&= \sum_s^T \{H(x_t, u_t, \lambda_{t+1}, \mu, \nu_t, t) - H(\tilde{x}_t, \tilde{u}_t, \lambda_{t+1}, \mu, \nu_t, t)\} \\
&\quad - \sum_s^T \lambda_{t+1} \cdot [x_{t+1} - \tilde{x}_{t+1}] \\
&\geq \sum_s^T H_x(x_t, u_t, \lambda_{t+1}, \mu, \nu_t, t) \cdot [x_t - \tilde{x}_t] - \sum_s^T \lambda_{t+1} \cdot [x_{t+1} - \tilde{x}_{t+1}] \\
&= \sum_s^T [H_x(x_t, u_t, \lambda_{t+1}, \mu, \nu_t, t) - \lambda_t] \cdot [x_t - \tilde{x}_t] + \lambda_s \cdot [x_s - \tilde{x}_s] \\
&\quad - \lambda_{T+1} \cdot [x_{T+1} - \tilde{x}_{T+1}]
\end{aligned}$$

The following conditions on $(x_t, u_t, \lambda_t, \mu, \nu_t)$ are obviously sufficient for this expression to be $= 0$, and hence *sufficient conditions for (x_t, u_t) to be a solution to the maximization problem:*

(Ex^j) $\quad H_{x^j}(x_t, u_t, \lambda_{t+1}, \mu, \nu_t, t) - \lambda_t^j = 0 \quad t = s, \dots, T \quad j = 1, \dots, n$

(Mu^j) $\quad u_t \in \operatorname*{argmax}_{u \in \Gamma_t} H(x_t, u_t, \lambda_{t+1}, \mu, \nu_t, t) \quad t = 1, \dots, T \quad j = 1, \dots m^*$

for any $j = 1, \dots, n$, either x_j^{T+1} is given, or else

(TUx^j) $\quad \lambda_{T+1}^j = 0$

for any $j = 1, \dots, n$, either x_s^j is given, or else

(TLx^j) $\quad \lambda_s^j = 0$

* *In contrast to the continuous case, this is not a necessary condition if H is not concave.*

Infinite horizon

Let in the previous case $T = \infty$. The only impact this has on the analysis above is on the last expression $-\lambda_{T+1} \cdot [x_{T+1} - \tilde{x}_{T+1}]$. A sufficient condition for this expression to be ≥ 0 as $T \to \infty$ is, in addition to *(TLx^j)* above,

$$(TU^{\infty}x^j) \quad \liminf_{t\to\infty} \lambda_t \cdot \tilde{x}_t \geq 0 \quad \textit{for all feasible } \tilde{x}_t \textit{ and } \quad \lim_{t\to\infty} \lambda_t \cdot x_t = 0$$

The envelope theorem

Let $V = \max \sum_s^T f(x_t, u_t, t)$ and let α be a parameter which does not influence Q, s, T, Γ_t or any of the boundary values of x, if any are given as constraints. Assume that the sufficient conditions given above are satisfied and that u_t^j is uniquely determined by *(Mu^j)* and locally bounded in α. Then

$$\frac{dV}{d\alpha} = \sum_s^T H_\alpha(x_t, u_t, \lambda_{t+1}, \mu, \nu_t, t)$$

$$\frac{dV}{dx_s^j} = \lambda_s^j \quad \textit{if } x_s^j \textit{ is given as a constraint}$$

$$\frac{dV}{dx_{T+1}^j} = -\lambda_{T+1}^j \quad \textit{if } x_{T+1}^j \textit{ is given as a constraint}$$

