Collateralisation bubbles when investors disagree about risk

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Abstract

Survey respondents strongly disagree about return risks and, increasingly, macroeconomic uncertainty. This may have contributed to higher asset prices through increased use of collateralisation, which allows risk-neutral investors to realise perceived gains from trade. Investors with lower risk perceptions buy collateralised loans, whose downside-risk they perceive as small. Investors with higher risk perceptions buy upside-risk through asset purchase and collateralised loan issuance, raising prices. More complex collateralised contracts, like CDOs, can increase prices further. In contrast, disagreement about mean payoffs raises prices without collateralisation. And the latter may even discipline prices as risky loans must be sold to pessimists with lower collateral valuations.

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1 Introduction

From the mid-1990s to the beginning of the Great Recession, the world economy has seen an unprecedented wave of financial innovation, partly in the form of new collateralised debt products. At the same time, prices of collateral assets, such as real estate, but also stocks, experienced an unprecedented increase. This paper links these two phenomena to a third, less documented one: disagreement among investors about economic risk. Particularly, we provide evidence for this argument from three well-known US surveys. Thus, supplementary questions to the Michigan Survey of Consumer Sentiments administered between 2000 and 2005 (summarised in Amromin and Sharpe (2008)) show how retail investors disagree strongly about medium and long-term dispersion of stock returns. Also, we show that finance executives, who are known to miscalibrate their expectations of SP 500 returns (Ben-David et al. (2013)), show the same, or stronger, disagreement about the standard deviation of returns than about their means. Finally, to analyse a longer time horizon covering the Great Moderation period, we also show how, since 1980, near-term GDP forecasts from the Survey of Professional Forecasters show an increasing disagreement between forecasters about the dispersion of GDP growth, while disagreement about mean growth has fallen. We show how these heterogeneous risk perceptions, when combined with financial innovation in the form of collateralised debt products, can create large asset price bubbles. In the absence of collateralisation, risk-neutral investors trade assets at their common fundamental value even if they disagree about payoff risk. The introduction of simple collateralised loans increases asset prices above this common fundamental value by unleashing perceived gains from trade: while investors with a concentrated posterior distribution of payoffs are less afraid of the downside risk embodied in collateralised debt, those who perceive higher payoff risk value more highly the upside risk of leveraged asset purchases. In equilibrium, this raises the price of collateral assets. Finally, we also show how further financial innovation in the form of more complex collateralised debt products can strongly affect prices.

We think that this analysis is interesting, and in our view important, for two main reasons. First, we believe that heterogeneity in risk perceptions, or perceived second moments of asset returns, is a previously neglected case of disagreement that is a priori plausible and born out by the data. Thus, a priori, in line with the fact that "you need a mean estimate to estimate the variance" it is typically more difficult to estimate second moments from any given amount of data than first moments. And empirically, we show how, in two well-known surveys of, respectively, stock-market investors (Amromin and Sharpe, 2008) and finance executives (Ben-David et al., 2013), there is strong evidence that investors and managers both make mistakes and differ among each other in their assessment of return risks. We augment this evidence by showing that disagreement among professional forcasters about the variance of US GDP growth, a main determinant of asset returns, has increased since the 1980s, while that about mean payoffs has decreased. Second, we think that our analysis is relevant as it may help explain the increased use of collateralised debt products and the strong increase in the price of collateral assets during this period of diverging risk perceptions up until the recent crisis. This is because an investor who believes asset returns to be more dispersed than another perceives more upside potential at the same time as more downside risk than her counterpart. Buying the asset and using it as collateral for debt issuance then naturally distributes payoffs according to this absolute advantage: by using leverage, the investor with dispersed beliefs keeps the upside risk she values more highly, while her counterpart purchases debt that she views as relatively riskless. In contrast, an optimist whose beliefs first-order stochastically dominate those of another - the case analysed in all previous studies - perceives both less downside risk and more upside potential, and thus finds both debt and leveraged asset purchases to be more profitable. Only if she lacks funds, and if her optimism is more concentrated in upside risk, is there a comparative, rather than absolute, advantage that gives rise to incentives for issuing collateralised debt products (Simsek, 2013).

The literature on the consequences of heterogenous investor beliefs has infact entirely focused on disagreement about mean payoffs.[1] Thus, Miller (1977)'s seminal article shows how asset prices rise when investors disagree about future mean payoffs and the absence of short-selling makes the marginal investor become more optimistic. Geanakoplos (2003) and Geanakoplos

[1] See (Xiong, 2013) for a survey.

(2010) introduce leverage into this framework, whereby investors can issue debt collateralised by the assets they want to buy in order to increase the amount they can invest. This allows optimists to increase their asset purchases and thus makes the marginal investor more optimistic, increasing prices further. At the same time it leads to fluctuations in asset prices in response to changes in investor balance sheets. Simsek (2013) uses a similar model with two groups of investors to show how leveraged investment dampens the effect of belief disagreements on prices when when optimists have relatively positive views on the distribution of relatively bad realisations of shocks. Instead when optimists are particularly positive about the upside potential of the asset, leveraged investments amplify the effect of belief disagreements.

Interestingly, Miller (1977)'s original article associates higher payoff risk with stronger disagreement about mean payoffs. Neither his article, nor the literature that it precedes, however, has analysed disagreement in beliefs about risk per se.[2] In fact, in Miller (1977)'s original setup where risk-neutral investors simply buy or sell the asset, disagreement about payoff risk around common mean payoffs does not affect prices, which equal their common discounted expected payoff. The introduction of collateralised debt, however, enables investors to realise the perceived gains from trade that arise from heterogeneous risk perceptions by splitting asset payoffs non-linearly. Investors that perceive payoffs to be volatile are thus happy to sell the downside risk by issuing collateralised loans, and retain the upside risk they value highly. Investors who believe payoffs to be concentrated happily buy collateralised loans whose downside risk they perceive as small.

It is important to realise how this complementary relationship between collateralisation and disagreement about risk is different to that arising from disagreement about mean payoffs. In Geanakoplos (2003), with two discrete payoff realisations, optimists find it optimal to issue riskless collateralised loans in order to increase their available funds. Market clearing then implies a more optimistic marginal investor, and an increase in asset prices from a level that, absent

[2]In an early reaction to (Miller, 1977), (Jarrow, 1980) has pointed out the importance of the variance-covariance structure of asset returns for the effect of short-selling constraints on asset prices. His focus is very different to the one in this paper, however.

short-selling, exceeds the mean of investor valuations even in the absence of leverage. When collateralised loans are risky, the common assumption of first-order stochastic dominance implies a trade-off for optimists, as in Simsek (2013): issuing collateralised debt raises funds for investments, but means selling downside risk at unfavourable prices to pessimists. This is why collateralised contracts can discipline asset prices whenever optimism is about downside risks. Moreover, an increase in disagreement in the form of increased pessimism acts to lower the price of risky loans and assets.

With disagreement about payoff dispersion, in contrast, we show how the effect of collateralisation is fundamentally different. First, asset prices equal a common fundamental in the absence of collateralised contracts. In other words, a departure of prices from their fundamental requires financial innovation, e.g. in the form of collateralised debt. Second, collateralisation allows investors to realise perceived gains from trade only when debt is risky, by channeling upside and downside risk to those that value them more highly. This implies, third, that there is no trade-off, and no disciplining effect of collateralisation: issuing risky collateralised debt realises pure perceived gains from trade. Finally, an increase in disagreement makes collateralised loans and leveraged assets more valuable to those that hold them, and thus always raises asset prices.

In an extension to our framework, we show how further financial innovation in the form of more complex collateralised debt products can strongly affect prices. We show that this is, perhaps surprisingly, not the case with collateralised debt obligations (CDOs), whose introduction does not affect prices relative to trade in collateralised loans. Allowing agents to use CDOs to collateralise more junior "secondary" CDO contracts, however, strongly raises prices. This is because, by buying and issuing a pair of CDO contracts, investors can buy any sub-set of the payoff distribution. This drives up asset prices to the expectation of its payoffs evaluated at the maximum of all individual probability density functions. For example, with two types of investors that have disjoint payoff distributions, asset prices rise to twice their fundamental value with trade in secondary CDOs. Finally, we also analyse the effects of risk-aversion and study a simple example of a dynamic equilibrium in a scenario with learning that tries to capture the main features of the Great Moderation in the US. As a subset of investors adjusts their posterior estimate of volatility more quickly to the Great Moderation than the rest, increasing divergence of posteriors raises asset prices between 5 and 40 percent.

2 Heterogeneity in risk perceptions: evidence from US survey data

This section shows evidence from US surveys that documents the extent to which investors, or forecasters, disagree about risk, or the dispersion of outcomes around their expectations. For this we use three data sources: first, the forecasts for S&P 500 returns by a sample of Chief Financial Officers (CFOs) reported in Ben-David et al. (2013). Second, supplementary questions to the Michigan Survey of Consumer Attitudes that, between 2001 and 2005, ask stock market investors for the stock market returns they expect on average and the uncertainty around them in the medium and long-run. And third, a longer history of GDP forecasts elicited in the Survey of Professional Forecasters (SPF) that contains a fully specified histogram of near-term GDP growth. Relative to other investor surveys, the Michigan survey and the Ben-David et al. (2013) dataset have the advantage that, for an important asset class - US stocks - they ask actual investors and CFOs, respectively, not only for their mean expectations but also for the uncertainty around them.[3] The SPF, in contrast, asks for GDP growth, which is interesting as one of the main macroeconomic determinants of investment returns, if not a perfect predictor. Its advantage is that it contains a long history of histograms with a finer support than that of the other surveys.

2.1 Disagreement about US stock market returns

This section uses information from two US surveys to show how investors strongly disagree not only about expected returns, but also about return risks. Table 1 reports summary statistics

[3]See Greenwood and Shleifer (2013) for a description of surveys about return expectations in the US.

of the supplementary questions in the Michigan Survey of Consumer Sentiments, covering 22 surveys in the years 2000 to 2005, taken from Amromin and Sharpe (2008) [4]. The first row reports the distribution of investors' answer to the question about the "annual rate of return that you would expect a broadly diversified portfolio of U.S. stocks to earn, on average". The second row reports the probability "that the average return over the next 10 to 20 years will be within two percentage points of your guess", and the third one shows the corresponding standard deviation assuming normally distributed beliefs about stock market returns.

The first row of Table 1 shows that expected annual returns, averaged across respondents and surveys, equal 9 percent, which coincides almost exactly with the average 10 year annual returns on the S&P total returns index in the period before the last survey in 2005. Disagreement about future mean returns, however, is strong, with 10 percent of respondents expecting an average return of or below 5, and another 10 percent expecting above 16 percent. The perceived riskiness of stock investments, however, also varies strongly across investors: while 10 percent of respondents believe realised returns to fall within 2 percentage points of their expectation with a probability of at least 80 percent, another 10 percent expect returns to fall outside this range with at least 80 percent probability. Using a normality assumption to transform these assessments into standard deviations, the 90-10 percentile difference of standard deviations equals 6.3, compared to 11 for expected returns.

Table 1: Return Expectations in the Michigan Survey 2000-2005

	N	Mean	10th pct	25th pct	Median	75th pct	90th pct
Expected return R_e	3,046	10.4	5	7	10	12	16
Prob $\|R - R_e\| < 2pp$	3,015	43.3	20	25	50	50	80
Implied σ_{10-20} (in percent)	2,854	4.56	1.56	1.73	2.96	2.96	7.88

Ben-David et al. (2013) present similar survey evidence for a quarterly sample of senior finance executives, mainly Chief Financial Officers, whom they ask to forecast both the 1-

[4]The authors eliminate incomplete responses, those deemed by the interviewer to have a low level of understanding or a poor attitude toward the survey, and those that answered "50 percent" to all probability questions.

year return and average 10-year returns of the US SP 500 stock market index. Specifically, respondents are asked for their expected returns, and 80 percent confidence intervals around them. The authors show how their respondents' return forecasts are "miscalibrated", in the sense that respondents underestimate the uncertainty around their expected returns both relative to history and relative to subsequent outcomes. Interestingly for the present study, the survey also shows how respondents strongly disagree in their individual volatility estimates. To show this, figure 1 plots the standard deviations of both individual expected return and individual volatility (equal to the standard deviation of the individual-specific return distribution inferred from the confidence interval), for both 1-year returns (left panel) and 10-year returns (right panel, transformed from average to annualised returns).[5] As the left panel shows, the disagreement about volatility is of the same order of magnitude ($3-5$ percentage points) as that about means for the one-year returns, and highly correlated before the crisis, with peaks in 2001 and 2004. Both kinds of disagreement rise strongly in 2008 - disagreement about means more strongly so - and fall slowly back to common levels seen at the beginning of the 2000s at the end of the sample. Interestingly, for 10-year returns, the level of disagreement is about twice as high for individual volatility ($5-8$ pp before 2008) than for expected returns ($2-4$ pp). It is also higher than for short-term individual volatility, while before the crisis average disagreement about expected returns is about the same for short and long-term returns. Moreover, before 2008, disagreement about expected long-term returns is flat, while that about their volatility is slightly increasing. Finally, both rise strongly after 2008 and remain elevated relative to their previous means until the end of the sample.

Surveys of investors and finance executives thus show strong disagreement about return volatilities. Specifically, the disagreement about return standard deviations seems of similar magnitude to that about expected returns, and there is some evidence that it may be larger for 10-year returns. The time-series of both surveys we discussed, however, is relatively short. This is why the next section studies disagreement among forecasters about the distribution of a particularly important determinant of asset returns, aggregate output, for which we have longer

[5]We thank Izhak Ben-David for providing the series of standard deviations of volatility estimates.

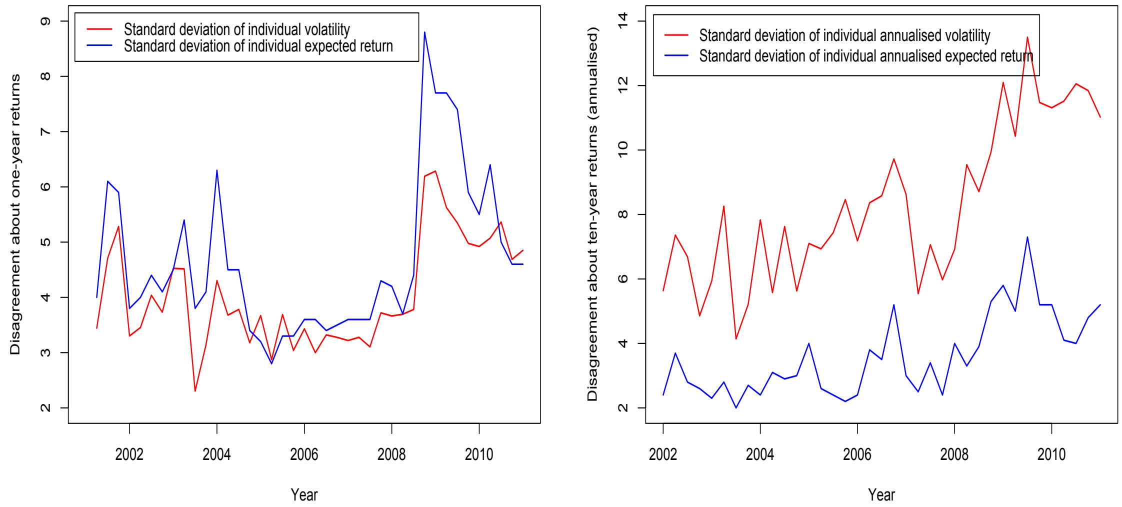


Figure 1: The left panel plots the time series of the standard deviations of individual means and volatilities for 1-year returns. The right panel plots the corresponding measure for 10-year returns.

time-series evidence.

