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ON THE TRANSVERSALITY CONDITION IN INFINITE HORIZON OPTIMAL PROBLEMS[1]

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There are many infinite horizon optimal problems in economic models. In such problems, the transversality condition may not be verified, as shown by Halkin's example. But we prove another property: the maximum of the Hamiltonian converges to zero when time goes to infinity. And, if along the optimal trajectory, after some time, changes of speed (by controls) in all directions at a given level are possible, then the transversality condition is verified. Examples show that the additional property proved here (i) allows exclusion of nonoptimal trajectories which verify the usual necessary conditions in an infinite horizon; (ii) is a result directly useful in some economic studies.

1. INTRODUCTION

IN FINITE HORIZON optimal problems without constraint on the final state, necessary conditions for optimality include the transversality condition: the final value of the shadow price-vector is zero. This means that one more unit of any good at final time gives no additional value to the criterion. Halkin's example [**8**] shows that this property is not necessarily true in an infinite horizon. In an infinite horizon, one more unit of a good, at any time, changes the whole future, and the zero value of the state becomes a limit property which is not necessarily verified.

Nevertheless the transversality condition in an infinite horizon is an important property: it intervenes in sufficient conditions for optimality [**1**] and in stability studies [**6**]. Moreover, it is generally verified in economic models. In the literature, studies of the infinite horizon transversality property have been made, which only give results in special cases: cases of linear evolution equations [**2**, **3**], cases of boundedness conditions involving "fast convergence" of the criterion [**9**, **10**], ... The concave case when using variational calculus has been studied in detail in [**4**].

We study a general discounted problem (Section 2) and we obtain the property of zero limit of the maximum of the Hamiltonian without any particular assumption; this may be obtained with the Hamilton-Jacobi equation but requires differentiability of the value function. In Section 3, we show that this property implies the transversality condition in the case where enough possibilities of changing the state's speed exist indefinitely, which means a long-run condition of tradeoff between the effects of the control on the criterion and on the state.

In Section 4 examples of application are given. Proofs are in the Appendix.

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2. NECESSARY CONDITIONS FOR OPTIMALITY IN INFINITE HORIZON PROBLEMS

A simple optimization problem with an infinite horizon is the following. *Maximize*

$$(1) \qquad \int_0^{\infty} e^{-rt} g(X(t), C(t))\, dt$$

subject to

$$(2) \qquad \dot{X}(t) = f(X(t), C(t)), \qquad \text{and} \qquad X(0) = X_0.$$

The state $X(t)$ belongs to an open subset E of R^n, and the control $C(t)$ is a piecewise continuous function valued in some topological space B. Functions f and g defined in $E \times B$ are valued in R^n and R respectively; we assume that f and g are continuous and continuously differentiable with respect to the state variable. The differentials are denoted by f_X and g_X.

The study will be limited to this simple problem. But there is no more difficulty in studying the case with additional constraints such as $h(X(t), C(t)) = 0$ and/or ≥ 0. Then the necessary conditions are valid in the subset of controls which verify the qualification condition (see [**13**]) and the additional constraints with the optimal state $\bar{X}(t)$.

DEFINITIONS: A trajectory $(X(t), C(t))$, $0 \leq t < \infty$ is *admissible* if $X(t)$ is a solution of equation (2) with control $C(t)$ on $0 \leq t < \infty$ and if integral (1) converges. A trajectory $(\bar{X}(t), \bar{C}(t))$ is an *optimal solution* of problem (1), (2) if it is admissible and it is optimal in the set of admissible trajectories, i.e., for any admissible trajectory $(X(t), C(t))$, the value of integral (1) is not greater than its value corresponding to $(\bar{X}(t), \bar{C}(t))$.

