1. Dynamic Optimization in Discrete Time

Last edited 1999-11-22

1.1. Non-Stochastic Dynamic Programming

Consider the dynamic problem

$$
\begin{aligned}
&\max_{c,k} \sum_{t=0}^{T} u(k_t, c_t, t) \\
&s.t. \qquad k_0 = \bar{k}_0, \\
&\qquad\qquad k_{t+1} = f(k_t, c_t),\ t = 0, \ldots, T, \\
&\qquad\qquad k_{T+1} = 0
\end{aligned}
\tag{1.1}
$$

Notice:

1. Per period payoff is time additive.

2. k_t cannot be changed in period t, but its future values, its *law-of-motion* can be changed by c_t,. i.e., k is a state variable and c is the control variable.

The direct way to solve this would be to form the Lagrangean

$$
L = \sum_{t=0}^{T} u(k_t, c_t, t) + \sum_{t=0}^{T} \lambda_t \left(\left(f(k_t, c_t) - k_{t+1} \right) \right. \tag{1.2}
$$

with first order conditions

$$
\begin{aligned}
&u_c(k_t, c_t, t) + \lambda_t f_c(k_t, c_t) = 0, \\
&u_k(k_t, c_t, t) + \lambda_t f_c(k_t, c_t) - \lambda_{t-1} = 0.
\end{aligned}
\tag{1.3}
$$

This works if T is finite.

An alternative way is to recognize that in a problem like this, each sub-section of the path must be optimal in itself. This means that the problem has a recursive formulation. For example, it the problem is over three periods, we can rewrite (1.1)

$$
\begin{aligned}
&\max_{c_0, k_1|_{k_0}} \left(u(k_0, c_0, 0) + \max_{c_1, k_2|_{k_1}} \left(u(k_1, c_1, 1) + \max_{c_2, k_3|_{k_2}} \left(u(k_2, c_2, 2) \right) \right) \right) \\
&s.t. \qquad k_0 = \bar{k}_0, \\
&\qquad\qquad k_{t+1} = f(k_t, c_t),\ t = 0, \ldots, 2, \\
&\qquad\qquad k_3 = 0
\end{aligned}
\tag{1.4}
$$

We then solve the problem backwards starting from the last period. In period T-1 the remaining problem only depends on earlier actions through k_{T-1}. In the final period, the problem is trivial; simply set c_2 so that k_3 =0. In period T-1, we then want to solve

$$\begin{aligned}
&\max_{k_2,c_2,c_1} u\left(k_1,c_1,1\right)+u\left(k_2,c_2,2\right) \\
&s.t. \\
&k_2 = f(k_1,c_1) \\
&0 = f(k_2,c_2)
\end{aligned} \tag{1.5}$$

We can here use the constraints to substitute

$$\max_{c_1} u\left(k_1,c_1,1\right)+u\left(f(k_1,c_1), f^{-1}(f(k_1,c_1),0),2\right) \tag{1.6}$$

where $f^{-1}(f(k_1,c_1),0)=f^{-1}(k_2,0)$ gives the value of c_2 that is consistent with k_3=0, given $k_2 = f(k_1,c_1)$. Clearly, the solution to (1.6) depends on k_1. Furthermore, the achieved maximized value of (1.6) certainly also depends on k_1. This means that we need to find the *function* $c_1 = c_1(k_1)$, i.e., for all possible values of k_1. Given this, we can define

$$\begin{aligned}
V(k_1,2) &\equiv \max_{c_1} u\left(k_1,c_1,1\right)+u\left(f(k_1,c_1), f^{-1}(f(k_1,c_1),0),2\right), \\
&\equiv u\left(k,c_1(k_1),1\right)+u\left(f(k_1,c_1(k_1)), f^{-1}(f(k_1,c_1(k_1)),0),2\right).
\end{aligned} \tag{1.7}$$

This is the *maximum* value of the objective that can be achieved with two periods left and the the state variable being k.