2.2 Disagreement about US Macro Risk 1980-2010

The Survey of Professional Forecasters SPF is a quarterly survey that asks forecasters to indicate, among other measures, their probability distribution for GDP growth in the current calendar year.[6] Specifically, forecasters report the probability that short-term growth falls in any of 6 brackets.[7] This allows us to study the evolution of disagreement between forecasters about short-term US growth prospects. Particularly, using a normal approximation of the distributions, as in Giordani and Söderlind (2003) we can look at the distribution across forecasters of forecaster-specific means μ_{it} and standard deviations σ_{it} for every quarter since 1980 (when the survey changed from nominal to real GDP projections). Based on this cross-sectional distribution, we look at two measures of disagreement about the mean and volatility of output growth across

[6]Since 1992, the survey also asks for the same distribution for the following year. We don't use this measure because of the short history.

[7]The brackets have changed slightly in 1990.

forecasters: first, the standard deviations of $\widehat{\mu_{it}}, \widehat{\sigma_{it}}$ defined as

$$\begin{aligned} \mu_{it} &= \widehat{\mu_{it}} + \mu_t \\ \sigma_{it} &= \widehat{\sigma_{it}} \sigma_t. \end{aligned} \tag{1}$$

[8] The second disagreement measure is based on the integral of absolute differences of any two forecaster-specific normal densities, averaged across forecasters.

$$d = \frac{1}{N_t^2} \sum_i \sum_j \int |f_i(g_y) - f_j(g_y)| dg_y, \tag{2}$$

where N_t is the time-varying number of forecasters in the sample. [9] We calculate the contribution of the heterogeneity in standard deviations to this average disagreement using the formula in (2) with the mean of the two normal distributions held constant ($\mu_{it} = \mu_{jt}$), and define the remaining difference with overall disagreement as the contribution of heterogeneous means.

Figure 2 shows how the dispersion of means and standard deviations of short-term growth forecasts has evolved over time in the survey. In the early 1980s, the standard deviations of means (in the left panel) was about twice that of standard deviations (in the right panel). But while mean forecasts converged - with their standard deviation falling to less than half their initial value before rising abruptly at the beginning of the recent 'great recession' - the dispersion of forecast standard deviations has increased strongly, amid noticeable cyclical swings. Figure 3 shows the contributions to the overall disagreement measure d of heterogeneity in forecaster-specific means (in the left panel) and standard deviations (in the right panel).[10] While overall disagreement (not shown) does not follow any trend over the sample, the (smoothed) contribution

[8] For the positive variable σ_{it} we use the normalised standard deviation to prevent it from falling to zero mechanically as the mean of σ_{it} falls.

[9] This measure equals zero for any two identical distributions and is bounded above by 2 (for two disjoint distributions).

[10] We only use the first quarter of every year to keep the forecast horizon constant and equal to the remainder of the current year.

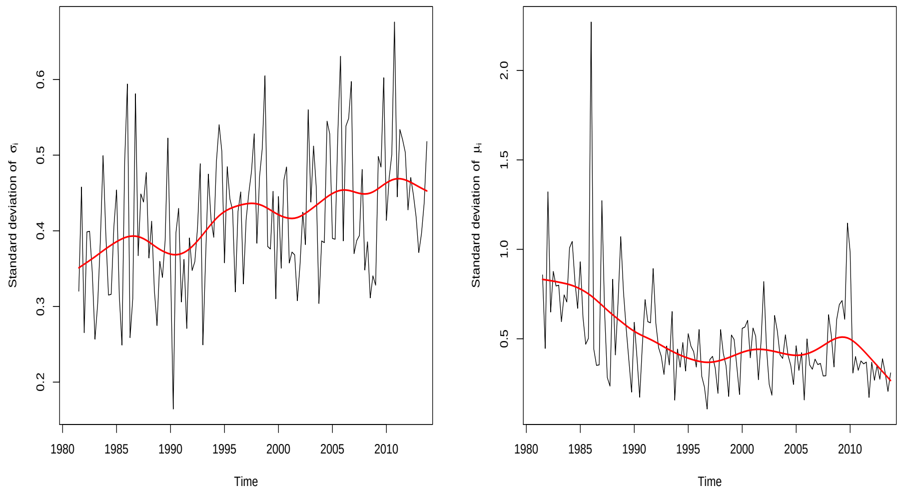


Figure 2: The left panel plots the time series of the standard deviations of $\widehat{\sigma_{it}}$. The right panel plots the corresponding measure for $\widehat{\mu_{it}}$ and as defined in equation (1). The red line shows the trend from an hp-filter with smoothing parameter 1600.

of heterogeneous standard deviations increases by about 1/3 until the beginning of the recent recession. The contribution of mean growth dispersion, of about the same magnitude at the beginning of the sample, falls by about 1/3 until the recession. Therefore the evidence from the SPF suggests that the contribution of heterogeneous perceptions of growth-dispersion has risen strongly since the early 1980s, while disagreement about mean growth has become less important.

Both the evidence from the Michigan Survey and the SPF thus suggest that there is strong beliefs' heterogeneity about the riskiness of stock market returns among US investors and about macroeconomic risk among professional forecasters. Finally, given that that for all the variables under consideration, in both surveys, there is ample public information, we believe that the evidence above reflect indeed agree-to-disagree type differences as opposed to informational differences.

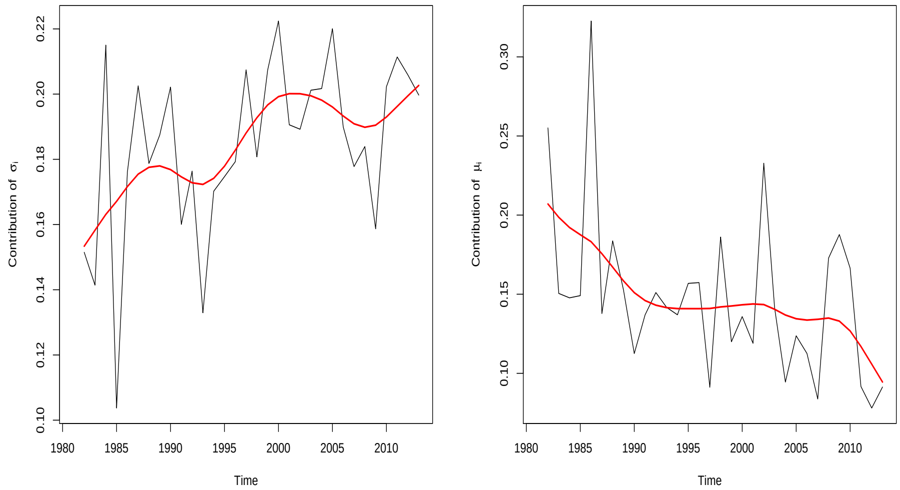


Figure 3: The left panel plots the contribution of heterogeneous standard deviations to overall disagreement d about current year GDP growth in the SPF as defined in equation (2). The right panel plots the corresponding contribution of heterogenous means. The red line shows the trend from an hp-filter with smoothing parameter 25 (to adjust for the annual frequency, see Ravn and Uhlig (2002).

3 Leveraged asset trade with disagreement about risk

This section presents the general environment and shows how disagreement about payoff risk implies a bubble in asset prices, in the sense that equilibrium asset prices increase the fundamental valuation that is shared by all investors. And a further increase in disagreement inflates the bubble. We show this for a simplified version of the economy with two investor types that differ in their risk perceptions. An appendix presents results for the general case with a continuum of types.

3.1 The General Environment

3.1.1 Preferences and beliefs

We study an economy that exists for two periods $t \in \{0, 1\}$. There is a unit-mass of investors indexed by i with $i \in I = [0, 1]$. The distribution of agents on I is defined by measure

$m : \mathbb{I} \longrightarrow [0,1]$, where $\mathbb{I}$ is the Borel-algebra of I and m has no mass-points. Denote as g the density function induced by m, and by G the cumulative density function of g with $G(0) = 0$ and $G(1) = 1$.

In period 0, agents of type i receive an endowment n_i of the unique perishable consumption good and $\overline{a}_i$ units of a risky asset (a "tree") that pays a stochastic amount $s \in S = [s_{min}, s_{max}]$, $s_{min} > 0$ in period 1. For simplicity, we assume all agents receive the same initial amount of assets, normalised to 1. All agents are assumed to be risk-neutral, so they maximise the present discounted sum of expected consumption in both periods i.e. $U_i = c_i + \frac{1}{R}E_i(c_i')$, where E_i is the mathematical expectation of agent i, c_i (resp. c_i') denotes consumption in period 0 (resp. 1) and $\frac{1}{R} < 1$ is the discount factor.

We assume that types differ in their beliefs about the distribution of random payoffs s, summarised by distribution functions $f_i : S \longrightarrow R^+$. We assume that all agents expect payoffs to be the same on average, but that any type $i : i > j$ believes them to be less tightly distributed than type j. In other words, f_j second-order stochastically dominates f_i whenever $i > j$, or formally:

Assumption A1 $E_i(s) = E_j(s) \equiv E_s,\ f_j \succ^2 f_i \Leftrightarrow j < i$,
where $\succ^2$ *denotes second-order stochastic dominance.*

Thus i is an index of belief dispersion.

3.1.2 Asset markets

Agents trade in 2 asset markets: In $t = 0$, agent i purchases $a_i - \overline{a}_i$ units of the physical asset in exchange of $p(a_i - \overline{a}_i)$ units of the consumption good. In addition, agents can borrow by pledging part of their future income. However, agents cannot commit to future payments, and therefore have to collateralise their borrowing. We assume that agents only trade the simplest form of these contracts, namely a debt contract. Debt contracts are characterised by a fixed promised face value. The absence of commitment means that agents transfer to their creditor the face value of the loan or the payoff of the assets that serve as collateral, whatever is smaller.

We normalise contracts to be secured by 1 unit of the asset as collateral.[11] Thus collateralised loan contracts have unit-payoffs equal to $\min\{s, \overline{s}\}$, where $\overline{s}$ is the promised face value. In $t = 0$, agents trade these contracts at competitive price $q(\bar{s})$. In the following we assume that this price function is Borel measurable. Given that borrowing is subject to a collateral constraint, agent i's positions of collateralized loans sold must satisfy the following condition:

$$b_i \geq -a_{i.} \tag{3}$$

Each unit of collateralized loans sold must be secured by at least one unit of the risky asset that agent i possesses and can be used as collateral.

In the special case of a given unique $\overline{s}$, the set of available assets implies that the budget constraints of agent i in $t = 0$ and $t = 1$ respectively are:

$$c_i + pa_i + qb_i \leq n_i + p\overline{a}_i, \tag{4}$$

$$c_i^{'} \leq a_i s + min\{s, \bar{s}\}b_i, \tag{5}$$

where a_i and b_i represent agent i's total holdings of risky assets, including the initial endowment, and of collateralised loans respectively.

3.1.3 Expected Returns

To simplify the general problem of an investor who chooses consumption and a portfolio of assets and loans with endogenous face value, it is useful to look at the profits an investor expects to make from his different investment options for a given vector of prices p, q and a given face value $\overline{s}$. Buyers of collateralised loans with face value $\overline{s}$ pay a sum q to their counterparty today, for a promise whose expected value is $E_i[\min\{s, \overline{s}\}]$. For a quantity of loans b_i, expected discounted

[11]Note that one unit of a bond with face value 1 collateralised by x units of the asset is payoff-equivalent to x units of a bond of face value 1/x collateralised by one unit of the asset.

profits are

$$\Pi_i^l = b_i[\frac{E_i[min\{s,\overline{s}\}]}{R} - q].$$

A second investment option is to buy the asset outright using consumption goods as payment, implying expected profits equal to $\Pi_i = [\frac{E_i(s)}{R} - p]a_i$. Agents can, however, also engage in leveraged asset purchases by using the asset as collateral for a loan that provides part of the invested funds. Then, for a given $\overline{s}$, the expected profits from buying a_i units of risky assets partly financed through a collateralised loan of equal size are

$$\Pi_i^a = [\frac{E_i(s) - E_i(\min(s,\bar{s}))}{R} - (p-q)]a_i. \tag{6}$$

Figure 4 illustrates how gross unit-profits in period 1 change as a function of the asset payoff s. The definition of profits implies that returns on collateralised loans are convex in s, while those on leveraged asset purchases are concave in s. Given the second order stochastic dominance relationship of beliefs, this immediately implies that investors with more (less) dispersed beliefs expect to make higher profits from investment in leveraged assets (collateralised loans). This is summarised in Proposition 1.

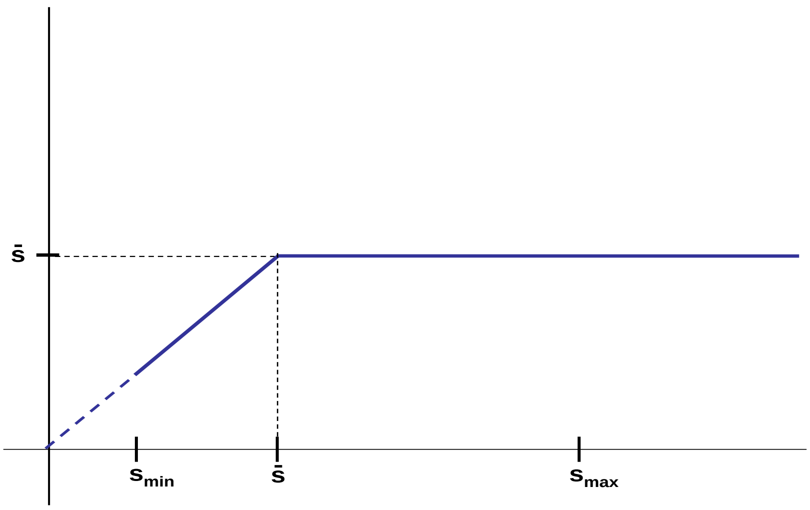
Proposition 1 - *Profits and risk perceptions*

Tight-prior agents (low i) have higher expected profits from investing in collateralised loans of a given face value $\bar{s}$ than those with dispersed priors (high i). The inverse is true for profits from leveraged asset purchases:

$$i > j \Rightarrow \Pi_i^l \leq \Pi_j^l \ \forall \ \overline{s} \in (s_{\min}, s_{\max}), \forall p, q, R,$$

$$i > j \Rightarrow \Pi_i^a \geq \Pi_j^a \ \forall \ \overline{s} \in (s_{\min}, s_{\max}), \forall p, q, R.$$

Unit-Profits from collateralised loans



Unit-Profits from leveraged asset purchases

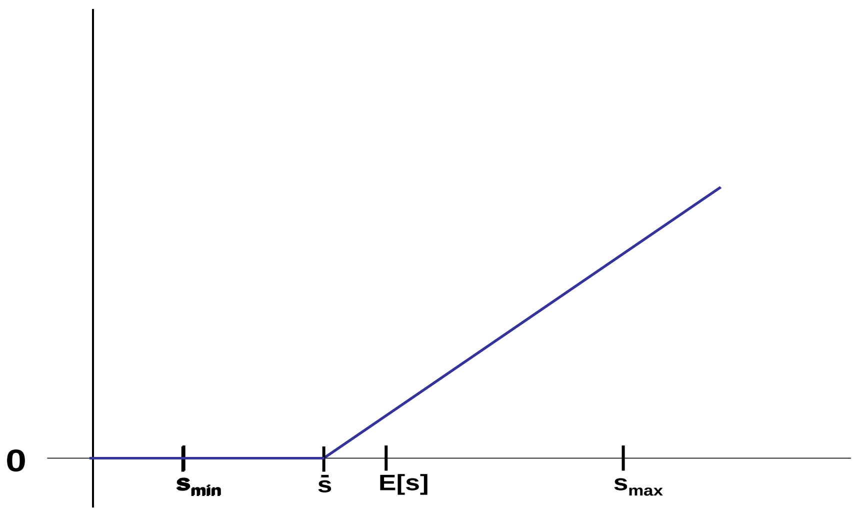


Figure 4: The upper panel plots the profits from collateralised loans that are concave in s and the lower panel plots those from leverage asset purchases, which are convex in s.

