The Optimal Quantity of Capital and Debt[1]

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Abstract: In this paper we solve the dynamic optimal Ramsey taxation problem in a model with incomplete markets, where the government commits itself ex-ante to a time path of labor taxes, capital taxes and debt to maximize the discounted sum of agents' utility starting from today. Whereas the literature has largely been limited to choosing policies that maximize steady state welfare only, we instead characterize the optimal policy along the full transition path.

We show theoretically that in the long run the capital stock satisfies the modified golden-rule. More importantly, we prove that in contrast to complete markets the long run steady state resulting from an infinite sequence of optimal policy choices is independent of initial conditions. This result is not only of theoretical interest but enables us to compute the long-run optimum independently from the transition path such that a quantitative analysis becomes tractable.

Quantitatively we find, robustly across various calibrations, that in the long-run the government debt-to-GDP ratio is high, capital is taxed at a low rate and labor income is taxed at a high rate when compared to current U.S. values. In our benchmark calibration, aimed at resembling the high income inequality in the U.S. and with a Frisch elasticity of labor supply equal to one, the long-run taxes on capital and labor are around 11 and 77 percent and the debt-to-GDP ratio is about 4.

Along the optimal transition to the steady-state, labor taxes are initially lowered, financed by issuing more debt, before they are eventually increased to their steady-state level.

Keywords: Optimal Debt, Incomplete Markets, Capital Taxation

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1 Introduction

What is the optimal amount of capital and government debt? Should capital be taxed and if yes how much? What is the optimal amount of redistribution? We study these classic questions in a heterogenous agents incomplete markets Aiyagari (1995) economy. In this economy households are exposed to idiosyncratic income shocks but no aggregate risk. They face exogenous credit constraints and the only assets are physical capital and government debt. The Ramsey planner commits itself ex-ante to a path of linear labor and capital taxes and government debt to maximize agents' discounted present value of lifetime utility.

We prove two main theoretical findings on optimal policies. First, we show that it is optimal to equalize the pre-tax return on capital and the rate of time preference in the long-run, i.e. the capital stock satisfies the modified golden-rule. Our second theoretical result shows that the long run steady state allocations and policies are independent of initial conditions. In particular, the long-run level of government debt is uniquely determined and does not depend on the initial value of debt or capital. Similarly steady-state tax rates on capital and labor are unique and independent of initial conditions.

A comparison with the optimal Ramsey taxation results in representative agents complete markets economies without aggregate risk as in Lucas (1990) and Chari and Kehoe (1999) helps to understand our findings. As is well known the steady state Ramsey planner solution depends on initial conditions such as the initial government debt level in this complete markets environment. The intuition for this result is straightforward. As in Barro (1979) the planner aims to smooth distortions over time using government debt. In the absence of any exogenous fluctuations it is optimal (after perhaps some initial periods) to keep government debt and labor taxes constant over time. This policy provides higher welfare than a deviating policy where for example labor taxes and distortions are lowered initially, additional debt is issued to finance this tax cut, and then eventually labor taxes and distortions are increased to cover the higher interest rate burden on government debt. This alternative policy would reduce welfare since the gain of lower distortions in the beginning is outweighed by the loss of higher distortions later on, since distortions are "convex" as in Barro (1979).

If markets are incomplete this reasoning is only one part of the story. Lowering taxes today still means higher debt (as in the complete markets case) but now more debt has a welfare-enhancing element as it enables households to better smooth consumption in response to income shocks. The costs of having higher debt - higher future taxes - is still present if markets are incomplete instead of complete. But with incomplete markets there is now an additional benefit, better consumption smoothing. As a result the planner lowers taxes initially as there are two benefits - lower distortions today and higher debt (more liquidity) - and still just one cost (higher distortions tomorrow). Of course there are limits to how high debt can become as eventually future distortions become too big and outweigh the initial lower distortions and the benefits of higher liquidity. The optimal level of government debt is determined as equalizing the benefits and costs at the margin.