This simplifies the problem in period 1 to

$$\begin{aligned}
&\max_{c_0}\left(u(k_0,c_0,0)+V(k_1,2)\right), \\
&s.t. k_1 = f(c_0,k_0),
\end{aligned} \tag{1.8}$$

or

$$\max_{c_0}\left(u(k_0,c_0,0)+V(f(c_0,k_0),2)\right), \tag{1.9}$$

with one first order condition

$$u_c(k_0,c_0,0)+V_k\left(f(c_0,k_0),2\right) f_c(c_0,k_0)=0. \tag{1.10}$$

The equations in (1.8) and (1.9) are called *Bellman equations*. It is of course straightforward to extend the analysis to any finite horizon problem.

Time consistency

As we have seen, a dynamic problem can sometimes be separated in sub-problems, where the solution to each sub-problem is optimal in itself (compare to sub-game perfection, if you know basic game theory). If the objective function changes over time, this property may no longer hold. Consider the following problem where individuals discount future utility at a faster rate for close dates (as shown to be consistent with empirical psychological evidence)

$$u(c_t)+\beta\sum_{s=1}^{T}u(c_{t+s}), \tag{1.11}$$

with $0<\beta<1$. Then, in period $t+1$ the objective is

$$u(c_{t+1})+\beta\sum_{s=1}^{T}u(c_{t+s+1}), \tag{1.12}$$

which is not a sub problem of (1.11). In particular, the marginal rate of substitution between $t+1$ and $t+2$ is $u'(c_{t+1})/u'(c_{t+2})$ in period t, but is it $u'(c_{t+1})/\beta u'(c_{t+2})$ in period t+1.

Take a three period example with exponential utility. Then if the individual could control all future consumption levels (commitment) in period 1, the individual maximizes

$$\begin{aligned}
&-e^{-c_1}-\beta e^{-c_2}-\beta e^{-c_3},\\
&s.t\, c_1+c_2+c_3=A\\
&\Rightarrow\;\; c_{c_1}=\frac{A-2\ln\beta}{3},\\
&\qquad c_2=c_3=\frac{A-c_1}{2}=\frac{A+\ln\beta}{3}
\end{aligned} \tag{1.13}$$

However, this is not time consistent since in period 2 the individual would solve

$$\begin{aligned}
&e^{-c_2}-\beta e^{-c_3},\\
&s.t\, c_2+c_3=A-c_1\\
&\Rightarrow c_2=\frac{A-c_1-\ln\beta}{2}>\frac{A-c_1}{2}
\end{aligned} \tag{1.14}$$

To get the time-consistent solution, we solve the problem under the constraint that second period consumption is given by $(A-c_1-\ln\beta)/2$

$$\begin{aligned}
&-e^{-c_1}-\beta e^{-c_2}-\beta e^{-c_3},\\
&s.t\, c_1+c_2+c_3=A\\
&c_2=(A-c_1-\ln\beta)/2\\
&\Rightarrow\;\; c_1=\frac{A-\ln\beta-2(\ln(1+\beta)-\ln 2)}{3}.
\end{aligned} \tag{1.15}$$

With *geometric* discounting, the objective function does change over time, but only by a linear transformation which does not affect the optimal solution. Suppose the objective function in period t is

$$\sum_{s=0}^{T} \beta^s u(c_{t+s}) \tag{1.16}$$

which can be separated in

$$\sum_{s=0}^{a-1} \beta^s u(c_{t+s}) + \sum_{s=a}^{T} \beta^s u(c_{t+s}) . \tag{1.17}$$

At period a, the latter term has changed, but only by a linear transformation since it is now

$$\sum_{s=0}^{T} \beta^s u(c_{t+s+a}) = \beta^{-\alpha} \sum_{s=a}^{T} \beta^s u(c_{t+s}), \tag{1.18}$$

which has the same solution. Thus, if we let the agent re-optimize in each period, he will not change his mind, if discounting is *geometric*.