3.2 Equilibrium with two investor types

The remainder of this section looks at a simplified version of the economy with two types $i \in \{0, 1\}$. Appendix A presents the model with a general continuous distribution of agents.

We assume that the distribution g has two mass points at $i \in \{0, 1\}$, corresponding to two groups of agents, for simplicity each of unit mass, whose beliefs satisfy $f_0 \succ^2 f_1$. Type 0 agents have less dispersed beliefs about payoffs than type 1 agents. Since $\Pi_1^l <= \Pi_0^l, \Pi_1^a > \Pi_0^a \ \forall \ \overline{s} \in (s_{\min}, s_{\max}), \forall p, q, R$, the agents less dispersed beliefs are the natural buyers of collateralised loans, and the agents with more dispersed beliefs are the natural investors in leveraged assets. In other words, if there is trade in collateralised loans in equilibrium $-b_1 = b_0 > 0$. We assume that type 0 agents who have less dispersed beliefs are cash-rich.

Assumption A2 - $n_0 \geq \frac{E_s}{R}$.

Under Assumption A2 any asset price below the fundamental value $\frac{E_s}{R}$ would lead to excess asset demand for loans, so $p \geq \frac{E_s}{R}$. Moreover, given Assumption A2, the total value of type 0 agents' endowment equals $n_0 + p \geq 2\frac{E_s}{R} \geq 2max_{\overline{s}}q(\overline{s})$. So type 0 agents can afford to buy all collateralised loans at their maximum expected value. In turn, this implies that 0 types bid up the price of any collateralised loan issued by type 1 agents to their expected discounted value, where they are indifferent between investing and consuming, implying a bond price function

$$q(\overline{s}) = \frac{E_0[\min\{s, \overline{s}\}]}{R}. \tag{7}$$

3.2.1 Type 1's problem and the choice of $\overline{s}$

Since investors with tighter priors buy all collateralised loans at their reservation price $q(\overline{s})$, the problem of type 1 agents who have more disperse priors simplifies to the choice of current consumption, which through the budget constraint determines their investments, and the level of leverage $\overline{s}$ given p and the price function $q(\overline{s})$.

$$\begin{aligned}\max_{c_1,\overline{s}} \quad U_1 &= c_1 + \frac{(n_1+p-c_1)}{R}\frac{[E_s - E_1(\min\{s,\overline{s}\})]}{p - \frac{E_0\{\min\{s,\overline{s}\}\}}{R}} \\ &= c_1 + \frac{(n_1+p-c_1)}{R}R_1^a. \end{aligned} \tag{8}$$

where $R_1^a \doteq \frac{[E_s - E_1(\min\{s,\overline{s}\})]}{p - \frac{E_0\{\min\{s,\overline{s}\}\}}{R}}$ is the leveraged gross return of the asset using a loan with riskiness $\overline{s}$. The first order condition for $\overline{s}$ can be written as:

$$\frac{(n_1+p)}{Rp - E_0\{min\{s,\overline{s}\}\}}[(1-F_1(\overline{s})) - \frac{R_1^a}{R}(1-F_0(\overline{s}))] = 0. \tag{9}$$

Proposition 2 - *Interior choice of* $\overline{s}$.

Suppose that p *is such that* $\frac{E_s}{R} = \underline{p} < p < \overline{p} \doteq \frac{E_s + E_0(min(s,\overline{s})) - E_1(min\{s,\overline{s}\})}{R}$ *holds for some* $\overline{s} \in (s_{min}, s_{max})$*, such that agent 1 expects to make profits for some* $\overline{s}$ *when she buys assets at* p *that exceeds the fundamental value. Then* $R_1^a(p,\overline{s})$ *has an interior maximum at some* $\overline{s}^\star \in (s_{min}, s_{max})$.

Proof of Proposition 2.

Note that $R_1^a(p, s_{max}) = 0$. Also, if $p > \frac{E_s}{R}$, $R_1^a(s_{min}) = \underline{R_1^a} < R$. But if at some $\overline{s}'$, $p < \frac{E_s + E_0(min\{s,\overline{s}'\}) - E_1(min(s,\overline{s}'))}{R}$, then $R_1^a(\overline{s}') > R$. The statement then follows from continuity of R_1^a. ■

3.2.2 Equilibrium Characterisation

Definition 1 - *A general equilibrium is an endogenous face value of collateralised loans* $\overline{s}$*, a set of prices (p, q) and allocations* $(c_i, c_i', a_i, b_i)_{i\in\{0,1\}}$*, such that* (7) *holds, agent* 1 *solves the optimization problem* (8)*, the demand for assets equals the fixed supply,*

$$a_0 + a_1 = 2,$$

and the collateralized loan market clears,

$$b_1(\overline{s}) + b_0(\overline{s}) = 0, \ \forall \overline{s}.$$

The following proposition shows that equilibrium is defined by two conditions: first, the optimal choice of leverage $\overline{s}$; and second, the market clearing for leveraged assets, which defines the price such that type 1 agents either exhaust all their wealth buying assets, or are indifferent between investing and consuming. Intuitively, as agent 1 wealth rises, their increasing demand for assets bids up the price until it reaches indifference level $\overline{p}$.

Proposition 3 - *Existence and uniqueness of equilibrium.*

Denote as $n_1^{\max}(\overline{s}) = n_1 + 2\frac{E_0[\min\{s,\overline{s}\}]}{R}$ *the resources available to type* 1 *agents for net purchases of assets when they issue collateralised loans backed by the whole asset endowment of the economy.* p *and* $\overline{s}$ *are given by the unique solution of the following equations:*

$$\mathbb{C} \equiv [E_s - E_1(\min\{s,\overline{s}\})](1-F_0) - (1-F_1)(Rp - E_0[min\{s,\overline{s}\}]) = 0, \tag{10}$$

$$p = max\{\overline{p}, p^\star\}, \tag{11}$$

$$p^\star = n_1^{\max}(\overline{s}), \tag{12}$$

where the left hand side of (12) *are the net purchases of assets and the right-hand side equals the available resources, both weighed by the mass of type* 1 *agents.*

As Proposition 3 shows, with heterogeneous risk perceptions, collateralised contracts lead to a bubble in asset prices, in the sense that equilibrium prices exceed the common fundamental value of the asset, shared by all investors. Moreover, it is easy to see from (12) that a rise in resources of type 1 agents (weakly) increases prices. In addition, as Proposition 2 has shown, there is a unique endogenous choice for leverage $\overline{s}$.[12]

[12] Appendix B shows that similarly to the two type economy, with heterogeneity in perceived risks across a continuum of types, the equilibrium prices of risky assets are necessarily above their common fundamental valuation. Unlike the two type economy, however, these results require an exogenous upper bound for the face value $\overline{s}$.

3.3 Comparative statics

This section looks at the effect of "belief-divergence", in the sense of a further mean-preserving contraction to f_0, or equivalently a dispersion of f_1. For this, we concentrate on economies where funds of type 1 agents are large either because their endowments are high, or because they can raise enough funds from issuing collateralised loans.

Assumption A3 - $n_1 \geq \underline{n_1} = \frac{E_s - E_0(min\{s,s^\star\}) - E_1(min\{s,s^\star\})}{R}$.

Assumption A3 implies that asset prices are at their upper bound, as the following lemma states.

Lemma 1 - *Assumption A3 implies* $p = \overline{p}$.

Proof of Lemma 1.

Under Assumption A3 we have

$$\begin{aligned} n_1^{\max}(s^\star) &\geq \frac{E_s - E_0(min\{s,s^\star\}) - E_1(min\{s,s^\star\}) + 2E_0[\min\{s,s^\star\}]}{R} \\ &\geq \frac{E_s + E_0[\min\{s,\overline{s}\}] - E_1(min\{s,s^\star\})}{R} = \overline{p} \end{aligned} \tag{13}$$

which implies that type 1 agents have resources larger than the value of assets evaluated at any $p \leq \overline{p}$. So equilibrium requires type 1 agents to be indifferent between consuming and investing in leveraged assets, implying an equilibrium price equal to $\overline{p}$. ■

Corollary 1 - *For any symmetric distributions* f_1, f_0*, we have* $\underline{n_1} < 0$*. So Assumption A3 trivially holds.*

Proof of Corollary 1.

Note that for any symmetric distribution $E_s = s^\star = \frac{1}{2}(s_{max} + s_{min})$. So

$$\begin{aligned} \underline{n_1} = \frac{E_s - E_0(min\{s, s^\star\}) - E_1(min\{s, s^\star\})}{R} \\ \leq \frac{E_s - 2s^\star(1 - F_0(s^\star))}{R} \\ = \frac{E_s - 2E_s\frac{1}{2}}{R} = 0, \end{aligned} \tag{14}$$

where the last line follows from $s^\star = E_s$ and $1 - F_0(s^\star) = 1 - F_1(s^\star) = \frac{1}{2}$ due to symmetry. ■

To look at belief divergence we assume that the distribution function f_i is parameterised by a variable v such that:

1. f_i is continuous in v for all s, $i = 0, 1$.
2. $E_{i,v}(s) = E_s, \forall v$, $i = 0, 1$.
3. $f_0(v)$ second order dominates $f_0(v')$ whenever $v > v'$.
4. $F_0(v, s) - F_0(v', s)$ is downward sloping in s whenever $v > v'$ and crosses the zero line once at $s^\star$.

In the following we define 'belief-divergence' as small changes in the beliefs of type 0 and 1 agents, f_0, f_1, through a pair of small changes in their corresponding values of v, $dv_0 \geq 0, dv_1 \leq 0$ with at least one strict inequality, corresponding to a mean-preserving contraction to f_0 and a mean-preserving spread to f_1.

From the pricing equation for bonds (7), it is immediately clear that $dv_0 > 0$ increases the valuation of collateralised loans by type 0 agents, and thus their price.

Lemma 2 - *A fall in risk perceived by type* 0 *increases prices of collateralised loans* $\frac{\delta q}{\delta v_0} > 0$.

Also, under Assumption A3 we can identify the effect of belief-divergence on asset prices.

Proposition 4 - *Belief-divergence increases asset prices.*

Proof of Proposition 4.

Assumption A3 implies $p = \overline{p} = \frac{E_s + E_0(min(s,\overline{s})) - E_1(min\{s,\overline{s}\})}{R}$, and $F_0(\overline{s}) = F_1(\overline{s})$ from (10), so $\overline{s} = s^\star$ from single-crossing. Since $s^\star$ does not change in response to dv_i, neither does $\overline{s}$. But $\overline{p}$ rises with a mean preserving spread $dv_0 \geq 0, dv_1 \leq 0$ as $\frac{\delta E_0[min\{s,s^\star\}]}{\delta v_0} \geq 0$ and $\frac{\delta E_0[min\{s,s^\star\}]}{\delta v_1} \leq 0$.

■

As Proposition 4 shows, divergence of risk-perceptions across investors leads to a further increase in asset prices. On the one hand, a mean-preserving spread to f_1 is equivalent to an increase in the upside potential of asset payoffs. This increases the profits expected by leveraged type 1 agents, and thus the reservation price $\overline{p}$. A mean-preserving contraction in f_0, on the other hand, is equivalent to lenders updating their beliefs to a lower level of risk. This increases the expected payoff from a collateralised loan of given riskyness, and thus increases the amount they are willing to pay for collateralised loans. For investors, this always increases expected profits at a given asset price and level of loan riskyness, and thus raises $\overline{p}$. Under Assumption A3, type 1 agents have enough resources to drive up the equilibrium price to $p = \overline{p}$, so increased attractiveness of leveraged investments immediately raises asset prices. When Assumption A3 does not hold, a mean preserving spread in beliefs has an ambiguous effect on marginal profits and thus the optimal value of riskyness $\overline{s}$. Specifically, while a rise in v_0 increases the return at any given riskyness, in an asymmetric equilibrium it can increase or decrease $1 - F_0$, the marginal effect of a change in $\overline{s}$ on profits at given returns. Under Assumption A3 this effect is absent as $p = \overline{p}$.

4 Collateralised Debt Obligations[13]

This section analyses the equilibrium of the two-type economy when agents can trade a more generalised set of collateralised contracts. Particularly, we look at trade in Collateralised Debt

[13]The results in this section have benefited from a conversation with Julian Kolm.

Obligations (CDOs). CDOs are debt-like instruments, whose payments are, typically, derived on the basis of an underlying pool consisting of a large number of credit contracts (unsecured credit to households, mortgage contracts, etc). Specifically, the issuer of a CDO sells the cash-flow of the credit-pool in 'tranches' that correspond to ordered percentiles of credit repayments. Thus, a unit-debt obligation collateralised by the xth percentile of credit payments is a promise to pay 1 dollar to the buyer if at least x percent of the creditors pay, and 0 otherwise. Tranches with low x are called 'senior', while those with high (highest) x are 'junior' ('equity') tranches. Note that as before, in the absence of a cash technology, all claims are ultimately collateralised by the asset, which is in fixed supply. This explains the difference of the results in this section with respect to those in, for example, Fostel and Geanakoplos (2012) or section 6 in Simsek (2013), where agents can also use a cash technology to collateralise claims.

In our framework, for $s \in [s_{min}, s_{max}] = [0, 1]$, $f_i(s)$ is the probability that exactly a fraction s of the underlying credit pool pays off. A CDO is a payment $\Psi(x)$ that equals 1 unit of consumption when $s \geq x$ and 0 otherwise. We call x the 'seniority' of the CDO (although it is strictly its 'juniority'). Its expected payoff to agent i equals $E_i[\Psi(x)] = 1 - F_i(x)$. From the single-crossing property of second-order stochastic dominance, we know that there is a value $s^\star$ such that $E_1[\Psi(x)] \geq (<) E_0[\Psi(x)]$ when $x \geq (<) s^\star$. Thus, agents with more volatile beliefs, are the natural buyers of the junior and equity tranches. We denote CDO prices as $Q(x)$. Note that when the asset is tranched fully into CDOs, the total payout for a realisation s equal $\int_0^s 1 \times dx = s$. So CDO payments exhaust the total payoffs.