$$\frac{dV}{dQ_j} = -\mu^j \quad j = 1, \ldots, J_1$$

Proof. Using the "usual" envelope theorem as to the variation in u,

$$\begin{aligned}
\frac{dV}{d\alpha} &= \frac{d}{d\alpha} \sum_s^T f(x_t, u_t, t) \\
&= \frac{d}{d\alpha} \Big\{ \sum_s^T [H(x_t, u_t, \lambda_{t+1}, \mu, \nu_t, t) - \lambda_{t+1} \cdot x_{t+1}] - \mu \cdot Q \Big\} \\
&= \sum_s^T H_\alpha(x_t, u_t, \lambda_{t+1}, \mu, \nu_t, t) + \sum_s^T \Big\{ H_x \cdot \frac{dx_t}{d\alpha} + H_\lambda \cdot \frac{d\lambda_{t+1}}{d\alpha} + H_\mu \cdot \frac{d\mu}{d\alpha} \\
&\quad + H_\nu \cdot \frac{d\nu_t}{d\alpha} - \frac{d\lambda_{t+1}}{d\alpha} \cdot x_{t+1} - \lambda_{t+1} \cdot \frac{dx_{t+1}}{d\alpha} \Big\} - \frac{d\mu}{d\alpha} \cdot Q \\
&= \sum_s^T H_\alpha(x_t, u_t, \lambda_{t+1}, \mu, \nu_t, t) + \sum_s^T [H_x - \lambda_t] \cdot \frac{dx_t}{d\alpha} + \sum_s^T [H_\lambda - x_{t+1}] \cdot \frac{d\lambda_{t+1}}{d\alpha} \\
&\quad + \Big\{ \Big(\sum_s^T H_\mu \Big) - Q \Big\} \cdot \frac{d\mu}{d\alpha} + \sum_s^T H_\nu \cdot \frac{d\nu_t}{d\alpha} + \lambda_s \cdot \frac{dx_s}{d\alpha} - \lambda_{T+1} \cdot \frac{dx_{T+1}}{d\alpha}
\end{aligned}$$

Using *(Ex)*, $H_\lambda - x_{t+1} = g(x_t, u_t, t) - x_{t+1} = 0$, $\sum_s^T H_\mu - Q = \sum_s^T k(x_t, u_t, t) - Q = 0$ and $H_\nu = h(\ldots) = 0$, we get

$$\frac{dV}{d\alpha} = \sum_s^T H_\alpha(x_t, u_t, \lambda_{t+1}, \mu, \nu_t, t) + \lambda_s \cdot \frac{dx_s}{d\alpha} - \lambda_{T+1} \cdot \frac{dx_{T+1}}{d\alpha}$$

Here, either $\lambda_s = 0$ (if s is a free boundary) or $dx_s/d\alpha = 0$ (if x_s is given); hence the λ_s-term $= 0$, and similarly the λ_{T+1}-term $= 0$. The other derivatives are shown similarly. *Q.E.D.*

Uncertainty

Let t be time, ξ_t a stochastic process, E_t the expectations operator conditional on I_t, the information set available at time t; in particular, $t' < t \Rightarrow I_{t'} \subseteq I_t$. Consider the problem of finding sequences $x_t = (x_t^1, \ldots, x_t^n)$ and $u_t = (u_t^1, \ldots, u_t^m)$ so as to maximize the sum

$$E_s \sum_s^T f(x_t, u_t, \xi_t, t) \quad \text{where } x_t, u_t \in I_t \text{, subject to the constraints}$$

$$x_{t+1}^j = g^j(x_t, u_t, \xi_t, t), \quad t = s, \ldots, T \quad j = 1, \ldots, n$$

$$\sum_s^T k^j(x_t, u_t, \xi_t, t) = Q^j \quad j = 1, \ldots, J_1$$

$$h^j(x_t, u_t, \xi_t, t) = 0 \quad t = s, \ldots, T \quad j = s, \ldots, J_2$$

Of course, $J_1 = 0$ means that there are no summation restrictions, and similarly if $J_2 = 0$. Define the *Hamiltonian*

$$\begin{aligned} H(x_t, u_t, \xi_t, \lambda_{t+1}, \mu, \nu_t, t) = f(x_t, u_t, \xi_t, t) + \lambda_{t+1} \cdot g(x_t, u_t, \xi_t, t) \\ + \mu \cdot k(x_t, u_t, \xi_t, t) + \nu_t \cdot h(x_t, u_t, \xi_t, t) \end{aligned}$$

where we of course have used vector notation: $g(\cdot) = (g_1(\cdot), \ldots, g_n(\cdot))$; $\lambda_t = (\lambda_t^1, \ldots, \lambda_t^n)$, etc. The *Lagrangean multipliers* λ^j and ν^j are *stochastic processes* and the $\mu^j : s$ are *stochastic variables* (i.e., independent of t) such that $\mu \in I_s$ and $\lambda_t \in I_t$. It is easy to verify that

$$E_s \sum_s^T f(x_t, u_t, \xi_t, t) = E_s \sum_s^T [H(\cdot) - \lambda_{t+1} \cdot x_{t+1}] - \mu Q$$