A conclusion common to both complete and incomplete markets is that the long-run capital stock satisfies the modified golden rule (see also Aiyagari (1995) and Acikgoz (2014)). In a representative agent economy distributional concerns are absent and investment efficiently transfers resources across time. If markets are incomplete distributional concerns are present but we show that they do not interfere with efficiency in investment, reminiscent of the production efficiency result in Diamond and Mirrlees (1971). A higher than the efficient capital stock could be used to achieve better consumption smoothing but we show that the planner issues more debt instead. A higher capital stock also increases wages which would benefit those depending primarily on labor income but we show that the planner uses labor taxes to increase the after-tax wage instead. On the other hand, if either of the instruments, issuing debt or taxing labor, is not available to the planner, then the capital stock will not satisfy the modified golden rule.

The result that the steady state is independent of initial conditions is not only of theoretical interest but also renders a quantitative analysis of the optimal taxation problem tractable. Whereas the literature has mainly focused on characterizing the steady state which maximizes welfare, we have to characterize the optimal policy along the full transition path. In particular our characterization has to take into account that the optimal policy at each point in time during transition depends on the full transition path of capital, debt and tax rates.

Computing the path of tax rates, government debt and transfers which maximizes welfare at date 0 is a huge computational challenge: Several hundred or thousands of variables have to be chosen in a highly nonlinear optimization problem. However, our result that the optimal long-run policy is independent of initial conditions turns this non-manageable optimization problem into a manageable one. From a computational point of view this independence of initial conditions means that we know the optimal long-run policies and allocations without having to compute the transition. Instead we know the initial conditions (economy calibrated to the US economy) and we know the terminal condition, the optimal long-run steady state characterized before. The (still huge) computational problem is then to find the policy path which satisfies all necessary first-order conditions along the transition and at the same time satisfies the initial and terminal conditions. This problem is still a challenge as it involves solving hundred or thousands of nonlinear equations but is significantly easier (and therefore tractable) than the original problem, which has to find the optimal transition and the optimal terminal point at the same time. Given the large number of variables involved there is no way to check whether a candidate solution is a global maximum in the original problem, a check which is not necessary in our approach.

In the optimal steady-state we find that capital taxes are always significantly positive in contrast to complete markets (see the seminal contributions of Chamley (1986) and Judd (1985)), although for all calibrations relatively low compared to most developed economies. In our benchmark calibration, aimed at resembling the high income inequality in the U.S. and with a Frisch elasticity of labor supply equal to one, the long-run taxes on capital and labor are around 11 and 77 percent. The optimal long run level of government debt equals 4 times GDP.

Our finding that government debt is high, capital is taxed at a low rate and labor income is taxed at a high rate when compared to current U.S. values is robust across various different alternative calibrations, although the precise numbers of course depend on the details of the calibration. Indeed, we reach the same conclusion for a low and a high Frisch elasticity of labor supply, for a low or high income elasticity of labor supply, for low and high income inequality and in a model with permanent income differences.

The high debt levels we find are a consequence of our assumption that the government always honors its debt so that elements such as a default premium are not present in our model and therefore do not restrict how much debt can be issued. Instead distortionary taxes is the only element which restricts debt from becoming infinitely large and thus maximizing the liquidity services. One conclusion from our result is that tax distortions by themselves restrict government debt to levels much larger than observed in developed countries.

Knowing the optimal path of policies allows us to compute the welfare gains of switching to the optimal policy and helps to better understand the properties of the optimal steady states policies as those are tightly linked to the policies chosen during the transition. The optimal transition is characterized by an initial period of high capital income taxation and low labor taxation. While the high initial capital tax rates are well known from complete markets and are a result of initially inelastically supplied capital, the low initial taxation (subsidization) of labor income is new to the incomplete markets environment. As a result labor market distortions are low initially and government debt is accumulated. Eventually labor taxes are increased to pay the interest rates on debt which converges to its high steady-state level.