Infinite Horizon

In an infinite horizon problem we cannot use the method of starting from the last period. Still, if the problem has a well-defined value function, it satisfies the Bellman equation. Furthermore, under conditions, which we will talk about later, there is only one function that solves the Bellman equation, so if we find one function that solves the Bellman equation, we have a solution to the dynamic optimization problem. Since geometric discounting will prove to be important for showing uniqueness, we will use that from now on.

To find a solution, we can use two different approaches.

1. Guess on a value function and make sure it satisfies the Bellman equation.

2. Iterate on the Bellman equation until it converges.

Guessing

Guessing is often feasible when the problem is autonomous (stationary). Then, the problem is independent of time in the sense that given in initial condition on the state variable(s), the solution and the maximized objective is independent of the starting date. This, requires that time is infinite, the law of motion for the state is independent of time and the per-period return function is the same over time, and that any restriction on the control variable is the same over time. (Think about what would happen if any of these conditions is not satisfied). Then, the value function is independent of time so we can write.

$$\begin{aligned} & V(k_t) = \max_{c_t} U(k_t, c_t) + \beta V(k_{t+1}) \\ & s.t. k_{t+1} = f(k_t, c_t). \end{aligned} \tag{1.19}$$

We can rewrite

$$V(k_t) = \max_{c_t} U(k_t, u_t) + \beta V(f(k_t, c_t)) \tag{1.20}$$

with first order conditions

$$U_c(k_t, u_t) + \beta V'(k_{t+1}) f_c(k_t, u_t) = 0. \tag{1.21}$$

Suppose we find a solution to (1.20) (and to (1.21) if the optimum is interior). This has to be a function $c(k_t)$, which is time-independent since *U, V* and *f* are . Plugging that into (1.20) we get rid of the max so we have

$$V(k_t) = U\left(k_t, c(k_t)\right) + \beta V\left(f(k_t, c(k_t))\right) \tag{1.22}$$

If (1.22) is satisfied *for all values of k* we have a solution to the value function, otherwise our guess was incorrect.

Note that the whole RHS of (1.19) is a functional of the unknown function *V*(.) for the given functions *U* and *g*. We can define this functional as *T*(*V*). The Bellman equation then defines a *fixed point* for *T* in the space of *functions V.* The Bellman equation can thus be written

$$V(k) = T(V(k)) \equiv \max_{c} U\left(k, c\right) + \beta V\left(f(k, c)\right) \tag{1.23}$$

The solution to the Bellman equation is thus a fixed point *in the space of functions* we are looking for, not a fixed point for *k.* I.e., if we plug in some function of *k* in the RHS of (1.22) we must get out the same function on the LHS. We will return this when we analyse the conditions under which we know that one and just one such fixed point exists (Contraction mapping theorem).

Typically the value function is of a similar for to the objective function. This is intuitive in the light of (1.22). For example if the utility function in (1.1) is logarithmic we guess that the value function is of the form $A \ln k + B$ for some constants *A,B*. For HARA utility functions (e.g., CRRA, CARA and quadratic) the value functions are generally of the same type as the utility function (Merton, 1971).

An example

In (1.1) let $U(k,c,t)=\ln(c)$ and $f(k,c,t)= k^{\alpha} - c$, with $0<\alpha<1$. We then have

$$\begin{aligned} &V\left(k_t\right) = \max_{c_t, k_{t+1}} U\left(c_t\right) + \beta V\left(k_{t+1}\right) \\ &s.t\; k_{t+1} = k_t^{\alpha} - c_t \\ &\Rightarrow V\left(k_t\right) = \max_{k_{t+1}} U\left(k_t^{\alpha} - k_{t+1}\right) + \beta V\left(k_{t+1}\right) \end{aligned} \tag{1.24}$$

Now, guess that *V* is of the same form as *U,* for example $A \ln k + B$, giving first order conditions

$$
\begin{aligned}
&U'(c_t) = \beta V'\left(k_t^\alpha - c_t\right) \\
&\frac{1}{c_t} = \beta A \frac{1}{k_t^\alpha - c_t} \Rightarrow c_t = \frac{1}{1+A\beta} k_t^\alpha \equiv c(k_t), k_{t+1} = \frac{A\beta}{1+A\beta} k_t^\alpha
\end{aligned} \tag{1.25}
$$