We now consider an optimal solution $(\bar{X}(t), \bar{C}(t))$, and a fixed time $T > 0$. Let us define

$$(3) \qquad h(x) = e^{-rx} \int_T^{\infty} e^{-rt} g\big(\bar{X}(t), \bar{C}(t)\big)\, dt = \int_{T+x}^{\infty} e^{-rt} g\big(\bar{X}(t-x), \bar{C}(t-x)\big)\, dt.$$

The following problem (P_T) will be considered, with state (Y, z) belonging to $E \times R$, and control (U, v) belonging to $B \times (1/2, \infty)$. Maximize:

$$(4) \qquad \int_0^T v(t) e^{-rz(t)} g(Y(t), U(t))\, dt + h(z(T) - T)$$

subject to

$$(5) \qquad \begin{cases} \dot{Y}(t) = v(t) f(Y(t), U(t)), & Y(0) = X_0 \quad \text{and} \quad Y(T) = \bar{X}(T), \\ \dot{z}(t) = v(t), & \text{and} \quad z(0) = 0. \end{cases}$$

LEMMA: *State* $(\overline{Y}(t), \overline{z}(t)) = (\overline{X}(t), t)$ *and control* $(\overline{U}(t), \overline{v}(t)) = (\overline{C}(t), 1)$, $0 \leq t \leq T$, *constitute an optimal solution of problem* (P_T).

PROOF: See the Appendix.

THEOREM: *A necessary condition for* $(\overline{X}(t), \overline{C}(t))$, $0 \leq t < \infty$, *to be an optimal solution of problem* (1, 2) *is that there exist a real number* a, *a vector* A *of* R^n, *and continuous functions* $P(t)$ *and* $q(t)$ *valued in* R^n *and* R *respectively such that* (i) (a, A) *is not zero, and* a *is nonnegative*; (ii) $P(t)$ *is the solution of*

$$(6) \qquad \begin{cases} \dot{P}(t) = -ae^{-rt}g_X\big(\overline{X}(t), \overline{C}(t)\big) - P(t) \cdot f_X\big(\overline{X}(t), \overline{C}(t)\big), \\ P(0) = A; \end{cases}$$

(iii) $q(t)$ *is the solution of*

$$(7) \qquad \begin{cases} \dot{q}(t) = rae^{-rt}g\big(\overline{X}(t), \overline{C}(t)\big), \\ limit_{t \to \infty} q(t) = 0; \end{cases}$$

(iv) *for each* t *at which* $\overline{C}(t)$ *is continuous, the Hamiltonian*

$$ae^{-rt}g\big(\overline{X}(t), c\big) + P(t) \cdot f\big(\overline{X}(t), c\big)$$

is maximum on the set B *at* $c = \overline{C}(t)$; (v) *the maximum of the Hamiltonian verifies for every* t

$$(8) \qquad M(t) = ae^{-rt}g\big(\overline{X}(t), \overline{C}(t)\big) + P(t) \cdot f\big(\overline{X}(t), \overline{C}(t)\big) = -q(t).$$

PROOF: See the Appendix.

REMARKS: In addition to the usual conclusions, the theorem gives at every time t_0 the value $M(t_0)$ of the maximum of the Hamiltonian which is equal to:

$$(9) \qquad -q(t_0) = ra\int_{t_0}^{\infty} e^{-rt}g\big(\overline{X}(t), \overline{C}(t)\big)\,dt.$$

The value function of problem (1), (2):

$$(10) \qquad V(X_0, t_0) = \sup \int_{t_0}^{\infty} e^{-rt}g(X(t), C(t))\,dt$$

(the upper bound is taken on the set of admissible trajectories such that $X(t_0) = X_0$) verifies:

$$V\big(\overline{X}(t_0), t_0\big) = \int_{t_0}^{\infty} e^{-rt}g\big(\overline{X}(t), \overline{C}(t)\big)\,dt = e^{-rt_0}V\big(\overline{X}(t_0), 0\big),$$

$$\frac{\partial V}{\partial t}\big(\overline{X}(t_0), t_0\big) = -r\int_{t_0}^{\infty} e^{-rt}g\big(\overline{X}(t), \overline{C}(t)\big)\,dt.$$