Although most of the literature either maximizes steady-state welfare or when considering transitions assumes fixed tax rates throughout the transition, there are a few papers which deviate from these restrictive assumptions. The paper most closely related to ours is Acikgoz (2014). He was the first to develop a methodology to compute the long-run optimal policy and we built on his work and extend it to different utility functions and income processes. In addition we prove independence of the steady-state Ramsey policies from initial conditions and compute the full transition path of optimal policy, including labor and capital income tax rates, debt, and capital. Dyrda and Pedroni (2015) also compute the optimal transition path in an incomplete markets economy, using however a quite different approach. In particular they do not characterize the optimal steady policies first before computing the transition but instead compute both jointly. Their findings for the optimal steady state policy differ from ours. In Dyrda and Pedroni (2015) capital income in the long-run is taxed at a high rate whereas labor income is taxed at a low rate only and government debt is negative. Aiyagari and McGrattan (1998) study the optimal level of debt in an incomplete markets model but under the alternative assumption that the planner maximizes the utility at the steady state instead of ex-ante welfare. They find that the optimal level of debt is two thirds of GDP in line with the current US level. Many of the follow-up works in this literature also maximizes the steady state welfare. For example, Röhrs and Winter (2014) find that if inequality is large, the optimal level of debt that maximizes the steady-state welfare is even lower and it should be negative, -0.8. One reason why the optimal level of debt is low or even negative when steady-state welfare is maximized is that this optimality criterion ignores the welfare loss of reducing debt along the transition path to a low debt steady state.

In a series of papers Bhandari et al. (2015, 2016a,b, 2017) also consider optimal taxation in incomplete market models building on the work of Aiyagari et al. (2002) who where the first to investigate the Ramsey policy in a Lucas and Stokey (1983) economy with incomplete markets (and aggregate risk). A key difference is that we follow Aiyagari (1995) and impose tight exogenous credit constraints which is necessary to match the joint distribution of earnings, consumption and wealth observed in the data and to generate a realistic distribution of MPCs. These credit constraints make the computational problem significantly more complicated as a fraction of households is not operating on their consumption Euler-equation, preventing us from using an easy backward shooting approach where we iterate backwards on the Euler equation.

Tight credit constraints also seem to render a characterization of optimal policy through sufficient statistics impossible (Piketty and Saez (2013)) since they induce different policies and different distributions of assets, labor income and consumption in the short-run, during the transition and in the long-run to be optimal. Indeed we show that tight credit constraints and precautionary savings demand imply that it is optimal to increase the level of government debt and lower labor taxes initially and increase them in the long-run, which induces a very different distribution of consumption, income and wealth in the long-run from what is currently observed in the U.S. A sufficient statistics approach, however, is necessarily based on the observable inequality measures for the U.S. while optimal policy in the long-run depends on the corresponding long-run statistics and their optimal evolution during the transition. Policy conclusions based on two very different statistics are likely to be very different. Furthermore it seems infeasible to solve the fixpoint problem - different policies lead to different wealth and income distribution which render different policies optimal and so on ... - within the sufficient statistic framework. Our results show that these considerations are not just a theoretical possibility but are key in determining the full transition path of optimal policies.[1]

It is also the presence of credit constraints which generates a large demand for precautionary savings and thus potentially a positive capital income tax rate. The reason why we nevertheless do not find high capital income tax rates is the large amount of debt which allows households to smooth consumption quite well but at the same time requires an after tax interest rate close to the rate of time preference. For a higher capital income tax rate and thus a lower pre-tax interest rate the private sector would just not be willing to absorb the capital stock and the large stock of debt. The planner finds it welfare-maximizing to reduce inequality through more debt and low capital income tax rates instead of low debt and high capital income taxes. Both a high level of debt and high capital tax rates are not possible since the asset market would not clear.

[1] One of the difficulties that arises if for example one is interested in finding the optimal capital income tax rate is that this requires to specify how the revenue from this tax is used: to lower the tax on labor, to pay higher transfers or to reduce government debt. This choice is not arbitrary but has to be optimal requiring to take into account its effects on the full transition path, which will in turn affect the optimal capital income tax rate, which ...