Plugging this into the Bellman equation yields

$$
\begin{aligned}
A \ln k_t + B &= \ln\left(\frac{1}{1+A\beta} k_t^\alpha\right) + \beta\left(A \ln \frac{A\beta}{1+A\beta} k_t^\alpha + B\right) \\
&= \alpha \ln k_t + \ln \frac{1}{1+A\beta} + \alpha\beta A \ln k_t + \beta A \ln \frac{A\beta}{1+A\beta} + \beta B \\
&= (\alpha + \alpha\beta A) \ln k_t + \ln \frac{1}{1+A\beta} + \beta A \ln \frac{A\beta}{1+A\beta} + \beta B.
\end{aligned} \tag{1.26}
$$

We see immediately that for this to be identically true for all values of k, we must have.

$$
\begin{aligned}
&A = (\alpha + \alpha\beta A) \\
&B = \ln \frac{1}{1+A\beta} + \beta A \ln \frac{A\beta}{1+A\beta} + \beta B
\end{aligned} \tag{1.27}
$$

This system is straightforward to solve, giving

$$
\begin{aligned}
&A = \frac{\alpha}{1-\alpha\beta} \\
&B = \frac{\ln(1-\alpha\beta)}{1-\beta} + \frac{\alpha\beta \ln \alpha\beta}{(1-\beta)(1-\alpha\beta)} \\
&\Rightarrow V(k) = \frac{\alpha}{1-\alpha\beta} \ln(k) + \frac{\ln(1-\alpha\beta)}{1-\beta} + \frac{\alpha\beta \ln \alpha\beta}{(1-\beta)(1-\alpha\beta)}.
\end{aligned} \tag{1.28}
$$

Having V, it is easy to find the optimal control from (1.25),

$$
c_t = \frac{1}{1+A\beta} k_t^\alpha = (1-\alpha\beta) k_t^\alpha. \tag{1.29}
$$

Iteration

An alternative way is try to find the limit of finite horizon Bellman equation as the horizon goes to infinity. Under for economical purposes quite general conditions this limit exists and is equal to the value function for the infinite horizon problem. Let $V_s(k)$ be the value function of the finite problem with s periods left. Then we try to find

$$
\lim_{s\to\infty} V_s(k) \equiv V(k) \tag{1.30}
$$

This method is usually done numerically, but it can (at some cost of messiness) be done also analytically. Using the notation in (1.23)

$$
\begin{aligned}
&V_{s+1}(k) = TV_s(k) \\
&V_{s+n}(k) = T^n V_s(k) \\
&\lim_{n\to\infty} V_n(k) = \lim_{n\to\infty} T^n V_0(k) = V(k).
\end{aligned} \tag{1.31}
$$

If the limit exists, it clearly satisfies the Bellman equation

$$
\begin{aligned}
&V(k) = T(V(k)) \\
&\lim_n T^n V(k) = T \lim_n T^n V(k) \\
&\lim_n T^n V(k) = \lim_n T^{n+1} V(k)
\end{aligned} \tag{1.32}
$$

The remaining issue is what function to plug in as $V_0(k)$ in (1.31). However, suppose that T discounts, i.e., β in (1.23) is strictly smaller than zero. Then, if we can show that

$$
\lim_n \beta^n V(k_n) = 0 \tag{1.33}
$$

and

$$
\lim_n \beta^n V_0(k_n) = 0 \tag{1.34}
$$

for all *permissible* values of k given relevant initial conditions, then it turns out that it does not matter what function V_0 we use. More importantly; if (1.33) holds, $\lim_{n\to\infty} T^n V_0(k)$ provides the *unique* solution to the Bellman equation so we have found the correct value function. A particularly simple case would be if U and thus V are bounded.

The iteration in (1.31) can easily be done numerically, either by specifying a functional form, if we know that, or by just choosing a grid. In the latter case we just a set of values for the state variable $\{k_0, k_1, \ldots, k_n\}$. $V_0(k)$ is then a set of preliminary values (numbers) for each of the state variables in the grid. We can also sometimes do it analytically.