The Hamilton-Jacobi equation

$$\frac{\partial V}{\partial t}\left(\bar{X}(t),t\right) + e^{-rt}g\left(\bar{X}(t),\bar{C}(t)\right) + \frac{\partial V}{\partial X}\left(\bar{X}(t),t\right)\cdot f\left(\bar{X}(t),\bar{C}(t)\right) = 0 \tag{11}$$

together with $\partial V/\partial X(\bar{X}(t),t) = (1/a)P(t)$, implies relation (8). But, as is well known, the assumption that V is C^1 does not hold in the general case studied here.

One may also remark that conclusion (v) of the theorem is equivalent to the property that the *limit of the maximum* $M(t)$ *of the Hamiltonian is zero when* t *goes to infinity*. The existence of the limit of $M(t)$ results from the equality between partial and total derivatives of the Hamiltonian with respect to t [**13**]:

$$\frac{dM(t)}{dt} = -rae^{-rt}g\left(\bar{X}(t),\bar{C}(t)\right),$$

$$M(t_1) = M(t_0) - ra\int_{t_0}^{t_1} e^{-rt}g\left(\bar{X}(t),\bar{C}(t)\right)dt,$$

$$\operatorname*{limit}_{t\to\infty} M(t) = M(t_0) + q(t_0).$$

Our theorem shows that this limit is 0, which extends to the infinite horizon case the property $M(T) = 0$ valid in a finite horizon optimal problem with free terminal time.

Note finally that the conclusion: $a \neq 0$ (allowing the choice of $a = 1$), which is always assumed in applications, cannot be obtained without an additional assumption, as shown in the following example.

EXAMPLE: Maximize $\int_0^\infty e^{-t}(2x(t) + c(t))\,dt$ subject to

$$\dot{x}(t) = 2x(t) + c(t), \qquad \text{and} \qquad x(0) = 0; \qquad c(t) \le 0.$$

The optimal solution is $\bar{c}(t) = 0, \bar{x}(t) = 0$. Any other control gives a negative value of the criterion. The Hamiltonian $H = ae^{-t}(2x + c) + p(t)(2x + c)$ is maximum at $c = 0$; this implies $p(t) + ae^{-t} \ge 0$. An easy calculation gives: $p(t) = -2ae^{-t} + be^{-2t}$, with $b = p(0) + 2a$. And the condition

$$-a + be^{-t} = e^{t}\left(p(t) + ae^{-t}\right) \ge 0$$

for every t, implies $a \le 0$. Then a nonnegative is necessarily equal to zero.

3. APPLICATION TO THE TRANSVERSALITY CONDITION

The usual formulation of the transversality condition refers to the property: $\text{limit}\, P(t)\cdot X(t) = 0$. More precisely to obtain sufficient conditions for optimality, one needs the following condition [**1**]:

$$\operatorname*{limit}_{t\to\infty} P(t)\cdot\bar{X}(t) - P(t)\cdot X(t) \le 0 \tag{13}$$

for any admissible trajectory $X(\cdot)$. In the case where every solution of (2) is bounded, the following condition is sufficient:

(14) $$\operatorname*{limit}_{t\to\infty} P(t) = 0.$$

This transversality condition is the simplest transcription of the finite horizon transversality condition: $P(T) = 0$, which holds for an optimal problem with free terminal state.

Often in economic models g is nonnegative, and if not, it is generally possible to modify it in such a way as to obtain a nonnegative g. Then, *to obtain the transversality property, it is sufficient that the set of admissible speeds* $f(\overline{X}(t), c)$, *for all controls* c *in* B, *contains enough possibilities*. Formally, we obtain the following property.