The paper is organized as follows. Section 2 presents our incomplete markets model and the Ramsey taxation problem. We provide our theoretical results in Section 3 before we move to the quantitative analysis. Section 4 shows optimal policy in the steady-state and the optimal transition path is presented in Section 5. Section 6 concludes.

2 The Model

In this section, we present the incomplete markets model with heterogenous agents and uninsurable idiosyncratic labor productivity shocks. The setup is similar to Aiyagari (1995), except that our utility function is more general, which allows for income effect of labor supply and government spending is exogenous.

2.1 The Environment

Time is discrete and infinite, denoted by $t \in \{0, 1, 2, ...\}$. There is a continuum of ex ante identical households, a representative firm and a government.

Endowment and Technology A household supplies labor $n_t \in [0, 1]$ in period t. She faces an idiosyncratic labor productivity shock $e_t \in E$, which follows a Markov process and is i.i.d. across households. She has access to an incomplete market and can only hold a non-state contingent one-period bond $a_t \in A$, subject to a constraint $a_t \geq -\underline{a}$.

A representative competitive firm produces final goods using capital K_t and labor N_t using the neoclassical constant-returns-to scale production function $F(K, N)$ which satisfies the standard conditions.[2] Capital depreciates at rate δ.

The government is a Ramsey planner with full commitment. It collects linear capital income tax at the rate τ_{kt} and linear labor income tax at the rate τ_{nt}. It issues government debt B_t to finance lump-sum transfer T_t and government expenditure G_t.

Preferences The instantaneous utility of a household is $u(c_t, n_t) = \frac{c^{1-\sigma}}{1-\sigma} - \chi \frac{n^{1+\frac{1}{\phi}}}{1+\frac{1}{\phi}}$. Her lifetime utility is the expected discounted sum of utilities $\mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t, n_t)$. This utility function allows for income effect of labor supply, namely, the labor supply decision of a household reacts not only to labor productivity and wage, but also the asset level. In the

[2] The production function is assumed to be twice contiunuously differentiable, strictly increasing and concave in each argument and satisfies the standard Inada conditions: $\lim_{K \to 0} F_K = \infty, \lim_{K \to \infty} F_K = 0$ and $\lim_{N \to 0} F_N = \infty$.

literature, to simplify the analysis, income effects of labor supply are usually shut down either by using for example GHH preferences or by allowing for home production. In the benchmark we make a more standard choice and allow for income effects but also report how our results depend on the strength of the income effect, including a specification without any income effects.

Markets There are competitive markets for labor, capital, final goods, and bonds.

2.2 Competitive Equilibrium

Firm The optimality conditions for the firm imply that in each period, the interest rate and the wage are equal to the marginal return of capital and the marginal return of labor respectively, as follows:

$$\begin{aligned} r_t &= F_K\left(K_t, N_t\right) - \delta, \\ w_t &= F_N\left(K_t, N_t\right). \end{aligned}$$

Government The government collects linear taxes on capital income and labor income. Denote the after-tax capital return and wage as $\bar{r}_t$ and $\bar{w}_t$, so that $\bar{r}_t = \left(1 - \tau_{kt}\right) r_t$ and $\bar{w}_t = \left(1 - \tau_{nt}\right) w_t$. The government's inter-temporal budget constraint is

$$G_t + \left(1 + \bar{r}_t\right) B_t + T_t \leq \tau_{kt} r_t A_t + \tau_{nt} w_t N_t + B_{t+1}, \tag{1}$$

where $A_t = K_t + B_t$ is the total amount of assets, the sum of physical capital and government debt. Standard arguments using the constant-return-to-scale assumption lead to the following equivalent resource constraint:

$$G_t + \left(1 + \bar{r}_t\right) B_t + \bar{r}_t K_t + \bar{w}_t N_t + T_t \leq F\left(K_t, N_t\right) - \delta K_t + B_{t+1}. \tag{2}$$