An Iteration Example

Suppose the final condition in the finite horizon analogue of the problem behind (1.24) that $k_{T+1}=0$. Then, with one period to go (in T-1), we have

$$
\begin{aligned}
&V_1(k_{T-1}) = \max_{k_T} \ln\left(k_{T-1}^{\alpha} - k_T\right) + \beta \ln k_T^{\alpha} \\
&FOC \quad \frac{1}{\left(k_{T-1}^{\alpha} - k_T\right)} = \frac{\beta\alpha}{k_T} \Rightarrow k_T = \frac{\beta\alpha k_{T-1}^{\alpha}}{1+\alpha\beta}
\end{aligned} \tag{1.35}
$$

Substitute into the value function

$$V_1(k_{T-1}) = \ln\left(\frac{1}{1+\alpha\beta}k_{T-1}^{\alpha}\right) + \beta\ln\left(\frac{k_{T-1}^{\alpha}}{1+\alpha\beta}\right)^{\alpha}$$
$$= -(1+\alpha\beta)\ln(1+\alpha\beta) + \alpha(1+\alpha\beta)\ln k_{T-1} \tag{1.36}$$

Then

$$V_2(k_{T-2}) = \max_{k_{T-1}} \ln\left(k_{T-2}^{\alpha} - k_{T-1}\right) + \beta V_1(k_{T-1})$$
$$FOC \quad \frac{1}{k_{T-2}^{\alpha} - k_{T-1}} = \beta V_1'(k_{T-1}) = \beta\alpha(1+\alpha\beta)\frac{1}{k_{T-1}} \tag{1.37}$$
$$\Rightarrow k_{T-1} = \frac{\beta\alpha(1+\alpha\beta)}{1+\beta\alpha(1+\alpha\beta)}k_{T-2}^{\alpha}$$

So

$$V_2(k_{T-2}) = \ln\left(\frac{k_{T-2}^{\alpha}}{1+\beta\alpha(1+\alpha\beta)}\right) + \beta V_1\left(\frac{\beta\alpha(1+\alpha\beta)}{1+\beta\alpha(1+\alpha\beta)}k_{T-2}^{\alpha}\right)$$
$$= \alpha\left(1+\alpha\beta+\alpha^2\beta^2\right)\ln k_{T-2}$$
$$+\left(\beta+\alpha\beta+\alpha^2\beta^2\right)\ln\alpha\beta \tag{1.38}$$
$$+(\alpha\beta+\alpha^2\beta^2-\beta-\beta^2)\ln(1+\alpha\beta)$$
$$-(1+\alpha\beta+\alpha^2\beta^2)\ln(1+\alpha\beta+\alpha^2\beta^2).$$

You might be able to see that the coefficient on k is a power series that converge to $\alpha/(1-\alpha\beta)$ when the horizon goes to infinity (provided $\alpha\beta$ <1). Also the constants converge if also 0<β <1 and the resulting function is

$$\lim_{s\to\infty} V_s(k_{T-s})$$
$$\equiv V(k) = \frac{\alpha}{1-\alpha\beta}\ln k + \frac{\ln(1-\alpha\beta)}{1-\beta} + \frac{\alpha\beta\ln\alpha\beta}{(1-\beta)(1-\alpha\beta)} \tag{1.39}$$
$$= A\ln k + B.$$

Note that the policy function is a stable difference equation under the assumptions about α,β.