COROLLARY: *Assume* g *is nonnegative and there exists a neighborhood* V *of* 0 *in* R^n *which is contained in the set of the possible speeds* $f(\overline{X}(t), c)$ *for* $c \in B$, *this for all* t *large enough. Then, an optimal solution in infinite horizon verifies, in addition to the conclusions of the theorem, the transversality condition* (14).

PROOF: See the Appendix.

REMARK: Without the assumption: nonnegative g, the transversality condition is verified if the set of speeds $f(\overline{X}(t), c)$ contains a neighborhood of 0 in R^n, for c belonging to a subset B' of B such that the infimum limit of $e^{-rt}g(\overline{X}(t), c)$ is nonnegative. For example, if $r > 0$, it is possible to consider a set B' in which $g(\overline{X}(t), c)$ is bounded from below. The assumption of the corollary implies that the set of speeds has a nonempty interior in R^n, which is an important restriction. This restriction being made, the significance of the corollary's assumption is that the optimal state is such that there always exist controls allowing changes of the speed in all directions at a given level. This can be interpreted as an "equilibrium" condition in the sense that tradeoffs between the effects of the control on the criterion and on the state have to be made indefinitely during the time.

4. EXAMPLES OF APPLICATIONS

EXAMPLE 1—*Halkin's example* [**8**]: Maximize

(15) $$\int_0^{\infty} (1 - x(t))c(t)\,dt$$

subject to

(16) $$\dot{x}(t) = (1 - x(t))c(t), \qquad \text{and} \qquad x(0) = 0,$$

the control $c(t)$ belonging to an interval (α, β), $\alpha < 1 < \beta$. The criterion is equal to $x(\infty)$; and because the solution of equation (16) is

$$x(t) = 1 - \exp\left(-\int_0^t c(u)\,du\right), \tag{17}$$

any control such that $\int_0^\infty c(t)\,dt = \infty$ is optimal. Choosing $\bar{c}(t) = 1$, the maximum of the Hamiltonian

$$a(1 - \bar{x}(t))c + p(t)(1 - \bar{x}(t))c$$

is reached at $c = 1$ only if $p(t) = -a$, and then $a \neq 0$. In this example, $\bar{x}(t)$ converges to 1, and the set of possible speeds $\{(1 - \bar{x}(t))c \mid \alpha < c < \beta\}$ contains a given neighborhood of 0 for t large enough, only if $\alpha = -\infty$ and $\beta = +\infty$; and in this case, the assumption $g \geq 0$ of the corollary is not verified.

EXAMPLE 2—*The one-sector optimal growth model* [**5, 12, 14**]: In its reduced form (variables per unit of labor), the problem is to maximize the discounted sum of the utility of consumption $c = (1 - s)f(k)$,

$$\int_0^\infty e^{-rt} U((1 - s(t))f(k(t)))\,dt, \tag{18}$$

subject to

$$\dot{k}(t) = s(t)f(k(t)) - dk(t), \qquad \text{and} \qquad k(0) = k_0, \tag{19}$$

where the rate of saving $s(t)$ belongs to $[0, 1]$. The utility function U and the production function f of the capital stock k (per unit of labor) are assumed to satisfy the usual concavity condition. For $r > 0$, the optimal solution $\bar{k}(t)$ converges to the modified golden rule k^* defined by $f'(k^*) = d + r$, and the limit of the set of possible speeds is the interval $[-dk^*, f(k^*) - dk^*]$ which is a neighborhood of 0. By continuity, the assumption of the corollary is verified and the zero limit of the shadow price of the capital is verified as well known. *The trajectory* $s(t) \equiv 0$, $k(t) = k_0 e^{-dt}$ *also verifies the usual necessary conditions in an infinite horizon, but it does not verify the new property obtained.* Let us prove this.

The function $p(t)k(t)$ is decreasing for $k(t) = k_0 e^{-dt}$ because its derivative $k_0 e^{-dt}(\dot{p} - dp)$ is negative. On the other hand, $p(t)$ becomes negative. This implies that the limit of the Hamiltonian which is the limit of the positive increasing function $-dp(t)k(t)$, for $t > T$, is not zero. Conclusion (v) of the theorem is not verified.