Households Starting from period 0 with asset a_0 and productivity e_0, a household solves the following problem

$$V_0\left(a_0, e_0\right) = \max_{\{a_{t+1}, c_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{n_t^{1+\frac{1}{\phi}}}{1+\frac{1}{\phi}} \right),$$

subject to

$$c_t + a_{t+1} \leq a_t (1 + \bar{r}_t) + \bar{w}_t e_t n_t + T_t, \tag{3}$$

$$a_{t+1} \geq -\underline{a}, \tag{4}$$

where $V_0(a_0, e_0)$ represents the lifetime utility of a household with initial state (a_0, e_0). The optimality condition of n_t is

$$u_c(c_t, n_t) e_t \bar{w}_t + u_n(c_t, n_t) = 0,$$

which implies that labor supply

$$n_t = \left(\chi^{-1} e_t \bar{w}_t c_t^{-\sigma}\right)^{\phi}, \tag{5}$$

and after tax labor income y_t

$$y_t = e_t n_t \bar{w}_t = (e_t \bar{w}_t)^{1+\phi} \left(\chi^{-1} c_t^{-\sigma}\right)^{\phi}. \tag{6}$$

In the rest of the paper, we can treat n_t and y_t as known functions of $\bar{w}_t$ and c_t, reducing the number of choice variables. The optimality condition for a_{t+1} and the borrowing constraint imply the necessary conditions:

$$u_c(c_t, n_t) \geq \beta (1 + \bar{r}_{t+1}) \mathbb{E}_t u_c(c_{t+1}, n_{t+1}), \tag{7}$$

$$0 = (a_{t+1} + \underline{a}) (u_c(c_t, n_t) - \beta (1 + \bar{r}_{t+1}) \mathbb{E}_t u_c(c_{t+1}, n_{t+1})). \tag{8}$$

Equation (7) is the standard Euler equation, and equation (8) is the Kuhn-Tucker condition for the borrowing constraint.

Equilibrium The distribution of households with productivity e_t and asset a_t in period t is denoted by μ_t, a measure on $S = E \times A$. The asset market clearing conditions for assets, labor and capital are,

$$A_t = \int_S a_t d\mu_t, \tag{9}$$

$$
\begin{aligned}
N_t &= \int_S e_t n_t d\mu_t, && (10) \\
K_t &= A_t - B_t. && (11)
\end{aligned}
$$

A sequence of prices and allocations and policies $\{\bar{r}_t, \bar{w}_t, T_t, B_{t+1}, K_{t+1}, a_{t+1}, c_t\}_{t=0}^{\infty}$ is a competitive equilibrium given initial conditions (B_0, K_0, μ_0) if

1. Households maximize utility (taking prices and policies as given).
2. Firms maximize profits (taking prices and policies as given).
3. Market clearing conditions (9), (10) and (11) hold.

2.3 The Optimal Taxation Problem

The Ramsey planner maximizes the sum of lifetime utilities of all households, by choosing time paths for $\bar{r}_t, \bar{w}_t$ and B_t consistent with equilibrium conditions described above. Later we allow the planner to also choose a path for transfers T_t. As explained choosing the full time path distinguishes this paper from many other studies on optimal taxation in the literature, which e.g. maximize steady-state welfare. The Ramsey problem is

$$
\max_{\{\bar{r}_t, \bar{w}_t, B_{t+1}, T_t, a_{t+1}, c_t\}} \int V_0(a_0, e_0) d\mu_0
$$

subject to the resource constraint (2), households budget constraints (3), households consumption Euler equation (7), and the credit constraint (8). The other unknowns, including $n_t, r_t, w_t, K_t, A_t, N_t$ can be all expressed as functions of the choice variables in the Ramsey problem, using the equations described in subsection 2.2.