4.1 Trade in primary CDOs

Consider the two type environment from the previous section, and assume, for now, that agents can only issue CDOs backed by the original asset payoffs. So $B_i(x) \geq -a_i$. This implies a budget constraint

$$c_i = n_i + p - pa_i - \int_0^1 Q(x)B_i(x)dx, \tag{15}$$

$$c'_i = a_i s + \int_0^s B_i(x)dx, \tag{16}$$

where now $B_i(x)$ denote i's purchases of CDOs of seniority x. Note that while seniority x usually increases in discrete steps, or tranches, we take it, for simplicity, to be a continuous variable. Type i's problem is thus

$$\max_{c_i, c'_i, a_i \geq 0, B_i(x) \geq -a_i} c_i + \frac{1}{R} E(c'_i) \tag{17}$$

subject to

(15) and (16)

Definition 2 - *A general equilibrium is a set of prices of the asset p and of CDOs with seniority x $Q(x)$, and an allocation $(c_i, c'_i, a_i, B_i(x))_{i \in \{0,1\}}$, such that given prices $(c_i, c'_i, a_i, B_i(x))_{i \in \{0,1\}}$ solve agent i's problem* (18)*, the demand for assets equals the fixed supply,*

$$a_0 + a_1 = 2$$

and the market for primary CDOs clears,

$$B_1(x) + B_0(x) = 0 \ \forall x.$$

Note that buying all CDOs with seniority $x : x \in X$ for a set $X \subset [0, 1]$ is equivalent to buying the asset and selling all CDOs in the complementary set $X^{-1} = [0, 1] \backslash X$. We will normalise contracts such that type 1 agents buy the asset and sell CDOs to type 0 agents. The latter will want to buy all CDOs with $Q(x) < (1 - F_0(x))$. Under Assumption A2, they can afford this because $n_0 + p \geq 2\frac{E_s}{R} = 2\int_0^1 (1 - F_0(s))ds \geq 2\int_{x:Q(x)<(1-F(0))} 1 \times Q(x)dx$. Thus, market clearing requires $Q(x) \geq (1 - F_0(x))$. In other words, type 0 agents drive up the price of all

CDOs to at least their reservation value. Since $1 - F_0(x) > 1 - F_1(x)\ \forall x < s^\star$, type 0 agents with tighter priors value CDOs with seniority below the single-crossing point $x < s^\star$ strictly more than type 1 agents with more disperse beliefs. Therefore, type 1 agents do not buy any CDOs with $x < s^\star$ at $Q(x) \geq (1 - F_0(x))$, as this implies an expected loss. Note, however, that at price $Q(x) = (1 - F_0(x))$ type 1 agents expect to make a strict profit from keeping CDOs with seniority $x > s^\star$, which they value more than type 0. Note also that for any realisation of the asset payoff $s \in [0, 1]$ the second period payments on CDOs with seniority $x > s^\star$ in the second period equal exactly the payments on a collateralised loan with face value $s^\star$, since $\int_0^{min\{s,s^\star\}} 1 \times dx = \min\{s, s^\star\}$. This implies, if Assumptions A2 and A3 are satisfied, that the asset price with primary CDO trade equals that with trade in collateralised loans.

Proposition 5 - *Under Assumptions A2 and A3, the equilibrium asset price, p, in the economy with trade in CDOs equals that in an economy with trade in collateralised loans.*

Proposition 5 shows how 'horizontal' tranching of payoffs through CDOs does not allow agents to exploit more gains from trade than with trade in collateralised loans. This is because the decision by type 1 agents to issue a CDO of seniority x has exactly the same marginal cost (in terms of additional payments in the second period) and benefits (in terms of funds raised today) as a marginal increase in the face value of a loan at $\overline{s} = x$. Costs are proportional to $1 - F_1(x)$ while benefits are proportional to $1 - F_0(x)$. So single-crossing implies that the additional flexibility that CDOs allow is not used in equilibrium.

4.2 Trade in primary and secondary CDOs

A buyer of a CDO tranche of seniority x can easily use the cash-flow from this asset to collateralise an additional asset that pays 1 dollar whenever $s \geq x' \geq x$. In practice, these assets are often debt-like, implying that the collateral-CDO backs the promised payment in all states of nature. In our framework, investors have trivially no incentive to issue debt backed by CDOs, as a unit of debt collateralised by a CDO is payoff-equivalent to the CDO itself. Here, we analyse equilibrium prices when allowing primary CDOs of seniority x to be used as collateral for any more junior CDO of seniority $x' < x$.[14] We call such an asset equivalently a 'secondary CDO' of seniority x'. Note that unlike so-called 'synthetic' CDOs, which are pure derivative contracts, secondary CDOs are not sold 'naked', but collateralised by more senior primary or secondary CDOs. Denoting by $\widehat{B}_i(x)$ the amount of secondary CDOs of seniority x held by agent 1, this implies a 'downward collateral constraint'

$$\widehat{B}_i(x) \geq -\int_0^x (B_i(s) + \widehat{B}_i(s))ds, \ \forall x \in [0,1] \tag{18}$$

and budget constraints

$$c_i = n_i + p - pa_i - \int_0^1 Q(x)(B_i(x) + \widehat{B_i}(x))dx, \tag{19}$$

$$c_i' = a_i s + \int_0^s (B_i(x) + \widehat{B_i}(x))dx, \tag{20}$$

which exploit the arbitrage condition that primary and secondary CDOs of seniority x must have the same price $Q(x)$, as they represent the same claims.

Trade in secondary CDOs allows agents to trade claims to any range of payoff in S. To see how this increases expected profits, consider the equilibrium prices with primary CDO trade of the previous section and a type 0 agent who has purchased a primary CDO of seniority $x < s^\star$ from type 1 agents. At the price $Q(x) = 1 - F_0(x)$, she is exactly indifferent between consuming or buying the CDO. Introducing secondary CDOs, however, allows her to sell of the tail risk $1 - F_0(x'), x' > s^\star$ by issuing a new CDO collateralised by the original primary CDO. Type 1 agents, who expect to make 0 profits on their asset portfolio, are happy to buy this secondary CDO at price $Q(x') = 1 - F_1(x')$. Expected profits for type 0 agents from this pair of trades are strictly positive as $1 - F_0(x) - (1 - F_0(x')) - (Q(x) - Q(x')) = F_0(x') - F_1(x') > 0$.

[14]Note that issuing a junior CDO collateralised by a more senior CDO is equivalent to borrowing a junior CDO against the senior collateral and selling it on the market. We do not use this interpretation here as we do not allow short-selling of the underlying asset.

More formally, with trade in primary and secondary CDOs, type i's problem becomes

$$\max_{c_i, c_i', a_i \geq 0, B_i(x) \geq -a_i, \widehat{B_i}} c_i + \frac{1}{R} E(c_i') \tag{21}$$

subject to

(19), (20) and (18)

Definition 3 - *A general equilibrium is a price of the asset, p, and of CDOs with seniority x, $Q(x)$, and an allocation $(c_i, c_i', a_i, B_i(x), \widehat{B_i}(x))_{i\in\{0,1\}}$, such that given prices $(c_i, c_i', a_i, B_i(x), , \widehat{B_i}(x))_{i\in\{0,1\}}$ solve agent i's problem* (21)*, the demand for assets equals the fixed supply,*

$$a_0 + a_1 = 2$$

and the market for both primary and secondary CDOs clears,

$$B_1(x) + B_0(x) = 0 \ \forall x,$$
$$\widehat{B_1}(x) + \widehat{B_0}(x) = 0 \ \forall x.$$

Note, again, how we have imposed the arbitrage condition that, for a given seniority x, the prices of primary CDOs need to equal those of secondary CDOs, in our definition of equilibrium. It is easy to see how trade in secondary CDOs completes the asset market, as the joint purchase and sale of CDOs with seniority x and $x + dx$ respectively is equivalent to the purchase of an asset that is contingent on state x as $dx \longrightarrow 0$. Proposition 6 shows how, even in the presence of collateral constraints, this implies asset prices equal to those that would pertain with unconstrained trade of Arrow-Debreu securities. For this to be true, we have to replace Assumption A3 by an alternative condition that ensures type 1 agents have enough resources to buy the assets that they value highly at their own valuation. The new condition is:

$$n_1 \geq E_s - \int_{X_0} s f_0(s) ds \tag{22}$$

where X_0 is defined as the set of states that type 0 agents perceive as more likely $X_0 = \{x \in [0,1] : f_0(x) > f_1(x)\}$, and X_1 is its complement $X_1 = [0,1] \backslash X_0$.

Proposition 6 - *There is an equilibrium with trade in secondary CDOs where the asset price equals*

$$p^{SCDO} = \frac{\int_0^1 s \;\; max_i\{f_i(s)\}}{R} ds. \tag{23}$$

Proposition 6 shows how trade in secondary CDOs allows agents to exploit all perceived gains from trade, as they can separately trade claims to any subset of S. When the economy is sufficiently cash-rich, and given the collateral constraints that restrict the asset supply, this drives up the price of a portfolio of primary and secondary CDOs that has a unit net payoff in state s to the maximum valuation across agents. The following corollary shows how this equilibrium drives up the price of debt in line with that of assets.

Corollary 2 - *There is an equilibrium with trade in secondary CDOs where the price of CDO contracts equals*

$$Q^{SCDO}(x) = \frac{\int_x^1 \; max_i\{f_i(s)\}}{R} ds. \tag{24}$$

The next proposition compares the asset price with trade in secondary CDOs to that with collateralised trade or primary CDO trade.

Proposition 7 - *When f_1, f_0 are symmetric and have no mass points, the asset price bubble $p^{SCDO} - \frac{E_s}{R}$, with trade in secondary CDOs is at least twice as large as that with trade in collateralised loans or primary CDOs.*

Proposition 7 shows how trade in secondary CDOs allows agents to at least double the bubble in asset prices, as they can exploit differences in beliefs more efficiently. Particularly, while both collateralised loans and primary CDOs only allow to trade on differences in agents' CDFs, secondary CDOs allow trade on divergent beliefs regarding the PDF. An appendix shows how, at the price of a significantly more cumbersome proof, the condition of symmetry can be relaxed in favour of the more general assumption that $s^{\star} \geq \frac{1}{2}$.

Proposition 8 - *Whenever f_1 and f_0 are disjoint, the equilibrium price of the asset and that of the most senior CDO equal twice their expected discounted payoff $p^{SCDO} = 2\frac{E_s}{R}$, $Q^{SCDO}(0) = 2$.*

Proof of Proposition 8.

$$\begin{aligned} p^{SCDO} * R = \int_0^1 s \; max\{f_i(s)\} \\ = \int_0^1 s \; f_0(s)ds + \int_0^1 s \; f_1(s)ds \\ = 2E_s, \end{aligned} \tag{25}$$

where the second line follows from the fact that f_1, f_0 are disjoint. The proof for the price of the most senior CDO is equivalent. ■

With disjoint distributions f_0, f_1, type 1 perceives no cost from selling claims to agent 0, as for any x in $X_0 = \{x : f_0(x) > 0\}$ she expects to make 0 payments in the second period. Proposition 8 shows how this, intuitively, drives up asset prices to twice their fundamental value.

5 Extensions of the simple model

5.1 Risk-aversion

Similar to many previous studies, such as Miller (1977), Geanakoplos (2010), Fostel and Geanakoplos (2012), or Simsek (2013), we assume that investors maximise expected profits, and are therefore risk-neutral. How restrictive is this assumption for our results? In this section we briefly consider how risk-aversion changes the relative attractiveness of the three alternative risky investments (outright asset purchases, collateralised loans, leveraged asset purchases) in our setting, and also compare the effects to those that arise in a more standard framework where investors disagree about mean payoffs.

A first thing to note is that leverage 'spreads' asset payoffs. Specifically, while returns from outright asset purchase are distributed between $\frac{s^{min}}{p}$ and $\frac{s^{max}}{p}$, leveraged returns are riskier, located between $max\{0, \frac{s^{min}-\bar{s}}{p-q}\}$ and $\frac{s^{max}-\bar{s}}{p-q}$. In fact, for a sufficiently smooth distribution $f(s)$, returns from outright purchase have a second-order stochastic dominance relationship with leverage returns.[15] Introducing risk-aversion therefore trivially reduces the attractiveness of leveraged investments relative to outright purchases, independently of the particular form of belief disagreement under study.

A second effect of introducing risk-aversion is particular to the kind of disagreement we look at, and consists of breaking the consensus on what we have called the 'fundamental value' of assets, as investors with high perceived payoff dispersion now value the risky asset less than their counterparts.

Both the reduced attractiveness of leveraged assets relative to outright purchases, and the disagreement about 'fundamental value', however, will have a limited effect on the equilibrium as long as both investors continue to prefer the same investments as under risk-neutrality (in leveraged assets and collateralised loans respectively), and continue not to invest in outright asset

[15] It is easy to prove that, for given p,q, profits from outright purchase second order stochastically dominate leveraged profits whenever $f(s) < \frac{p}{p-q}f(s')\forall s, s' \in S$, which is, for example, trivially true for a uniform distribution.

purchases. The most important question is thus whether the pattern of equilibrium investments by different types changes with the introduction of moderate levels of risk-aversion. It seems instructive to compare the effect of risk-aversion on investment patterns in our setting, with different risk-perceptions but equal perceived mean payoffs, to that in a more standard environment where the beliefs of an "optimist" investor first-order stochastically dominate that of a "pessimist", as in Simsek (2013). Remember that, with different risk-perceptions, investors that perceive high risk have an absolute advantage - in the sense of higher expected returns at given prices - in buying leveraged assets, but a disadvantage in collateralised loans (whose downside risk they perceive as higher). Optimists whose beliefs first-order stochastically dominate those of pessimists, in contrast, have an absolute advantage in both investments, as they expect higher returns from both collateralised loans and leveraged investments. As Simsek (2013) illustrates, in equilibrium, optimists may be happy to 'overpay' on collateralised loans to raise funds for investing in the upside potential of assets, but disagreement on the downside, or the riskyness of collateralised loans, reduces the asset price in this environment. Importantly, this discussion implies that, in the framework of the present paper that considers heterogeneous risk perceptions, collateralised loans are held by the agents that perceive them to be relatively riskless, while the opposite is true in an alternative environment with first-order stochastic dominance in beliefs. This may amplify the effect of introducing risk-aversion on the equilibrium with first-order stochastic dominance, where it reinforces the dampening effect on asset prices through risk-premia on collateralised loans. We illustrate these effects of risk-aversion in a simple numerical example. Consider a standard utility function that exhibits risk-aversion and satisfies Inada conditions

$$U = u(c) + \frac{1}{R}u(c'), \tag{26}$$

where $u' > 0,\ u'' < 0,\ lim_{c \longrightarrow 0} u(') = \infty$, and we abstract from discounting for simplicity by setting $R = 1$.