In the case $r = 0$, the integral (18) is not converging. To solve the problem, we consider the difference

$$\int_0^\infty [U((1 - s(t))f(k(t))) - U(c^*)]\,dt, \tag{20}$$

c^* being the stationary consumption level $f(k^*) - dk^*$ corresponding to the golden rule $f'(k^*) = d$. In this case, the shadow price of capital does not converge to zero, but it converges to $U'(c^*)$. It is the assumption $g \geq 0$ of the corollary which is not verified, but *conclusion* (v) *of the theorem implies then that* $\dot{k}(t)$ *converges to zero.*

EXAMPLE 3—*Optimal growth with a convex-concave production function* **[15]**: In his study of optimal growth with a nonconvex production function, A. K. Skiba uses the following property:

$$(21)\qquad \int_0^{\infty} e^{-rt} g\big(\overline{X}(t), \overline{C}(t)\big)\,dt = \frac{1}{r}\Big[g\big(\overline{X}(0), \overline{C}(0)\big) + P(0)\cdot f\big(\overline{X}(0), \overline{C}(0)\big)\Big].$$

To prove this property, he assumes that

$$(22)\qquad \operatorname*{limit}_{t\to\infty} P(t)\cdot F\big(\overline{X}(t), \overline{C}(t)\big) = 0.$$

Property (21) results from conclusion (v) of the theorem and is verified *without any assumption other than* $a \neq 0$. And the "transversality condition" (22) is verified if $e^{-rt}g(\overline{X}(t), \overline{C}(t))$ converges to 0 when t goes to infinity. With these assumptions (verified by Skiba's model), properties (21) and (22) are both equivalent to conclusion (v) of the theorem.

4. CONCLUSION

It is possible to obtain the transversality condition in infinite horizon optimal problems with particular assumptions like those of the corollary. But these assumptions are based on properties of the optimal trajectory to be determined. As a matter of fact, the transversality condition then simply is a consequence of the additional property obtained in the theorem. The way to apply the new result is simple: the application of the theorem selects trajectories. It is then easy to see if for this selected set, which is smaller than the usual one, the transversality condition is verified. The transversality condition becomes a "by-product" which can be used then, for example, for sufficiency conditions. On the other hand, the new result may be directly used to obtain an additional property of optimal solutions in economics models.

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APPENDIX

PROOF OF THE LEMMA: The state $(\overline{Y}(t), \bar{z}(t)) = (\overline{X}(t), t)$ and control $(\overline{U}(t), \bar{v}(t)) = (\overline{C}(t), 1)$ verify equations (5) because $\overline{X}(t)$ and $\overline{C}(t)$ verify equation (2). The corresponding value of integral (4) is:

$$\int_0^T e^{-rt} g\big(\overline{X}(t), \overline{C}(t)\big)\,dt + h(0) = \int_0^{\infty} e^{-rt} g\big(\overline{X}(t), \overline{C}(t)\big)\,dt.$$

For any admissible trajectory $(Y(t), z(t); U(t), v(t))$, $0 \leq t \leq T$ (i.e., which verifies (5)), the function

$$z(t) = \int_0^t v(s)\,ds$$

is continuous, strictly increasing, and reversible. Let

$$C(s) = U(z^{-1}(s)), \qquad X(s) = Y(z^{-1}(s)), \qquad \text{and} \qquad S = z(T).$$