An envelope result

We will later have use for the envelope result, that we can evaluate $V'(k)$ as the partial derivative holding u constant, i.e.,

$$
\begin{aligned}
V'(k_t) &= U_k(k_t,u_t) + \frac{du_t}{dk_t}\left(\overbrace{U_c(k_t,c_t) + \beta V'(k_{t+1}) f(k_t,c_t)}^{=0} \right) \\
&+ \beta V'(f(k,c)) f_k(k,c) \\
&= U_k(k_t,c_t) + \beta V'(f(k,c)) f_k(k,c)
\end{aligned} \quad . \tag{1.40}
$$

Combining this with (1.21) gives

$$
V'(k_t) = U_k(k_t,c_t) - U_c(k_t,u_t)\frac{f_k(k_t,c_t)}{f_c(k_t,u_t)}. \tag{1.41}
$$

So, if, for example, $U(k,c,t) = U(c)$ and $f(k,c,t) = f(k) - c$.

$$
V'(k_t) = U_c(k_t,u_t) f_k(k_t). \tag{1.42}
$$

In other words, along the optimal path, we can evaluate $V'(k)$ by looking at what happens to the objective function an additional unit of k is "consumed" today and all future values of the state variable are unchanged.

State Variables

We often solve the dynamics programming problem by guessing a form of the value function. The first thing to determine is then which variables should enter, i.e., which variables are the state variables. The state variables must satisfy both following conditions

1. To enter the value function at time they must be realized at t.

   Note, however, that it sometimes may be convenient to use $E_t\,(z_{t+s})$ as a state variable. The expectation as of t is certainly realized at t even if the stochastic variable is not realized.

2. The set of variables chosen as state variables must together give sufficient information so that the value of the program from t and onwards when the optimal control is chosen can be calculated.

What do we need if the per period utility function in (1.1) was $U(c_t, c_{t-1})$?

Note, we should try to find the smallest such set. Look for example on the following problem.

$$
\begin{aligned}
&\max_{c,k} \sum_{t=0}^{T} \beta^t U(c_t) \\
&s.t. \qquad k_0 = \bar{k}_0, \\
&\qquad\qquad l_0 = \quad \bar{k}_0 \\
&\qquad\qquad k_{t+1} + l_{t+1} = f(k_t,l_t) - c_t,\ t = 0,\ldots,T, \\
&\qquad\qquad k_{T+1} = 0
\end{aligned} \tag{1.43}
$$

In general we need both k, and l in the value function but if f is linear we may only need a linear combination. If $f(k_t,l_t) = a(k_t + l_t)$ we could define a new state variable $w = k+l$ and use $V(w)$ as our value function. The reason is that to compute the value of the program we only need to know the sum of *k* and *l*, their share are superfluous information.

1.2. Stochastic Dynamic Programming

As long as the recursive structure of the problem is intact adding a stochastic element to the transition equation does not change the Bellman equation. Consider the problem

$$\begin{aligned}
&\max_{\{u_t\}_0^\infty} E\sum_{t=0}^{\infty}\beta^t r(k_t,u_t) \\
&s.t. \qquad k_0=\bar{k}_0, \\
&\qquad\qquad k_{t+1}=g(k_t,u_t,\varepsilon_{t+1}),\ \forall t\geq 0. \\
&\qquad\qquad \varepsilon_t=\begin{cases}\bar{\varepsilon}, & \text{with probability } p \\ \underline{\varepsilon}, & \text{with probability (1-}p\text{)}\end{cases}
\end{aligned} \tag{1.44}$$

where *E* is the expectations operator. Note that we have to specify the set of information that u_t can be conditioned on. Clearly it will in general be optimal to condition for example consumption on observed realizations of ε_t. If the agent may condition on information available at *t* we get the Bellman equation with first order conditions

$$\begin{aligned}
&V(k_t)\equiv\max_{u_t}\left\{r(k_t,u_t)+\beta\left[pV(g(k_t,u_t,\bar{\varepsilon}))+(1-p)V(g(k_t,u_t,\underline{\varepsilon}))\right]\right\} \\
&FOC \\
&r_u(k_t,u_t)+ \\
&\beta\left[pV'(g(k_t,u_t,\bar{\varepsilon}))g_u(k_t,u_t,\bar{\varepsilon})+(1-p)V'(g(k_t,u_t,\underline{\varepsilon}))g_u(k_t,u_t,\underline{\varepsilon})\right] \\
&=0
\end{aligned} \tag{1.45}$$

or for a general distribution *F* of ε

$$\begin{aligned}
&V(k_t)\equiv\max_{u_t}\left\{r(k_t,u_t)+\beta EV(g(k_t,u_t,\bar{\varepsilon}))\right\} \\
&FOC \quad r_u(k_t,u_t)+\beta E\left(V'(g(k_t,u_t,\varepsilon))g_u(k_t,u_t,\varepsilon)\right)=0
\end{aligned} \tag{1.46}$$

where *E* denotes the expectations operator. Note that $V(k_t)$ in (1.45) and (1.46) is a current value function.