Suppose beliefs are such that $f_1(1-\epsilon) = f_1(1+\epsilon) = \frac{1}{2}$, and consider two specifications for the beliefs of type 0 agents f_0: first, $f_0(1) = 1$, an extreme example of second-order stochastic dominance, where agent 0 regards the asset payoff as riskless. And second, $f_0(1 + 3\epsilon) = \frac{1}{2}$ $f_0(1) = \frac{1}{2}$, an example of first order stochastic dominance. Suppose both agents are endowed in period 0 with 0.5 units of the asset. Moreover, agent 1 is endowed with 3 units of the consumption good, while agent 0 has 0.5 units. The difference in consumption endowments has no effect on the results for the case of heterogeneous risk perception (where prices would be the same if endowments were symmetric, as agent 0 expects to make riskless investments). We make it to generate an incentive for leveraged asset trade in the case when agent 0's beliefs first-order stochastically dominate those of agent 1. Both agents can transfer resources to period 1 through one of three ways: issuing loans by providing collateral (as there is no commitment to promises), buying assets, or using a storage technology with gross return of 1.[16]

Consider first the case of second order stochastic dominance. Note that agent 0 regards all collateralised loans that promise $\overline{s} \leq 1$ as riskless, but will not pay any more for unit promises greater than 1. Agent 1 expects to make profits from issuing collateralised loans at the riskless price, as she perceives an upside potential of asset investments, whose payoffs she expects to equal $1 + \epsilon$ in 50 percent of the cases. The rest of this section shows that there exists an equilibrium where $q = 1$, $\overline{s} = 1$, $p > q$ and $a_0 = 0$, $a_1 = 1$, $b_0 = 1 = -b_1$. To see how this is an equilibrium, note that the first-order conditions for investments in, respectively, storage, outright asset purchases, leveraged assets and collateralised loans are

$$1 \geq E_i[\frac{u'(c_i')}{u'(c_i)}] \tag{Store}$$

$$1 \geq E_i[\frac{u'(c_i')}{u'(c_i)}\frac{s}{p}] \tag{Outright}$$

$$1 \geq E_i[\frac{u'(c_i')}{u'(c_i)}\frac{max\{s - \overline{s}, 0\}}{p - q}] \tag{Leverage}$$

$$1 \geq E_i[\frac{u'(c_i')}{u'(c_i)}\frac{min\{s, \overline{s}\}}{q}] \tag{Coll Loan}$$

[16]We do not include storage in the benchmark framework as with risk-neutrality the marginal rate of substitution across periods is pinned down by the discount factor.

which hold with equality whenever asset holdings are interior.

It is easy to see that, at $q = 1$, agent 1 will never buy collateralised loans, as she expects them to pay the same amount, but with only half the probability, as storage. Similarly, at the conjectured price of $p > q$, agent 0 will not invest in assets outright (as she prefers storage or loans that for her are payoff equivalent and cheaper) nor through leverage at $\bar{s} = 1$ as she expects to make 0 profits on her down payment $p - q > 0$. Denoting the value of agent i's endowment as $e_i(p)$, type 0 thus expects consumption to equal $c_0 = c_0' = \frac{e_0(p)}{2}$ in equilibrium, and she is happy to hold all collateralised loans $b_1 = 1$, and store $d_0 = e(p) - b_1 > 0$, where d_i denotes storage by agent i in period 0. For low enough $p > q$, agent 1 will not want to buy assets outright (as they are more expensive and riskier than storage) but makes strict profits from buying all assets using leverage and storing some of her endowment on top. Her consumption in period 0, the price of assets and her storage are thus determined by the first-order conditions for storage and leveraged investment evaluated at type 1 expectations,

$$1 = \frac{1}{2}[\frac{u'(d_1)}{u'(c_1)} + \frac{u'(d_1 + \frac{\epsilon}{p-1})}{u'(c_1)}]$$

$$1 = \frac{1}{2}[\frac{u'(d_1)}{u'(c_1)} * 0 + \frac{u'(d_1 + \frac{\epsilon}{p-1})}{u'(c_1)} * \frac{\epsilon}{p-1}]$$

and type 1's period 1 budget constraint

$$c_1 + d_1 + (p - 1) = 3 + \frac{1}{2}p,$$

where $p - 1$ is the equilibrium down payment on leveraged assets. In other words, Type 1 agents drive up the asset price to the point where they are indifferent between leveraged asset purchases and storage.

For example, with $\epsilon = 0.5$, the risk-neutral price equals 1.25. With log-preferences, corresponding to a constant relative risk aversion equal to 1, it is easily shown that the equilibrium asset price drops to $p = 1.22$, and agent 1's consumption and storage equal, respectively, $c_1 = 1.8$ and $d_1 = 1.59$. But importantly, agent 1 continues to hold all assets using leverage ($a_1 = 1$).

Despite risk-aversion and Inada conditions on utility, agent 1 takes a lot of risk through leveraged investments, as she expects to make an expected return of $\frac{1}{2}\frac{1+\epsilon}{p-1}$ equal to about 3.5 times that of investing in storage. To see that the comparative static effect of belief dispersion also pertains in this example, increase belief dispersion by raising ϵ to 1, which increases the asset price to $p = 1.37$, compared to an increase to 1.5 of the risk-neutral price.

To contrast this result briefly to the case of first-order stochastic dominance, consider agent 0 beliefs of $f_0(1+3\epsilon) = \frac{1}{2}$, $f_0(1) = \frac{1}{2}$ for $\epsilon = \frac{1}{2}$, and look at an example similar to the one before, but where it is now agent 0 that considers funding asset purchases by issuing collateralised loans of face value 1 to agent 1, as her endowment of 0.5 is too small to buy all assets at their face value outright. Under risk-neutrality, agent 0 thus issues collateralised loans at a reduced price of $1 - \frac{1}{2}\epsilon = 0.75$ that corresponds to their payoff expected by type 1. Particularly, she is happy to 'overpay' on what, in her eyes, are riskless loans in order to raise funds for investment in the upside potential of the asset. The risk-neutral equilibrium price of the asset is now equal to 1.5. With log-preferences, however, this equilibrium, where investors specialise in collateralised loans and leveraged assets respectively, breaks down, for two reasons. First, type 1 investors now require a risk-premium to hold collateralised loans, whose price would have to drop to 0.7 in order to make type 1s hold all collateralised loans at an unchanged value of their period 1 endowments. At this reduced price of loans, type 0 investors find it even less profitable to make risky asset investments using leverage. The second reason is the low consumption endowment of type 0 investors, which reduces their average consumption, and thus makes them more averse to assuming a given absolute amount of risk. The assumption, however, is necessary as, with a higher consumption endowment, type 0 investors would usually prefer to avoid issuing loans at unfavorable prices, and buy the asset without leverage. Note again the difference with heterogeneous beliefs about risk that the present paper focuses on: in our framework, investors are happy to issue collateralised loans to agents with less dispersed beliefs even when their consumption endowment is high, as they expect to make profits from doing so.

This example is, evidently, not more than a mere illustration of how the equilibrium may change with risk-aversion. In general, leveraged investments become less attractive as risk-aversion increases - the reason for the fall in asset prices relative to the risk-neutral price in the example above. But as long as belief disagreement is high enough and investors can diversify their investments to self-insure partly against the risk from leveraged investment, the equilibrium properties derived under risk-neutrality are likely to hold also under moderate levels of risk-aversion, as the example illustrates. The same is true, in principle, for the case of first-order stochastic dominance considered in previous studies, with the caveat that there, collateralised loans are bought by those who perceive down-side risks to be more severe. The dampening effect this has on asset prices will typically be reinforced with risk-aversion and can, as our example illustrates, contribute to changing the equilibrium properties under first-order stochastic dominance, as it becomes more expensive for optimists to raise funds from pessimists when the latter require a risk-premium.

5.2 A dynamic example with learning

This section takes a first step towards a dynamic analysis of belief differences in an environment with learning. Apart from the motivation of belief differences, there are, however, other factors at the heart of the analysis that have an important dynamic dimension. Thus, for all long-lived assets, future price movements are an important determinant of both the prices investors are ready to pay today, and of the return risk they face over and above fluctuations in payoffs. Also, the wealth distribution across agents of different beliefs - shown to be an important determinant of asset prices in the static environment - should be expected to vary over time and thus lead to fluctuations in prices. We leave an in-depth analysis of these dynamic considerations for future research. To illustrate some of the issues involved, however, we present in this section a simple example of a dynamic version of the two-type model where belief heterogeneity arises because some agents adjust their risk perception more quickly than others in reaction to an observed fall in macro-volatility. Specifically, we look at a version of the two-type economy where time is infinite $t = 0, 1, 2, ...$ and physical assets pay a random amount of the consumption good s_t that is independent across periods. Agents of both types are infinitely-lived and receive consumption endowment $n_i, \forall t$. Agents maximise the present discounted value of consumption through decisions on consumption c_{it} and on purchases of assets and collateralised loans, a_{it+1}, b_{it+1} respectively, every period. As before, they trade physical assets, whose quantity we normalise to 1, and collateralised loans whose face value for next period $\overline{s}_{t+1}$ is agreed on in t.

Suppose for now that agents specialise their investment in only 1 asset class, with i and j agents buying leveraged assets and collateralised loans, respectively. The budget constraint for leveraged investors is then

$$a_{it+1}(p_t - q_t) + c_{it} \leq n_i + max\{a_{it}(p_t + s_t - \overline{s}), 0\}). \tag{27}$$

The budget of agents that purchase collateralised loans is characterised by

$$b_{jt+1}q_t + c_{it} \leq n_i + b_{jt}(min\{p_t + s_t, \overline{s}\}). \tag{28}$$

Again we assume that type j buyers of collateralised loans make 0 surplus, so

$$q_t = \frac{E_{jt}[min\{p_{t+1} + s_{t+1}, \overline{s}]}{R}. \tag{29}$$

Agents invest in leveraged assets if

$$R_i^a = \frac{E_{it}[p_{t+1} + s_{t+1} - min\{p_{t+1} + s_{t+1}, \overline{s}\}]}{p_t - \frac{E_j[min\{p_{t+1}+s_{t+1},\overline{s}\}]}{R}} \tag{30}$$

$$= \frac{E_{it}[p_{t+1}] + E_s - E_{it}[min\{p_{t+1} + s_{t+1}, \overline{s}\}]}{p_t - \frac{E_j[min\{p_{t+1}+s_{t+1},\overline{s}\}]}{R}} > R, \tag{31}$$

where expectations are now allowed to vary both across type and time as agents learn from empirical evidence. In Broer and Kero (2013), we analyse how risk-averse homogeneous investors price assets under different Bayesian and ad-hoc learning rules in a model with time-varying volatility but without leverage. Here, we use an anticipated-utility approach with a particularly simple learning rule to illustrate the asset price dynamics that can result from introducing leverage in a model where, absent leverage, prices would equal the constant common valuation of dividends by risk-neutral investors. Again, we concentrate on payoffs of a generic asset as the source of randomness in the environment, which agents believe to be distributed according to the distribution $f_{i,t+1}$ next period. Motivated by the fact that the Great Moderation was established as an empirical fact by the mid-1990s, we analyse a scenario at the beginning of which agents perfectly observe a change in the payoff distribution from h^{pre} to h^{post}, but differ in the way they adjust their beliefs about the future. Specifically, agents adjust last period's belief $f_{i,t}$ by a constant fraction g_i towards the current observed distribution of payoffs h_t

$$f_{i,t+1} = (1 - g_i)f_{i,t} + g_i h_t. \tag{32}$$

In line with the characterisation of type 0 agents as the ones with tighter beliefs than type 1 agents in the static environment, we assume $g_0 > g_1$, such that type 0 adjust believe quicker in the new environment of low volatility. Finally, we adopt an anticipated utility approach to behaviour as in Cogley and Sargent (2008), whereby agents update their beliefs in line with the simple rule in (32), but do not anticipate to learn further in the future.

We define a competitive equilibrium in this environment as sequences of prices and quantities as functions of the state of the economy $(s_t, \overline{s}_t, a_{it}, a_{jt})$ such that agents optimise given belief f_{it} and markets for consumption and assets clear. In other words we focus on an equilibrium where agents agree on the price but disagree on the distribution of exogenous shocks. Their beliefs change over time, but, in line with our anticipated utility approach, agents use constant beliefs to calculate expectations for the future.

The following proposition shows the equilibrium with trade in collateralised loans when both h^{pre} and h^{post} are continuous and symmetric, implying symmetric distributions $f_{0,t}, f_{1,t}$. For this, we assume that assets are initially endowed to type 0 agents.[17]

[17] This is convenient as it implies that the present discounted value of tomorrow's asset holdings $\frac{\overline{p}}{R}$ nets out in type 0 agents' budget, who sell 2 units of the asset but by 2 units of collateralised loans. Alternative assumptions that are consistent with type 0 agents being able to afford all collateralised loans, however,

Proposition 9 - *Stationary Dynamic Equilibrium*

If $n_1 \geq 2\frac{E_s - E_1[min\{s,s^{\star}\}]}{R}$*, there is an equilibrium with the following properties*

- $p_t = \overline{p} = \frac{E_s + E_0(min(s,\overline{s})) - E_1(min\{s,\overline{s}\})}{R-1}\ \forall\ t \geq 0,$

- $q_t = \frac{E_0(min(s,\overline{s})) + \overline{p}}{R}\ \forall\ t \geq 0,$

- $\overline{s} = s^{\star} + \overline{p}\ \forall\ t \geq 0,$

where again $s^{\star}$ *is the single crossing point of type* 1 *and* 0*'s CDFs.*

5.2.1 A calibrated example

To illustrate the quantitative effect of leverage on asset prices in this simple example economy, we normalise expected payoffs to 2 and the standard deviation during the pre-Great Moderation period to 1 by choosing h^{pre} to have a uniform distribution on $[0.285, 3.715]$. In line with the fall in the standard deviation of both US GDP and consumption growth during the Great Moderation to half their previous values, we choose h^{post} to be a triangular distribution with standard deviation $\frac{1}{2}$. Figure 5 illustrates the two distributions.[18] Finally, we look at two calibrations of the learning parameters. First, we analyse a scenario of "temporary disagreement", where $g_1 = 1\%$ and $g_0 = 3\%$, chosen to have asset prices peak after 55 periods, in line with the US experience where the Great Moderation started in the second half of the 1980s and prices peaked towards the end of the 1990s. A second calibration is based on the evidence from the SPF, where disagreement about risk continued to rise throughout the Great Moderation period. In this second calibration we thus set $g_1 = 0$, implying that type 1 agents continue to have 0 confidence in the fall in macro-volatility. We set quarterly interest rates to 1 percent (which affects the level of asset prices, but not the normalised levels presented below).

would imply the same equilibrium prices.

[18]We opt for a triangular distribution for h^{post}, rather than a uniform distribution, because, for infinitesimally small disagreement, the latter implies that $f_{0,t}(s) > f_{1,t}(s)$ for all $s \in (0.285, 3.715)$, potentially leading to extreme swings in asset holdings for small changes in beliefs with trade in secondary CDOs.