We obtain the following equalities:

$$\int_0^T v(t)e^{-rz(t)}g(Y(t), U(t))\,dt$$

$$= \int_0^T \dot{z}(t)e^{-rz(t)}g(X(z(t)), C(z(t)))\,dt$$

$$= \int_0^S e^{-rs}g(X(s), C(s))\,ds;$$

$$X(z(t)) = Y(t) = X_0 + \int_0^t \dot{Y}(s)\,ds$$

$$= X_0 + \int_0^t \dot{z}(s)f(X(z(s)), C(z(s)))\,ds$$

$$= X_0 + \int_0^{z(t)} f(X(u), C(u))\,du;$$

$$X(s) = X_0 + \int_0^s f(X(u), C(u))\,du.$$

The last equality implies that function $X(t)$ is almost everywhere derivable with derivative $f(X(t), C(t))$. The final condition on $Y(t)$ gives:

$$X(S) = X(z(T)) = Y(T) = \overline{X}(T).$$

For $s \geq S$, trajectory $(X(s), C(s))$ is defined by:

$$X(s) = \overline{X}(s - S + T), \qquad \text{and} \qquad C(s) = \overline{C}(s - S + T).$$

This trajectory is continuous and it verifies equation (2): for $s \leq S$, this has been shown; for $s \geq S$, the last definition gives:

$$\dot{X}(s) = \dot{\overline{X}}(s - S + T) = f(\overline{X}(s - S + T), \overline{C}(s - S + T)),$$

$$\dot{X}(s) = f(X(s), C(s)).$$

On the other hand, definition (3) of h implies:

$$h(z(T) - T) = h(S - T)$$

$$= \int_S^\infty e^{-rt}g(\overline{X}(t - S + T), \overline{C}(t - S + T))\,dt$$

$$= \int_S^\infty e^{-rt}g(X(t), C(t))\,dt.$$

Consequently, the value of the criterion (4) of problem (P_T) is:

$$\int_0^T v(t)e^{-rz(t)}g(Y(t), U(t))\,dt + h(z(T) - T)$$

$$= \int_0^\infty e^{-rt}g(X(t), C(t))\,dt.$$

The trajectory $(X(t), C(t))$, $0 \leq t < \infty$, is admissible, and optimality of $(\bar{X}(t), \bar{C}(t))$ implies:

$$\int_0^\infty e^{-rt} g(X(t), C(t))\, dt \leq \int_0^\infty e^{-rt} g(\bar{X}(t), \bar{C}(t))\, dt.$$

The left term is equal to the value of criterion (4) corresponding to the trajectory $(Y(t), z(t); U(t), v(t))$, $0 \leq t \leq T$; and the right term is equal to its value corresponding to $(\bar{Y}(t), \bar{z}(t); \bar{U}(t), v(t))$. This shows that the latter is an optimal solution of problem (P_T). The proof of the lemma is complete.

PROOF OF THE THEOREM: Let us apply the usual necessary conditions without constraint qualification condition (see [**7** or **11**]) to problem (P_T). For this problem, the Hamiltonian is:

$$H_T = a_T v e^{-rz} g(Y, U) + v P_T \cdot f(Y, U) + q_T v.$$

The necessary conditions are: there exist a real number $a_T \geq 0$, a vector D_T belonging to R^n, and continuous functions $P_T(t)$ and $q_T(t)$ such that: $(a_T, D_T) \neq 0$,

$$\dot{P}_T(t) = -a_T e^{-rt} g_X(\bar{X}(t), \bar{C}(t)) - P_T(t) \cdot f_X(\bar{X}(t), \bar{C}(t)),$$

$$P_T(T) = D_T;$$

$$\dot{q}_T(t) = re^{-rt} a_T g(\bar{X}(t), \bar{C}(t)),$$

$$q_T(T) = a_T h_x(0) = -ra_T \int_T^\infty e^{-rt} g(\bar{X}(t), \bar{C}(t))\, dt.$$

The Hamiltonian is maximum with respect to (U, v) on the set $B \times (1/2, \infty)$ at $(\bar{C}(t), 1)$. The maximum with respect to v for $U = \bar{C}(t)$ gives:

$$a_T e^{-rt} g(\bar{X}(t), \bar{C}(t)) + P_T(t) \cdot f(\bar{X}(t), \bar{C}(t)) + q_T(t) = 0;$$

and the maximum with respect to U for $v = 1$ implies that:

$$a_T e^{-rt} g(\bar{X}(t), c) + P_T(t) \cdot f(\bar{X}(t), c)$$

is maximum on the set B at $c = \bar{C}(t)$.