Assumption: *The Hamiltonian $H(x_t, \ldots, t)$ is concave in the variables (x_t, u_t) and continuously differentiable wrt x.*

By lemma 1, if $u_t \in \underset{u}{\operatorname{argmax}} \in \Gamma_t E_t H(x_t, u_t, \lambda_{t+1}, \mu, \nu_t, t)$ and $(\tilde{x}_t, \tilde{u}_t$ is any feasible pair, then

$$\begin{aligned} E_t H(x_t, u_t, \xi_t, \lambda_{t+1}, \mu, \nu_t, t) - E_t H(\tilde{x}_t, \tilde{u}_t, \xi_t, \lambda_{t+1}, \mu, \nu_t, t) \\ \geq E_t H_x(x_t, u_t, \xi_t, \lambda_{t+1}, \mu, \nu_t, t) \cdot [x_t - \tilde{x}_t] \end{aligned}$$

Hence, if (x_t, u_t) is a fixed feasible pair such that

$$u_t \in \underset{u \in \Gamma_t}{\operatorname{argmax}} E_t H(x_t, u_t, \lambda_{t+1}, \mu, \nu_t, t)$$

and $(\tilde{x}_t, \tilde{u}_t)$ is any feasible pair, then

$$
\begin{aligned}
&E_s \sum_s^T f(x_t, u_t, \xi_t, t) - E_s \sum_s^T f(\tilde{x}_t, \tilde{u}_t, \xi_t, t) \\
&= E_s \sum_s^T \big\{ H(x_t, u_t, \xi_t, \lambda_{t+1}, \mu, \nu_t, t) - H(\tilde{x}_t, \tilde{u}_t, \xi_t, \lambda_{t+1}, \mu, \nu_t, t) \big\} \\
&\quad - E_s \sum_s^T \lambda_{t+1} \cdot [x_{t+1} - \tilde{x}_{t+1}] \\
&= E_s \sum_s^T \big\{ E_t H(x_t, u_t, \xi_t, \lambda_{t+1}, \mu, \nu_t, t) - E_t H(\tilde{x}_t, \tilde{u}_t, \xi_t, \lambda_{t+1}, \mu, \nu_t, t) \big\} \\
&\quad - E_s \sum_s^T \lambda_{t+1} \cdot [x_{t+1} - \tilde{x}_{t+1}] \\
&\geq E_s \sum_s^T E_t H_x(x_t, u_t, \xi_t, \lambda_{t+1}, \mu, \nu_t, t) \cdot [x_t - \tilde{x}_t] - E_s \sum_s^T \lambda_{t+1} \cdot [x_{t+1} - \tilde{x}_{t+1}] \\
&= E_s \sum_s^T E_t [H_x(x_t, u_t, \xi_t, \lambda_{t+1}, \mu, \nu_t, t) - \lambda_t] \cdot [x_t - \tilde{x}_t] \\
&\quad + \lambda_s \cdot [x_s - \tilde{x}_s] - E_s \big\{ \lambda_{T+1} \cdot [x_{T+1} - \tilde{x}_{T+1}] \big\}
\end{aligned}
$$

The following conditions on $(x_t, u_t, \lambda_t, \mu, \nu_t)$ are obviously sufficient for this expression to be ≥ 0, and hence *sufficient conditions for* (x_t, u_t) *to be a solution to the maximization problem:*

(Ex^j) $\quad E_t H_{x^j}(x_t, u_t, \lambda_{t+1}, \mu, \nu_t, t) - \lambda_t^j = 0 \quad t = s, \ldots, T \quad j = 1, \ldots, n$

(Mu^j) $\quad u_t \in \operatorname*{argmax}_{u \in \Gamma_t} E_t H(x_t, u_t, \lambda_{t+1}, \mu, \nu_t, t) \quad t = 1, \ldots, T \quad j = 1, \ldots m$

for any $j = 1, \ldots, n$, *either* x_j^{T+1} *is given, or else*

(TUx^j) $\quad \lambda_{T+1}^j = 0$

for any $j = 1, \ldots, n$, *either* x_s^j *is given, or else*

(TLx^j) $\quad \lambda_s^j = 0$

Infinite horizon

Let in the previous case $T = \infty$. The following transversality condition replaces *(TUx^j)* as a sufficient condition, and the derivation parallels that of the case with no uncertainty:

(TU^∞x^j) $\quad \liminf_{t \to \infty} E_\tau \lambda_t \cdot \tilde{x}_t \geq 0 \quad$ *for all feasible* $\tilde{x}_t$ *and* $\quad \lim_{t \to \infty} E_\tau \lambda_t \cdot x_t = 0 \ \forall \tau$

The envelope theorem

Let $V = \max E_s \sum_s^T f(x_t, u_t, \xi_t, t)$ and let α be a parameter which does not influence Q, s, T, Γ_t, ξ_t or any of the boundary values of x, if any are given as constraints. Assume also that the probability measures of $x_t, u_t, \xi_t, \lambda_{t+1}, \mu, \nu_t$ conditional on I_t are independent of α, of x_s^j if x_s^j is given as a constraint, and of x_{T+1}^j if x_{T+1}^j is given as a constraint. Assume that the sufficient conditions given above are satisfied and that u_t is uniquely determined by *(Mu^j)*. Then

$$\frac{dV}{d\alpha} = E_s \sum_s^T H_\alpha(x_t, u_t, \lambda_{t+1}, \mu, \nu_t, t)$$

$$\frac{dV}{dx_s^j} = \lambda_s^j \quad \textit{if } x_s^j \textit{ is given as a constraint}$$

$$\frac{dV}{dx_{T+1}^j} = -E_s \lambda_{T+1}^j \quad \textit{if } x_{T+1}^j \textit{ is given as a constraint}$$

$$\frac{dV}{dQ_j} = -\mu^j \quad j = 1, \ldots, J_1$$

The proof parallels that of the certainty case, so we omit it.

Bellman's approach to the uncertainty case

A popular way to treat the uncertainty case is to use Bellman's equation. We now show that Bellman's approach often leads to the same equations as Pontryagin's, i.e., those we have derived. Let us look at the maximization problem in the uncertainty case again, where we only have the first type of constraint, i.e., $x_{t+1}^j = g^j(x_t, u_t, \xi_t, t)$, and x_s is given. Let

$$V(x_s; s) \equiv \max E_s \sum_s^T f(x_t, u_t, \xi_t, t)$$

Then *Bellman's principle* states that V satisfies the functional equation

$$V(x_t; t) = \max E_t\{f(x_t, u_t, \xi_t, t) + V(x_{t+1}; t+1)\} \qquad \textit{(FE)}$$
$$\text{where } \; x_{t+1}^j = g^j(x_t, u_t, \xi_t, t)$$

We assume that the probability measure of ξ_t conditional on I_t is independent of u_t and x_t. The first order condition for this maximization problem is then

$$E_t\{f_u(x_t, u_t, \xi_t, t) + V_x(x_{t+1}; t+1) g_u^j(x_t, u_t, \xi_t, t)\} = 0$$

We introduce the notation $\lambda_{t+1} \equiv V_x(x_{t+1}; t+1)$ (cf. the envelope theorem), so this equation becomes

$$E_t\{f_u(x_t, u_t, \xi_t, t) + \lambda_{t+1} g_u^j(x_t, u_t, \xi_t, t)\}$$

which is the same as our equation *(Mu)*: $E_t H_u(x_t, u_t, \xi_t, \lambda_{t+1}, t) = 0$. If we differentiate the functional equation *(FE)* wrt x_t, using the envelope theorem, we get

$$V_x(x_t; t) = E_t\{f_x(x_t, u_t, \xi_t, t) + V_x(x_{t+1}; t+1) g_x^j(x_t, u_t, \xi_t, t)\}$$

which with our λ-notation becomes

$$\lambda_t = E\{f_x(x_t, u_t, \xi_t, t) + \lambda_{t+1} g_x^j(x_t, u_t, \xi_t, t)\}$$

which is precisely our equation *(Ex)*: $E_t H_x(x_t, u_t, \xi_t, \lambda_{t+1}, t) - \lambda_t = 0$.