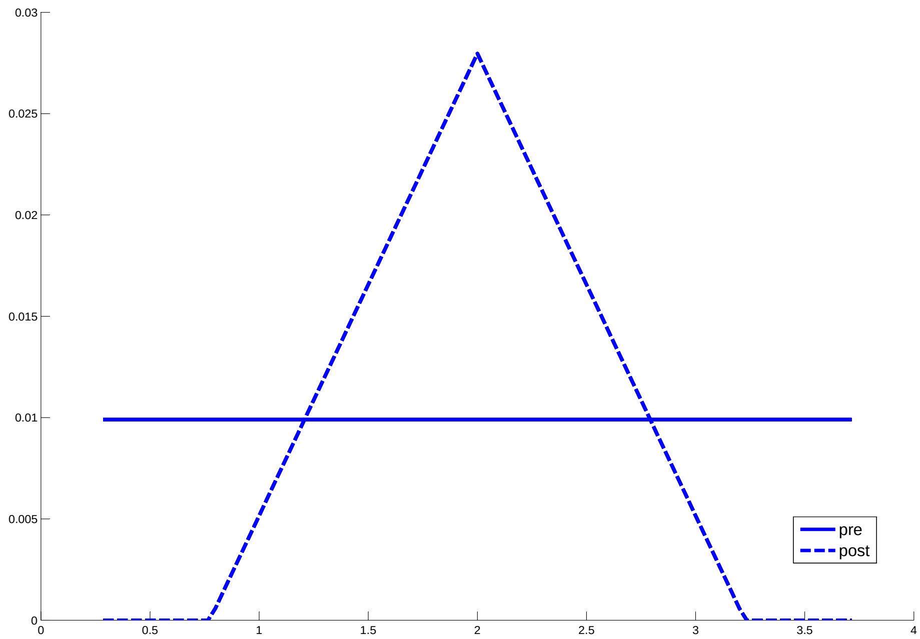


Figure 5: The figure plots the distribution of payoffs before (solid line) and during the Great Moderation (dashed lined).

Figure 6 presents the results for the first scenario. The top panel shows how, after the onset of the Great Moderation in $t = 0$, the standard deviations of the posterior payoff distribution first diverge before slowly re-converging. Mean asset prices with trade in collateralised loans (or primary CDOs), in the central panel, follow a hump-shaped pattern and peak at about 5 percent above their initial value. The price of collateralised loans follows closely that of the collateral asset. The bottom panel of figure 6 shows how, with trade in primary and secondary CDOs, the hump-shape of asset prices is more pronounced and the magnitude of the asset price boom, which peaks at about 15 percent, larger.

Figure 7 presents the results for the second scenario, where type 1 agents continue to believe in the high volatility of the pre-Great Moderation era. As type 0 agents' beliefs slowly converge to the new observed distribution, disagreement, as well as asset prices, increase monotonically. With trade in collateralised debt, prices of assets and collateralised loans rise by 11 and 13 percent, respectively. But with trade in primary and secondary CDOs, the rise in asset prices

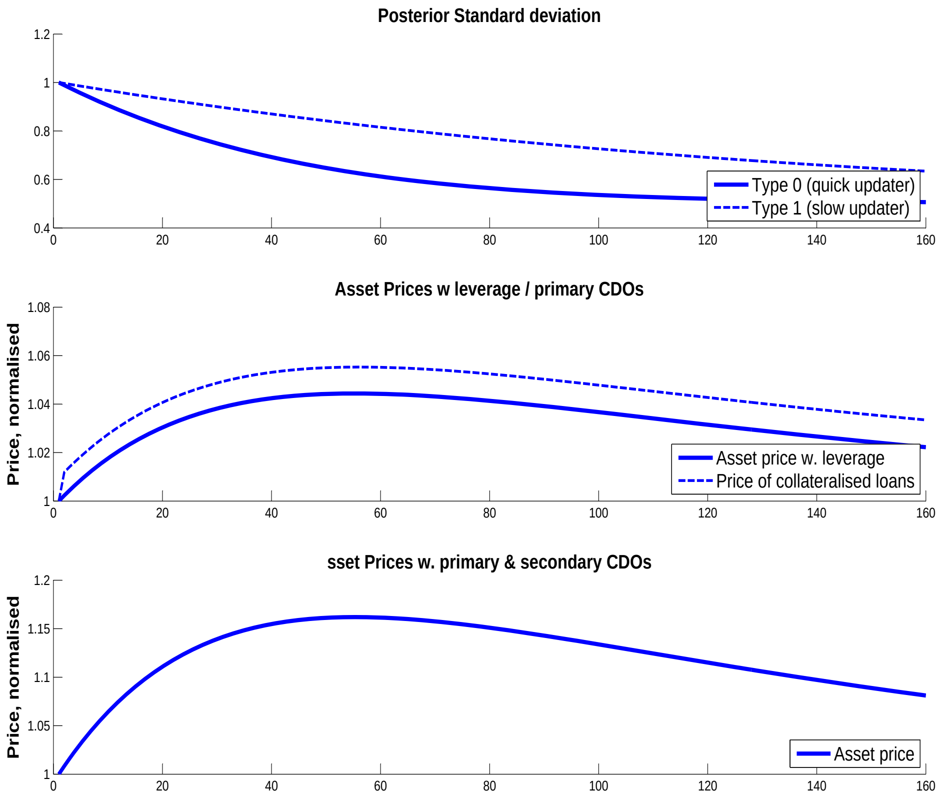


Figure 6: The top panel plots the standard deviations of payoffs for both types. The central panel plots the prices of collateralised loans and assets and bottom panel plots the asset price with trade in primary and secondary CDOs.

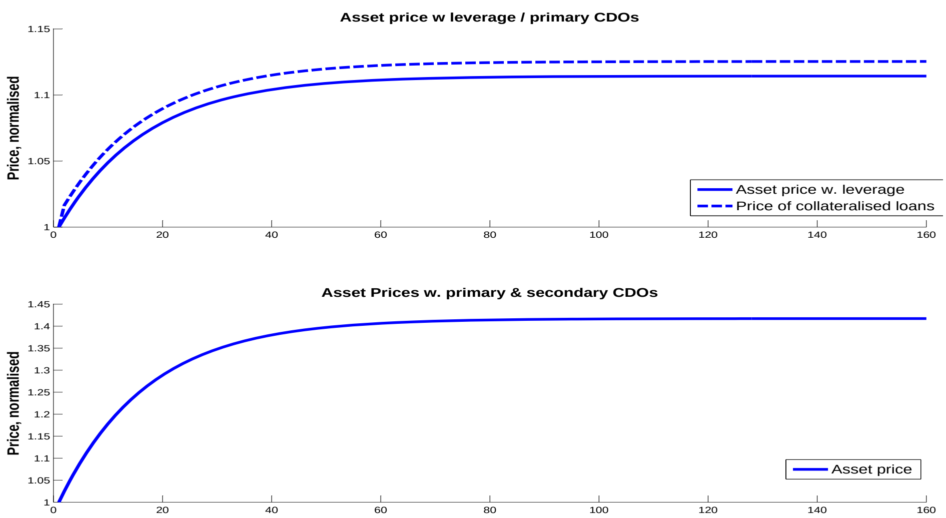


Figure 7: The top panel plots the prices of collateralised loans and assets and the bottom panel plots the asset price with trade in primary and secondary CDOs.

is increased strongly, to more than 40 percent.

We see these quantitative results as illustrative. In our view they show how, potentially, the financial innovation of the 1990s and 2000s may have contributed to a significant rise in asset prices, as the introduction of secondary CDOs raises the asset price bubble to between 3 and 4 times its level in the scenario with trade in collateralised loans only. We would like to stress, however, that the highly stylised nature of this example makes it, more than anything, an illustration of promising avenues for future research.

6 Conclusion

This paper has looked at the role of collateralised asset trade in economies where investors disagree about risk, rather than mean payoffs considered in the literature. The analysis was motivated by the fact that US surveys on expectations of stock returns and GDP growth showed strong, and in the case of GDP growth rising disagreement about the dispersion of outcomes around their mean value. We presented a simple static model of investor disagreement, where in the absence of collateralisation, risk-neutral investors trade assets at their common fundamental value even if they disagree about payoff risk. The introduction of simple collateralised loans increased asset prices above this common fundamental value by unleashing perceived gains from trade. In addition, allowing agents to use CDOs to collateralise more junior, "secondary" CDO contracts, strongly raised prices even more. Therefore, the paper underlines that disagreement about risk and collateralised contracts of different degrees of sophistication are strongly complementary in their effects on asset prices. Finally we illustrated this mechanism in a highly stylised quantitative example with learning at different speeds about an abrupt fall in volatility, as in the Great Moderation of macro-volatility. The rise in asset prices due to the resulting disagreement about risk was modest with trade in collateralised loans, but substantial with trade in primary and secondary CDOs. We hope that our analysis opens some avenues for further research. Particularly interesting seems a more complete dynamic analysis than that of Section 5.2. In addition, one of the implications of our model is that agents disagree about default probabilities in equilibrium. Interestingly, this is in line with recent evidence of disagreement among credit rating agencies. [19] Thus it would be interesting to study empirically, how credit rating disagreement affects the resulting asset prices. Finally, it is important to analyse the reasons behind disagreement about the distributions of payoffs, taken as given in this paper, and how perceived risk varies with investor characteristics.

[19] For example, Norden and Roscovan (2014) use a large sample of US and European firms to document important differences in ratings, and the implied default probabilities, among the three major credit rating agencies, Moodys, S&P and Fitch.

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7 Appendix A: Omitted Proofs

Proof of Proposition 3.

Equation (10) is simply the optimality condition for leverage choice. To understand equations (11) and (12), note that for any $p < \frac{E_s}{R}$ all agents would like to buy risky assets, which cannot be an equilibrium. Equivalently, for any $p > \overline{p} \doteq \frac{E_s + E_0(min(s,\overline{s})) - E_1(min\{s,\overline{s}\})}{R}$ both type 0 and type 1 agents would like to sell their risky assets, again contradicting equilibrium. Agent 1 optimality implies that they invest all resources in leveraged assets when $\frac{E_s}{R} \leq p < \overline{p}$, but are indifferent between buying leveraged assets and consuming at $p = \overline{p}$. Thus, for $\overline{s}(\overline{p})$ the value of $\overline{s}$ that solves (10) when $p = \overline{p}$, if $n_1^{\max}(\overline{s}(\overline{p})) \geq \overline{p}$, type 1's endowment is large enough to buy type 0's assets at the maximum price $\overline{p}$ that ensures her participation. There is thus an equilibrium price $\overline{p}$ at which type 1 agents are happy to consume in period 0 any resources that remain after purchasing all of type 0's assets.

If for some price $p : \frac{E_s}{R} \leq p < \overline{p}$ $n_1^{\max}(\overline{s}(p)) < p$, type 1 agents cannot buy all assets at that price but expect to make strictly positive profits $R_1^a > R$, so invest all their resources to buy type 0's assets, implying market clearing condition (12).

Finally, to prove uniqueness, since (12) is trivially strictly upward-sloping, it suffices to show that (10) is downward-sloping. This follows by differentiating (10) totally

$$\frac{dp}{d\overline{s}} = -\frac{\frac{d\mathbb{C}}{d\overline{s}}}{\frac{d\mathbb{C}}{d\overline{p}}} \tag{33}$$

Weak concavity of $R_1^a(\overline{s})$ at the optimum choice of $\overline{s}$ implies that the numerator is weakly negative. Since $\frac{\mathbb{C}}{d\overline{p}} < 0, \forall p, \overline{s}$ the result follows. ■

Proof of Proposition 5.

According to Lemma 1 the asset price with trade in collateralised loans equals $\overline{p}$. At this price, according to (10) the optimal choice of face value for loans $\overline{s}$ is equal to the single crossing point of the CDFs $s^\star$.

With trade in CDOs, $Q(x) = (1 - F_0(x))$ for $x < s^\star$ as type 0 agents are cash-rich. When type

1 agents buy assets, they thus find it optimal to issue all CDOs with seniority $x > s^\star$ to type 0 agents. This implies type 1's return on funds invested in the asset and partly financed through CDO issuance equals

$$R_1^{aCDO} = \frac{E_s - \int_0^{s^\star}(\int_s^1 f_1(x)dx)ds}{p - \frac{\int_0^{s^\star}(1-F_0)(x)dx}{R}}$$
$$= \frac{E_s - E_1[min\{s, s^\star\}}{p - \frac{E_0[min\{s,s^\star\}]}{R}} \tag{34}$$

where the last line follows since $\int_s^1 f_1(x)dx = 1 - F_1(s)$ and

$$\int_0^{s^\star}(1 - F_i(x))dx = s^\star - [xF_i(x)|_0^{s^\star} - \int_0^{s^\star} xf_i(x)dx]$$
$$= (1 - F_i(s^\star))s^\star + \int_0^{s^\star} xf_i(x)dx = E_i[min\{s, s^\star\}] \tag{35}$$

Equation (35) states that both the proceeds from CDO issuance in the first period and the expected payments by type 1 agents in the second period are equal to those from the issuance of collateralised loans in the previous section. The indifference condition $R_1^{aCDO} = R$ thus implies a reservation price of $\overline{p} = \frac{E_s+E_0(min(s,\overline{s}))-E_1(min\{s,\overline{s}\})}{R}$, exactly equal to that with collateralised loan trade. Since the payments in period 1 are exactly the same as with collateralised loan trade, type 1 agents can afford to buy all assets at this price. ∎

Proof of Proposition 6. [20]

Note that, as in the previous section with trade in primary CDOs only, portfolios are not uniquely defined in equilibrium. So, again, we normalise portfolios such that agents of type 1 buy all the assets. Suppose that they also buy all secondary CDOs with seniority $x \in X_1$ and issue primary CDOs (backed by their asset holdings) and secondary CDOs (backed by more senior secondary CDOs) for any $x \in X_0$. Suppose also that type 0 agents buy all primary CDOs with seniority $x \in X_0$ and sell secondary CDOs (backed by any more senior CDOs they hold) with seniority

[20]The proof considers the case where the mass of $x : f_1(x) = f_0(x)$ is zero, such that agents almost surely disagree on the pdf. This is without loss of generality, as it is trivial to account for 'regions of agreement'.

$x \in X_1$. This leaves type 1 agents with claims from their asset holdings and CDO purchase equal to $2s + \int_0^s B_1(x) + \widehat{B_1(x)}dx = 2s$ for all $s \in X_1$, and 0 otherwise. This is because for any $s \in X_1$ type 1 agents buy secondary CDOs that exactly equal their previous issuance of more senior CDOs, leaving a 0 net claim from CDOs, such that total net claims equal those from initial asset holdings. For $s \in X_0$, in contrast, claims from primary CDO issuance are equal to $-2s$ and claims from secondary CDOs bought and sold cancel, leaving a zero claim to payoffs overall. Similarly, net claims by type 0 agents due to their purchase and sale of CDOs equal $\int_0^s B_1(x) + \widehat{B_1(x)}dx = 2s$ for all $s \in X_0$, and 0 otherwise.