The vector $A_T = P_T(0)$ verifies: $(a_T, A_T) \neq 0$. If not, then $a_T = 0$ and $P_T(0) = 0$; and $P_T(t)$ being solution of:

$$\dot{P}_T(t) = -P_T(t) \cdot f_X(\bar{X}(t), \bar{C}(t)),$$

this would imply $P_T(T) = 0$, and $(a_T, D_T) = 0$. There is no change in the conclusions if we multiply a_T, D_T, P_T, and q_T by some positive constant. And we choose such a constant which gives the norm of (a_T, A_T) equal to 1.

All these properties are true for every $T > 0$. All the (a_T, A_T) being of norm 1, there exists some sequence (a_{T_n}, A_{T_n}) which admits a limit $(a, A) \neq 0$ (conclusion (i) of the theorem). Let us define $P(t)$ and $q(t)$ by:

$$\begin{cases} \dot{P}(t) = -ae^{-rt} g_X(\bar{X}(t), \bar{C}(t)) - P(t) \cdot f_X(\bar{X}(t), \bar{C}(t)), \\ P(0) = A, \end{cases}$$

$$q(t) = -ra \int_t^\infty e^{-rs} g(\bar{X}(s), \bar{C}(s))\, ds.$$

These functions verify conclusions (ii) and (iii) of the theorem. For every $t, q(t)$ is the limit of $q_{T_n}(t)$, because a is the limit of a_{T_n}. And $P(t)$ is the limit of $P_{T_n}(t)$: this follows from the definition of $P(t)$

which implies:

$$P(t) = R(t,0) \cdot A - a\int_0^t R(t,s)e^{-rs}g_X(\overline{X}(s), \overline{C}(s))\,ds$$

with $R(t,s)$ the fundamental matrix of the linear equation

$$\dot{Z}(t) = -Z(t) \cdot f_X(\overline{X}(t), \overline{C}(t)).$$

These limit properties imply conclusion (v) of the theorem; and for any fixed c and t the following inequality holds:

$$ae^{-rt}g(\overline{X}(t), c) + P(t) \cdot f(\overline{X}(t), c)$$
$$\leq ae^{-rt}g(\overline{X}(t), \overline{C}(t)) + P(t) \cdot f(\overline{X}(t), \overline{C}(t)).$$

This implies conclusion (iv) of the theorem and its proof is complete.

PROOF OF THE COROLLARY: g being nonnegative, conclusions (iv) and (v) of the theorem imply, for all c in B:

$$P(t) \cdot f(\overline{X}(t), c) \leq -q(t).$$

We define:

$$Q(t) = \frac{1}{\max\{1, \|P(t)\|\}} P(t),$$

$$l = \limsup_{t\to\infty} \| Q(t)\|.$$

If l is zero, the corollary is verified. Assume $l > 0$ and consider a sequence t_n converging to infinity such that $\| Q(t_n)\| > l/2$. For n large enough, the set of $f(\overline{X}(t_n), c)$, $c \in B$, contains all points in R^n of norm less than $\epsilon > 0$, and $-q(t_n)$ is not greater than $\epsilon l/2$. Then, there exists $c_n \in B$ such that $f(\overline{X}(t_n), c_n) = (2\epsilon/l)Q(t_n)$, and we obtain:

$$P(t_n) \cdot f(\overline{X}(t_n), c_n) = \max\{1, \|P(t_n)\|\}(2\epsilon/l)\| Q(t_n)\|^2,$$

$$P(t_n) \cdot f(\overline{X}(t_n), c_n) > \epsilon l/2 \geq -q(t_n).$$

This is the contradiction. The proof of the corollary is complete.

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