A Stochastic Consumption Example

Consider the following program

$$
\begin{aligned}
&\max_{c,\omega} \sum_{t=0}^{\infty} \beta^t \ln c_t \\
&s.t. \qquad A_{t+1} = (A_t - c_t)\left((1+r)\omega + (1+z_t)(1-\omega)\right).
\end{aligned}
\tag{1.47}
$$

The consumer decides how much to consume each period. The share ω of here assets is placed in a riskless asset yielding r in return and (1-ω) in a risky asset with return z_t, that is i.i.d.

The problem is autonomous so we write the current value Bellman equation with time independent value function V

$$
V(A_t) = \max_{c_t,\omega}\left[\ln c_t + \beta E_t V\left((A_t - c_t)\left((1+r)\omega + (1+z_t)(1-\omega)\right)\right)\right]
\tag{1.48}
$$

Necessary first order conditions yield

$$
\begin{aligned}
&c_t; \qquad \frac{1}{c_t} - \beta E_t V'(A_{t+1})\left((1+r)\omega + (1+z_t)(1-\omega)\right) = 0, \\
&\omega_t; \qquad E_t V'(A_{t+1})(A_t - c_t)(r - z_t) = 0.
\end{aligned}
\tag{1.49}
$$

Now we use Merton's result and guess that the value function is

$$
V(A_t) = a \ln A_t + B
\tag{1.50}
$$

for some constants a and B. Substituting into (1.49) we get

$$
\begin{aligned}
&\frac{1}{c_t} = \beta E_t a \frac{1}{A_{t+1}}\left((1+r)\omega + (1+z_t)(1-\omega)\right) \\
&= \beta a \frac{1}{(A_t - c_t)} \Rightarrow c_t = \frac{A_t}{1 + a\beta}.
\end{aligned}
\tag{1.51}
$$

and

$$
\begin{aligned}
&E_t V'(A_{t+1})(A_t - c_t)(r - z_t) \\
&= E_t \frac{\cancel{(A_t - c_t)}(r - z_t)}{\cancel{(A_t - c_t)}\left((1+r)\omega + (1+z_t)(1-\omega)\right)} \\
&= E_t \frac{(r - z_t)}{\left((1+r)\omega + (1+z_t)(1-\omega)\right)} = 0.
\end{aligned}
\tag{1.52}
$$

Note that (1.52) implies that ω is constant since z_t is i.i.d.

Now we have to solve for the constant a. This is done by substituting the solutions to the first order conditions and the guess into the Bellman equations.

$$
\begin{aligned}
& a \ln A_t + B \\
& = \ln A_t - \ln(1+a\beta) + \beta E_t \left(a \ln A_{t+1} + B\right) \\
& = \ln A_t - \ln(1+a\beta) + \beta a \ln\left(A_t - c_t\right) + \beta B \\
& \quad + \beta a E_t \ln\left((1+r)\omega + (1+z_t)(1-\omega)\right) \\
& = \ln A_t - \ln(1+a\beta) + \beta a\left(\ln A_t + \ln a\beta - \ln(1+a\beta)\right) \\
& \quad + \beta B + \beta a E_t \ln\left((1+r)\omega + (1+z_t)(1-\omega)\right) \\
& = (1+a\beta)\ln A_t + k \\
& \Rightarrow a = \frac{1}{1-\beta}, \quad c_t = (1-\beta)A_t
\end{aligned}
\tag{1.53}
$$

1.3. Contraction mappings

In the previous section we discussed guessing on solutions to the Bellman equation. However, we would like to know whether there exists a solution and whether it is unique. If the latter is not the case, it is not in principle sufficient to guess and the verify the solution since we might have other value functions that also satisfy the Bellman equation. To prove existence and uniqueness we will apply a contraction mapping argument.