Suppose $Q(1) = 0$, $\frac{\delta Q}{\delta s} = -\frac{max_i\{f_i(s)\}}{R}$, and that the asset is priced by arbitrage $p = p^{SCDO}$. To show that this is an equilibrium, note that all agents expect to make 0 profits from their trading strategies. Moreover, any deviation from agent 1's strategy by buying and selling CDOs of seniority $x \in X_0$ and $x' > x$ respectively, implies a loss as expected profits equal $E_1[s \in [x, x']] - [Q(x) - Q(x')] = \int_x^{x'} f_1(x)dx + \int_x^{x'} \frac{\delta Q(x)}{\delta x}dx = \int_x^{x'} (f_1(x) - max\{f_0(x), f_1(x)\})dx < 0$ (and equivalently for type 0 agents). Finally, agents are indifferent between consuming and buying the asset at p^{SCDO}, partly financed through appropriate issuance of CDOs. So trading strategies are optimal. To show that they are also affordable, write the period 1 budget as endowment plus net claims sold minus net claims purchased

$$\begin{aligned} n_0 + p^{SCDO} - \frac{2\int_{X_0} sf_0(s)ds}{R} &\geq \frac{E_s + \int_{X_1} sf_1(s)ds - \int_{X_0} sf_0(s)ds}{R} \geq 0, \\ n_1 + \frac{2\int_{X_0} sf_0(s)ds}{R} - p^{SCDO} &= n_1 + \frac{\int_{X_0} sf_0(s)ds - \int_{X_1} sf_1(s)ds}{R} \geq 0, \end{aligned} \tag{36}$$

where the last inequality follows from Eq. (22). ■

Proof of Proposition 7.

$$\begin{aligned}
p^{SCDO} * R &= \int_0^1 s\ max\{f_i(s)\}ds = E_s + \int_0^1 s(max\{f_i(s)\} - f_1(s))ds \\
&= E_s + \int_0^{\frac{1}{2}} (s + (1-s))(max\{f_i(s)\} - f_1(s))ds \\
&\geq E_s + 2[\int_0^{\frac{1}{2}} s(f_0 - f_1))ds + \int_{\frac{1}{2}}^1 \frac{1}{2}(f_0 - f_1)ds \\
&= E_s + 2[E_0[min\{\frac{1}{2}, s\} - E_1[min\{\frac{1}{2}, s\}] \\
&\Rightarrow p^{SCDO} - \frac{Es}{R} \geq 2(\overline{p} - \frac{Es}{R}),
\end{aligned} \tag{37}$$

where the second line follows from symmetry, the third follows since $1 - s \geq s$ for $s \leq \frac{1}{2}$ and $fmax\{f_i(s)\} \geq f_0(s)$ as well as the fact that $\int_{\frac{1}{2}}^1 c(f_0 - f_1) = 0$ for any constant c when f_0, f_1 are symmetric around $\frac{1}{2}$. ∎

Proof of Proposition 9. To verify that the postulated investment rules and price process form an equilibrium, we must show that type 1's choices are optimal and affordable given the price process, and that markets clear.

Note that at the postulated price, type 1 agents are exactly indifferent between consuming and investing as $E_1[p_{t+1}] = E_0[p_{t+1}] = p_t = \overline{p}$ implies $R_1^a = R$. Moreover, $E_1[min\{p_{t+1} + s_{t+1}, \overline{s}\}] < E_0[min\{p_{t+1} + s_{t+1}, \overline{s}\}]$, so $R_1^a > R_0^a$ and agent 0 strictly prefers to consume or invest in collateralised loans, rather than invest in leveraged assets.

That $\overline{s} = s^\star + \overline{p}$ is an optimal choice for loan riskiness follows from $R_1^a = R$ and the first order condition (9) that applies unchanged as prices are constant over time.

Finally, the assumption that $n_1 \geq 2\frac{E_s - E_1[min\{s, s^\star\}]}{R}$ implies that type 1 agents with tighter priors have large enough consumption endowments to buy all the assets at the price $\overline{p}$. Note also that payments from type 1 agents to type 0 agents to buy the assets are greater than the value of collateralised loans in the economy, so type 0 agents can afford to buy all collateralised loans.

∎

8 Appendix B: Asset prices in a continuum economy with disagreement about payoff risk

This section considers an economy with a continuum of investor types who differ in their perception of the riskiness of a single asset. It shows how the introduction of collateralised debt contracts creates an asset price bubble, which we define as a positive deviation of the asset prices from its expected discounted payoff, identical across investors. Moreover, we show how an increase in belief dispersion raises asset prices further.

We now look at the general case with a continuum of agents of unit-mass indexed by i with $i \in I = [0, 1]$.

8.1 Type i's problem for given $\overline{s}$

Take $\overline{s}$ as given. Agent i's optimization problem is:

$$\max_{c_i, c_i', a_i \geq 0, b_i > -a_i} c_i + \frac{1}{R} E(c_i') \tag{38}$$

subject to

(4), (5) and (3).

8.2 General Equilibrium

In this section we look at the equilibrium of an economy with exogenous face value $\overline{s} \leq E_s$. For this, we normalise the asset supply to 1 and assume $\overline{a}_i = 1 \ \forall i$. Also, for simplicity, we assume constant consumption endowments $n_i = n \ \forall i$.

Definition 4 - *A general equilibrium, given $\overline{s}$, is a set of prices (p, q) and allocations $(c_i, c_i', a_i, b_i)_{i \in [0,1]}$, such that agent $i \in [0, 1]$ behaves optimally given p, q and $\overline{s}$, the demand for assets equals the* *fixed supply,*

$$\int_{i \in I} a_i = 1, \tag{39}$$

and the collateralised loan market clears,

$$\int_{i \in I} b_i = 0.$$

8.2.1 Uniqueness of equilibrium and asset price bubbles

Note that for any $p < \underline{p} \doteq \frac{E_0}{R}$ all agents would like to buy risky assets, which cannot be an equilibrium. Similarly, for any given $q > \overline{q} \doteq \frac{E_0(min\{s,\overline{s}\})}{R}$, no agent is willing to buy collateralised loans, but all agents who hold assets make a strict profit by using them as collateral for the issuance of collateralised loans. Again, this cannot be an equilibrium.

Proposition 10 - *Uniqueness of a bubble equilibrium.*

There is a unique equilibrium with trade in assets of given riskiness $\overline{s}$. This equilibrium has the following properties:

- $p: \frac{E_s - E_1[min(s,\overline{s})]}{R} + q \doteq \overline{p} \geq p > \underline{p} \doteq \frac{Es}{R}$
- $q < \overline{q} \doteq \frac{E_0(min\{s,\overline{s}\})}{R}$
- *There are cutoff values $i_q < i_p$ such that all agents with $i > i_p$ invest their whole endowment $n + p$ in leveraged asset purchases, while all agents with $i < i_q$ invest their whole endowment in collateralised loans. Agents with $i : i_q \leq i \leq i_p$ sell their asset endowment and consume the proceeds together with their endowment of consumption goods.*

Proof of Proposition 10. Take any $q < \overline{q}$ and define

$$i_q : \ q = \frac{E_{i=i_q}(min\{s,\overline{s}\})}{R}. \tag{40}$$

Take any $p \geq \underline{p}$ and define

$$i_p : \; R^a_{i_p} \equiv \frac{E_s - E_{i=i_p}(min\{s, \overline{s}\})}{p - q} = R, . \tag{41}$$

where $R^a_i = \frac{E_s - E_i(min\{s,\overline{s}\})}{p-q}$ is the gross return agent i expects from a unit of own funds invested in leveraged assets. Note that for $p = \frac{E_s}{R}$, $i_q = i_p$ and for $p > \frac{E_s}{R}$, $i_q < i_p$. Note that for any $p \geq \underline{p}$, all agents weakly prefer to sell their assets and consume the proceeds over holding them outright (i.e. without leverage). Since $E_i[min\{s, \overline{s}\}]$ is decreasing in i, agents with $i > i_p$ ($i < i_q$) expect to make strictly positive profits from leveraged asset (collateralised loan) purchases. So all agents with $i > i_p$ ($i < i_q$) sell their assets and invest the proceeds, together with their consumption endowments, in leveraged assets (collateralised loans). Moreover, since for $p = \frac{E_s}{R}$ $i_p = i_q$, and for $p > \frac{E_s}{R}$ any agent with $i : i_q < i < i_p$ strictly prefers selling her assets and consuming, it has to be that $i > i_p$ agents buy all assets, while $i < i_q$ agents buy all collateralised loans, both of which have supply equal to 1. The market clearing condition for leveraged assets thus becomes

$$\int_{i_p}^{1} a_i g(i) di = \int_{i_p}^{1} \frac{n+p}{p-q} g(i) di = \overline{a} = 1 \tag{42}$$

$$\Rightarrow \qquad (n + p)(1 - G(i_p)) = (p - q), \tag{43}$$

where the first equality substitutes for a_i from the budget constraint for $i > i_p$ agents with $c_i = 0$, $\overline{a} = 1$ and $b_i = -a_i$. Note that this immediately puts an upper bound $\overline{p}(q)$ on the asset price at the level where even agents of type $i = 1$, whose beliefs are most dispersed and who thus expect to make the highest profit from leveraged asset purchases, do not want to buy assets

$$\overline{p}(q) \leq \frac{E_s - E_1[min(s, \overline{s})]}{R} + q. \tag{44}$$

The market clearing condition for collateralised loans can be written as

$$\int_0^{i_q} b_i g(i)di = \int_0^{i_q} \frac{n+p}{q} g(i)di = 1 \tag{45}$$

$$\Rightarrow \quad (n+p)G(i_q) = q. \tag{46}$$

where the first equality substitutes for b_i from the budget constraint for $i < i_q$ agents with $a_i = 0$, $\overline{a} = 1$ and $c_i = 0$. Note that, since i_q is decreasing in q, so are $G(i_q)$ and the left-hand side of (46), which thus provides a unique mapping from the asset price p into a market clearing price $q^\star$, thus defining $i_q^\star$. From (46) and (43) we get

$$p = n(\frac{1}{G(i_p) - G(i_q)} - 1). \tag{47}$$

Clearly, this equation has no finite solution for $i_q = i_p$. Hence $i_q < i_p$ in equilibrium and thus $p > \frac{E_s}{R}$. Note that, without loss of generality, we have assumed a tie-breaking rule for agents with $i = i_q$ and $i = i_p$ both of mass zero.

■

Proposition 10 shows how the convexity of payoffs due to leverage allows to exploit perceived gains from trade arising from heterogenous beliefs about payoff dispersion. Investors who perceive risk to be high (low) expect to make strictly positive profits and invest all their funds in leveraged assets (collateralised loans). Market clearing for consumption goods requires that there be a "middle" interval (i_q, i_p) of agents who consume in the first period. For this to be the case, asset prices must exceed their fundamental value $\underline{p}$. In other words, collateralised debt causes a bubble in asset prices.

Figure 8 illustrates the equilibrium. Types $i \leq i_q$ invest the value of their whole endowment (equal to $n + p$) in collateralised loans, with a total demand equal to $b = G_1 \frac{n+p}{q}$ for $G_1 = \int_0^{i_q} dG(i)$. Similarly, the demand for consumption goods, by agents with $i : i_q \leq i < i_p$ equals $c = G_2(n + p)$ for $G_2 = \int_{i_q}^{i_p} dG(i)$. And finally, total asset demand, by agents with $i > i_p$ equals $a = G_3 \frac{n+p}{p-q}$ for $G_3 = \int_{i_p}^1 dG(i)$.

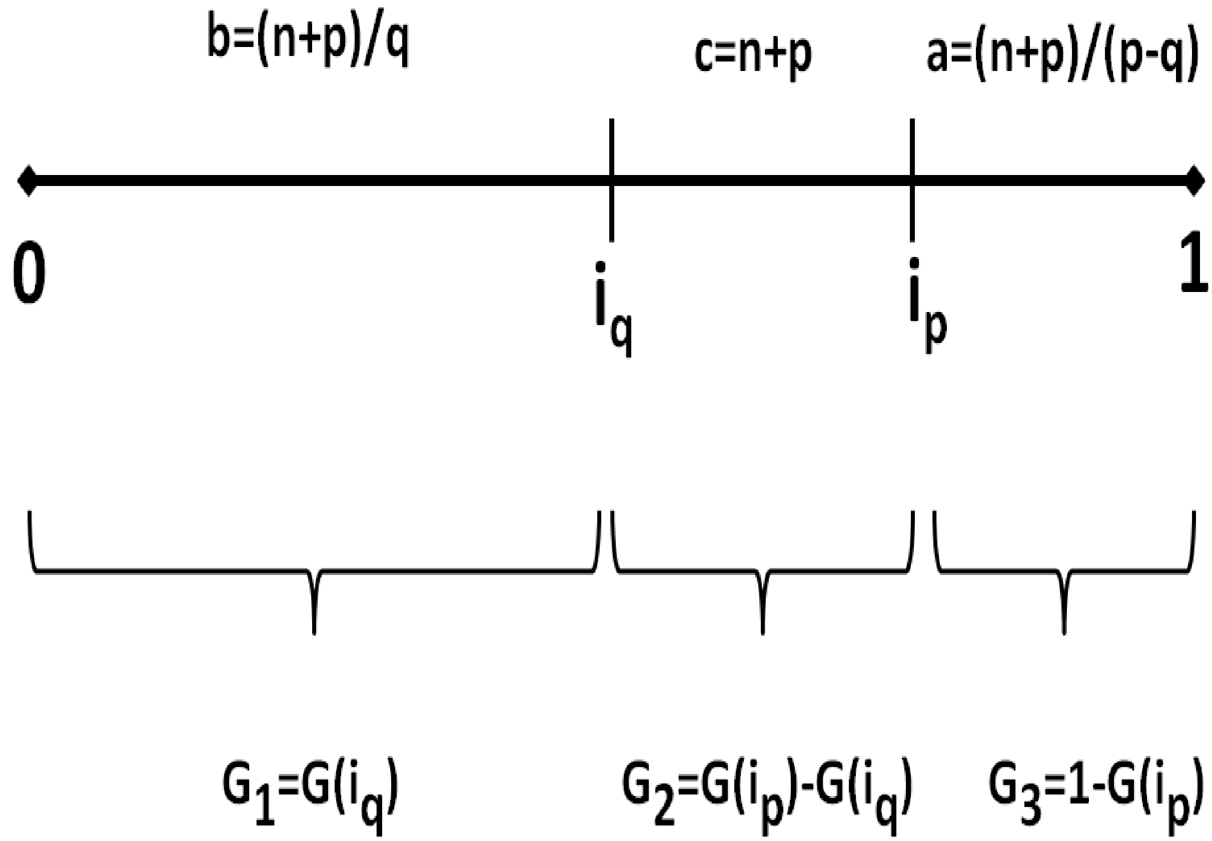

Figure 8: The upper panel plots the three intervals on [0,1] that correspond to: 1. investors with low belief dispersion ($i \leq i_q$), who prefer to buy collateralised loans to consuming or buying assets; 2. investors with medium belief dispersion ($i_q < i \leq i_p$), who prefer to consume, rather than invest; and 3. investors with high belief dispersion ($i > i_p$), who prefer to buy leveraged assets. The lower panel plots the corresponding mass for each investors' group.

8.2.2 Comparative statics: increased belief dispersion and asset prices

This section looks at the effect of increasing belief dispersion on asset prices. For this I define an increase in belief dispersion as a perturbation of the distribution of agents $dG(i)$ that reallocates mass from the middle interval $[i_q, i_p]$ to both extremes $[0, i_q], [i_p, 1]$. In other words, we look at a pair of exogenous small changes $dG_1, dG_3 > 0, dG_2 = -(dG_1 + dG_3) < 0$. The following proposition shows how market-clearing prices rise in response to this marginal increase in belief dispersion.