Complete Metric Spaces and Cauchy Sequences

Let X be a metric space, i.e., a set on which addition and scalar multiplication is defined. Also define an operator *d:* $X \times X \to \mathbf{R}$ which we can think of as measuring the (generalized) distance between any two elements of X. We call d a norm. It is assumed to satisfy

1. Positivity $d(x,y) \geq 0$, $d(x,y) = 0 \Leftrightarrow x = y$
2. Symmetry $d(x,y) = d(y,x)$ (1.54)
3. Triangle inequality $d(x,z) \leq d(x,y) + d(y,z)$

Now, we call (X,d) a normed vector space or a *metric space*. An example of such a space would be R^n together with the Euclidian norm $d(x,y) = \|x,y\|$. Another example is the space $C(S)$ of continuous and bounded functions where each element is a function from $S \subset \mathbf{R}^n \to \mathbf{R}$ together with the "sup-norm" defined as follows. For any two elements in $C(S)$, i.e., any two functions f and g, the distance d between them is the maximal euclidian distance, i.e,.,

$$
d(f,g) \equiv \sup_{y \in S} \|f(y), g(y)\| \tag{1.55}
$$

Now let us define a Cauchy sequence. This is a sequence of elements $\{x_n\}$ in a space X that come closer and closer to each other, using some particular norm. More precisely, for all ε>0, there exist a number n, such that for all $m,p \geq n$, $d(x_m,x_p) < \varepsilon$. An example of this would be the sequence $\{1,1/2,1/3,\ldots\}$ which is a Cauchy sequence using the Euclidian norm. A Cauchy sequence converges if there is an element in X such that $d(x_n,x)$ approach zero as n goes to infinity. It may, of course, be the case that the Cauchy sequence does not converge to a point in X. An eaxmple would be if we let X be the open interval $(0,\infty)$ and look at the Cauchy sequence $\{1,1/2,1/3,\ldots\}$ which converges to zero which is not in X.

Complete metric spaces

Now we are ready to define the complete metric space. This is a metric space where all Cauchy sequences in it are convergent, i.e., they converge to a point in the space.

Contraction Mapping

Consider the metric space (X,d) and look at the function T that maps each element in X to some element in X, $T: X \to X$. T is a contraction mapping if there exists a non-negative number ρ which is *strictly* smaller than unity, $0 \leq \rho < 1$, such that for all elements x, y in X,

$$d\left(T(x),T(y)\right) \leq \rho d\left(x,y\right) \tag{1.56}$$

An example of such a mapping whould we a map in say scale 1:10 000 put on top of a map in scale 1:1000 covering the same geographical area. The norm can be the distance between the points on the map. Clearly, (1.56) is satisfied for ρ =0.1.

The Contraction Mapping Theorem

Now we can state the very important contraction mapping theorem.

Result Consider a complete metric space, and let $T: X \to X$ be a contraction mapping. The T has one unique fixed point x, *i.e.*, $x=T(x)$.

Another very useful result is the following

Result Let S be a subset of $\mathbf{R}^n$ and $B(S)$ the space of all bounded functions from S to $\mathbf{R}$. Let T be a map that maps all elements of $B(S)$ into itself. Then, T is a contraction mapping if

1. For any functions $w(s)$ and $v(s)$ $w(s)-v(s) \geq 0, \forall s \in S \Rightarrow Tw(s)-Tv(s) \geq 0, \forall s \in S$, and

2. There is a non-negative $0 \leq \beta\, \beta$ strictly smaller than unity such that for any number c in $\mathbf{R}$, and any function w in $B(S)$, $T(w(s)+c) = T(w(s)) + \beta c, \forall s.$

Usually it is straightforward to apply the previous result to show that if we have positive discounting the Bellman equation is a contraction mapping. The only problem is that it is confined to bounded functions.