Proposition 11 - *Increased belief dispersion raises prices*
A small increase in belief dispersion $dG_1, dG_3 > 0$ *raises prices of both collateralised loans and assets.*

Proof of Proposition 11. Note that at given prices p, q, the change in excess demand for bonds and assets due to the increase in belief dispersion equals individual unit demands multiplied by the change in the mass of agents in the extreme intervals, respectively, $d\widetilde{a} = \frac{n+p}{q} dG_1$ and $d\widetilde{b} = \frac{n+p}{p-q} dG_3$. We are thus looking for a pair of price changes dp, dq that offsets this change to maintain asset market clearing

$$\begin{aligned} db &= -\frac{n+p}{p-q} dG_3 = \frac{n+p}{q}[\frac{\delta G_1}{\delta q} - G_1 \frac{1}{q}]dq \ + \ \frac{1}{q}[\frac{\delta G_1}{\delta p}(n+p) + G_1]dp < 0 \\ da &= -\frac{n+p}{q} dG_1 = \frac{n+p}{p-q}[\frac{\delta G_3}{\delta q} + G_3 \frac{1}{p-q}]dq + \frac{1}{p-q}[\frac{\delta G_3}{\delta p}(n+p) + G_3(1 - \frac{n+p}{p-q})]dp < 0 \end{aligned} \tag{48}$$

Note that, from the definition of G_1, G_3 as well as i_p and i_q in (41) and (40), we have $\frac{\delta G_1}{\delta q} = g(i_q)\frac{\delta i_q}{\delta q} < 0$, $\frac{\delta G_3}{\delta q} = -g(i_p)\frac{\delta i_p}{\delta p} < 0$, $\frac{\delta G_1}{\delta p} = 0$ and $\frac{\delta G_3}{\delta q} = -\frac{\delta G_3}{\delta p} > 0$. Use this, and the market-clearing conditions $G_1 \frac{n+p}{q} = G_3 \frac{n+p}{p-q} = 1$,to simplify (48)

$$\begin{aligned} db &= \ = [\frac{\delta G_1}{G_1 \delta q} - \frac{1}{q}]dq \ + \ \frac{1}{q} G_1 dp \\ da & \ = [\frac{\delta G_3}{G3 \delta q} + \frac{1}{p-q}]dq + [\frac{\delta G_3}{G_3 \delta p} + \frac{G_3 - 1}{p-q}]dp \end{aligned}$$

Denoting the vector of price and quantity changes as $\overline{dp}$ and $\overline{dx}$ respectively, and writing

$$\overline{dx} = A\overline{dp} \Rightarrow \overline{dp} = A^{-1}\overline{dx}$$

we can sign the row i-column j elements of A as $A_{11} < 0$, $A_{12} > 0$, $A_{22} < 0$, $A_{21} > 0$. In other words, the "own-price effects" on asset demand are negative, while the "cross-price effects" (the off-diagonal elements of A) are positive, implying cofactor matrix of A C_A with only negative entries. To conclude the proof, we thus have to show that D_A, the determinant of A, is positive.

$$\begin{aligned} D_A &= [\frac{\delta G_1}{G_1 \delta q} - \frac{1}{q}][\frac{\delta G_3}{G_3 \delta p} + \frac{G_3 - 1}{p-q}] \quad - \quad [\frac{\delta G_3}{G3\delta q} + \frac{1}{p-q}]\frac{1}{q}G_1 \\ &= \frac{1 - G_3 - G_1}{q(p-q)} + \frac{1}{q}\frac{g_3}{G_3}\frac{di_p}{dp}(1 - G_1) + \frac{g_1}{G_1}\frac{di_q}{dq}(-\frac{g_3}{G_3}\frac{di_p}{dp} - \frac{1-G_3}{p-q}) > 0 \end{aligned} \tag{49}$$

■

The proof of Proposition 11 shows how we need a rise in both prices to "undo" a rise in excess demand that results from an exogenous increase in belief dispersion. The challenge is three-fold: first, a change in prices changes both the unit demands as well as the size of the intervals G_1 and G_3; second, the unit demands comprise the asset endowment, leading to a positive effect of a rise in asset prices on both quantities; and finally, the cross-price effect of a rise in the price of collateralised loans $dq > 0$ on asset demand is positive, as it makes it cheaper to raise outside funds. The proof exploits market-clearing, and the fact that an equal increase in dp and dq leaves leveraged asset demand (excluding the endowment effect) unchanged to show that the own price effects dominate the endowment and cross-price effects.

8.2.3 Endogenous choice of $\overline{s}$

So far, we have taken $\overline{s}$, the face value of the loan, as exogenous. This section looks at the optimal choice of $\overline{s}$ subject to an upper bound: $s \leq \overline{s}^{max}$. In other words, we assume that there are some non-modelled features of the economy that put an upper bound to the riskiness of collateralised loans.

The net benefit of a marginal change $d\overline{s}$ to an investor in leveraged asset equals the additional returns on outside funds that increase when selling collateralised loans at a higher price, equal to $R_i^a \frac{\delta q(\overline{s})}{\delta \overline{s}}$, minus the increase in expected payments on the loan, equal to 1 minus the probability of default $(1 - F_i(\overline{s}))$.

$$\frac{d\Pi_i^a}{d\overline{s}} = \frac{n+p}{p-q}[R_i^a \frac{\delta q(\overline{s})}{\delta \overline{s}} - (1 - F_i(\overline{s}))] \tag{50}$$

Conversely, the net benefit to an agent j from increasing the $\overline{s}$ of the collateralised loan she purchases equals the expected rise in payments $(1 - F_j(\overline{s}))$ minus the loss in profits due to a reduced quantity of loans she can afford at the higher price, equal to $\frac{E_j[min(s,\overline{s})]}{q} \frac{\delta q(\overline{s})}{\delta \overline{s}}$

$$\frac{d\Pi_j^l}{d\overline{s}} = \frac{n+p}{q}[(1 - F_j(\overline{s})) - \frac{E_j[min(s,\overline{s})]}{q} \frac{\delta q(\overline{s})}{\delta \overline{s}}] \tag{51}$$

In order to characterise the equilibrium with endogenous leverage choice we make the following additional assumption:

Assumption A4 - $\overline{s}^{max} \leq s^\star = min_i min_j (s : F_i(s) = F_j(s); \quad j, i \in [0,1]) > s_{min}$.

Note that second-order stochastic dominance implies single-crossing of "adjacent" distributions F_i. The assumption ensures that the maximum leverage is smaller than the minimum of all single-crossing points. For example, if we were to restrict our attention to beliefs that are symmetric around E_s, then we would have $s^\star = E_s$ and $\overline{s}^{max} \leq E_s$, which is equivalent to assuming that the bankruptcy probability cannot exceed fifty percent. The following proposition shows that under this assumption there cannot be an equilibrium with trade in collateralised loans of face value below $\overline{s}^{max}$.

Proposition 12 - *Degenerate choice of* $\overline{s}$

Under Assumption A4, only one collateralised loan contract with $\overline{s} = \overline{s}^{max}$ *is traded in equilibrium.*

Proof of Proposition 12. The choice of $\overline{s}$ depends crucially on the slope of the equilibrium price function $q(\overline{s})$. Note that, for any $\overline{s}$ traded in equilibrium, it has to be that profits expected by the marginal buyer $i_q(\overline{s})$ are weakly decreasing from any $d\overline{s} > 0$. Thus

$$\frac{d\Pi_{i_q}(d\overline{s})}{d\overline{s}}^{+} \leq 0 \Rightarrow \frac{\delta q(\overline{s})}{\delta \overline{s}}^{+} \geq \frac{(1 - F_{i_q}(\overline{s}))}{R} \tag{52}$$

where $\frac{d\Pi_{i_q}(\overline{s})}{d\overline{s}}^{+}$ denotes the right-hand-side derivative of profits with respect to $\overline{s}$. We can substitute this into (50), to get

$$\frac{d\Pi_i^a}{\overline{s}}^{+} \geq \frac{n+p}{p-q}[\frac{R_i^a}{R}(1 - F_{i_q}(\overline{s})) - (1 - F_i(\overline{s}))] \geq \frac{n+p}{p-q}[(1 - F_{i_q}(\overline{s})) - (1 - F_{i_p}(\overline{s}))] > 0 \tag{53}$$

where the second-to-last inequality follows from $R_i^a \geq R \forall i \geq i_p$, and $(1 - F_i(s)) \leq (1 - F_{i_p}(s) \forall i \geq i_p$, and the last inequality follows from $(1 - F_{i_q}(\overline{s})) - (1 - F_{i_p}(\overline{s})) > 0 \forall \overline{s} < s^\star$. So agents only want to issue loans with maximum leverage $\overline{s}^{max}$. ■ The marginal buyer of a loan with face value $\overline{s}$ has to weakly prefer that face value to a slightly higher one. This puts a lower bound on the slope of the price function $q(\overline{s})$ at that point. Moreover, for $\overline{s} < \overline{s}^\star$, higher i implies a higher bankruptcy-probability, so the additional payment an issuer expects to make on collateralised loans from a small rise $d\overline{s} > 0$ falls with belief dispersion i. Since issuers of loans have higher i than buyers, this implies that at $\overline{s} < \overline{s}^\star$ issuers of collateralised loans gain more from a rise in prices $\frac{\delta q(\overline{s})}{\delta \overline{s}} > 0$ than they loose from higher expected payments. So they choose the maximum face value and leverage, equal to $\overline{s}^{max}$. Note that the assumption of an upper bound for the face value $\overline{s}$ is crucial here. Without it, issuers of collateralised loans would potentially choose different $\overline{s}$ and we would face a complicated assignment problem of issuers and buyers of collateralised loans to face values. Geerolf (2014) solves this assignment problem under one particular kind of disagreement about mean payoffs, namely point expectations. With disagreement about risk, this problem becomes, to our knowledge, untractable. This is why in the main text we look at a simplified environment with two types that allows to analyse endogenous leverage, and more complex collateralised debt contracts.

9 Appendix C: Asset price bubbles with trade in synthetic CDOs

This proposition generalises the conditions in Proposition 7 in the main text.

Proposition 13 - *When f_0 has no mass points and $s^\star \geq \frac{1}{2}$, the asset price bubble $p^{SCDO} - \frac{E_s}{R}$ with trade in secondary CDOs is at least twice as large as that with trade in collateralised loans or primary CDOs.*

Proof of Proposition 13.

$$
\begin{aligned}
p^{SCDO} * R &= \int_0^1 s\ max\{f_i(s)\}ds = E_s + \int_0^1 s(max\{f_i(s)\} - f_1(s))ds \\
&= E_s + \int_0^{s^\star} s(max\{f_i(s)\} - f_1(s))ds + \int_{s^\star}^1 s(max\{f_i(s) - f_1(s), 0\})ds \\
&\geq E_s + \int_0^{s^\star} s(f_0(s) - f_1(s))ds + s^\star \int_{s^\star}^1 max\{f_0(s) - f_1(s), 0\}ds \\
&\geq E_s + [\int_0^{s^\star} s(f_0(s) - f_1(s))ds + \int_{s^\star}^1 s^\star(f_0(s) - f_1(s))ds] + s^\star min_{\widehat{f_0},\widehat{f_1}}\{\int_{s^\star}^1 max\{\widehat{f}_0(s) - \widehat{f}_1(s), 0\}\}ds \\
&\geq E_s + \Delta E[min\{s, s^\star\} + s^\star \frac{\Delta E[min\{s, s^\star\}}{1 - s^\star} \\
&\Rightarrow\ p^{SCDO} - \frac{Es}{R} \geq 2(\overline{p} - \frac{Es}{R})\ \ \forall s^\star \geq \frac{1}{2}
\end{aligned}
$$

The fourth line follows by adding $\int_{s^\star}^1 c(f_0(s) - f_1(s))ds = 0$, which holds for any constant c, and because $\Gamma(f_0, f_1) \geq min_{\widehat{f}_0,\widehat{f}_1}\Gamma(\widehat{f}_0, \widehat{f}_1)$ for any function Γ and $\widehat{f}_0, \widehat{f}_1$ fulfilling the assumptions of equal expected payoffs $\int_{s^\star}^1 s(\widehat{f}_0(s) - \widehat{f}_1(s))ds = -\int_0^{s^\star} s(f_0(s) - f_1(s))ds +$ and given mass above the single-crossing point $\int_{s^\star}^1 \widehat{f}_i(s)ds = 1 - \int_0^{s^\star} f_i(s), i = 0, 1$. The minimisation problem can be written as

$$
\begin{aligned}
& min_{\widehat{f_0},\widehat{f_1}:[s^\star,1]\longrightarrow R+} \int_{s^\star}^1 (max\{\widehat{f}_0(s) - \widehat{f}_1(s), 0\})ds \\
& s.t.
\end{aligned}
$$

$$\int_{s^\star}^1 s(max\{\widehat{f}_1(s) - \widehat{f}_0(s), 0\})ds - \int_{s^\star}^1 s(max\{\widehat{f}_0(s) - \widehat{f}_1(s), 0\})ds = \int_0^{s^\star} s(f_0(s) - f_1(s))ds > 0 \tag{54}$$

$$\int_{s^\star}^1 \widehat{f}_i(s)ds = 1 - \int_0^{s^\star} f_i(s)ds, i = 0, 1 \tag{55}$$

Along the first and second constraint, disagreement can be reduced by concentrating s : $max\{\widehat{f}_1(s) - \widehat{f}_0(s), 0\} > 0$ at $s = 1$ and s : $max\{\widehat{f}_0(s) - \widehat{f}_1(s), 0\} > 0$ at $s = s^\star$. To see that this is the minimum disagreement consider a small reallocation $d\widehat{f}_1$ ($d\widehat{f}_0$) from 1 ($s^\star$)

to some other value $s \in (s^\star, 1)$. In order not to violate constraint (54), this requires an increase in $\int_{s^\star}^1 s(max\{\widehat{f}_1(s) - \widehat{f}_0(s), 0\})ds = \int_{s^\star}^1 \mathbf{1}(max\{\widehat{f}_1(s) - \widehat{f}_0(s), 0\})ds$ or a decrease in $\int_{s^\star}^1 s(max\{\widehat{f}_0(s) - \widehat{f}_1(s), 0\})ds = \int_{s^\star}^1 \mathbf{s}^\star(max\{\widehat{f}_0(s) - \widehat{f}_1(s), 0\})ds$. Since disagreement is concentrated at the extremes of the interval $[s^\star, 1]$, these changes necessitate an increase in disagreement, which increases the objective.

The resulting equal mass points of $\widehat{f}_0$ at $s^\star$ and of $\widehat{f}_1$ at 1 imply, after substituting the constraint of equal expected payoffs, $\widehat{f}_0(s^\star) = \widehat{f}_1(1) = \frac{\int_0^{s^\star}1 s(f_0(s)-f_1(s))ds}{1-s^\star} = \frac{\Delta E[min\{s,s^\star\}]}{1-s^\star}$. Together with $\int_0^{s^\star} s(f_0(s) - f_1(s))ds + \int_{s^\star}^1 s^\star(f_0(s) - f_1(s))ds = \Delta E[min\{s, s^\star\}]$ this implies the fifth line.

■