Following the notation of Acikgoz (2014), we assign present value Lagrangian multipliers γ_t, θ_{t+1} and η_{t+1} to constraints (2), (7) and (8), respectively. The Lagrangian can be written as

$$
\begin{aligned}
\mathcal{L} = & \int \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \\
& \left(u\left(c_t, n_t\right) + u_c\left(c_t, n_t\right)\left(\left(\eta_t\left(a_t - \underline{a}\right) - \theta_t\right)\left(1 + \bar{r}_t\right) - \left(\eta_{t+1}\left(a_{t+1} - \underline{a}\right) - \theta_{t+1}\right)\right)\right)
\end{aligned}
$$

$$+\gamma_t\Big(F\left(K_t,N_t\right)-\delta K_t+B_{t+1}-G_t-T_t-\left(1+\bar{r}_t\right)B_t-\bar{r}_tK_t-\bar{w}_tN_t\Big)\Bigg\}d\mu_0. \quad (12)$$

To simplify the notation, we define $\lambda_{t+1} := \eta_{t+1}\left(a_{t+1}+\underline{a}\right)-\theta_{t+1}$. We derive FOCs from the Lagrangian in the appendix and show that the interior solution of the Ramsey problem satisfies the following conditions:

$$\begin{aligned}
\lambda_{t+1}: &\quad u_{c,t} \geq \beta\left(1+\bar{r}_{t+1}\right)\mathbb{E}_t\left[u_{c,t+1}\right] \\
&\quad \text{with equality if } a_{t+1} > -\underline{a}, && (13)\\
a_{t+1}: &\quad u_{c,t}+\tfrac{\partial c_t}{\partial a_{t+1}}u_{cc,t}\left(\lambda_t\left(1+\bar{r}_t\right)-\lambda_{t+1}\right) \\
&= \beta\mathbb{E}_t\left[\left(1+\bar{r}_{t+1}\right)u_{c,t+1}+\tfrac{\partial c_{t+1}}{\partial a_{t+1}}u_{cc,t+1}\left(\lambda_{t+1}\left(1+\bar{r}_{t+1}\right)-\lambda_{t+2}\right)\right] \\
&\quad +\beta\gamma_{t+1}\left(F_K\left(K_{t+1},N_{t+1}\right)-\delta-\bar{r}_{t+1}\right) \\
&\quad \text{if } a_{t+1} > -\underline{a},\ \text{otherwise } \lambda_{t+1}=0, && (14)\\
B_{t+1}: &\quad \gamma_t = \beta\left(1+F_K\left(K_{t+1},N_{t+1}\right)-\delta\right)\gamma_{t+1}, && (15)\\
\bar{r}_t: &\quad \gamma_tA_t = \gamma_t\left(F_N\left(K_t,N_t\right)-\bar{w}_t\right)\tfrac{\partial N_t}{\partial \bar{r}_t} \\
&\quad +\mathbb{E}_t\left[u_c\left(c_t\right)\lambda_t+a_t\left(u_c\left(c_t\right)+u_{cc}\left(c_t\right)\left(\lambda_t\left(1+\bar{r}_t\right)-\lambda_{t+1}\right)\right)\right], && (16)\\
\bar{w}_t: &\quad \gamma_tN_t = \gamma_t\left(F_N\left(K_t,N_t\right)-\bar{w}_t\right)\tfrac{\partial N_t}{\partial \bar{w}_t} \\
&\quad +\mathbb{E}_t\left[e_tn_t\left(u_c\left(c_t\right)+u_{cc}\left(c_t\right)\left(\lambda_t\left(1+\bar{r}_t\right)-\lambda_{t+1}\right)\right)\right]. && (17)
\end{aligned}$$

$\frac{\partial c_t}{\partial a_{t+1}}, \frac{\partial c_{t+1}}{\partial a_{t+1}}$, etc. are known functions of control variables. The explicit expressions of these functions are shown in the appendix. If transfers T_t are a choice variable for the planner we obtain an additional FOC,

$$T_t: \quad \gamma_t = \mathbb{E}_t\left[u_c\left(c_t\right)+\tfrac{\partial c_t}{\partial T_t}u_{cc}\left(c_t\right)\left(\lambda_t\left(1+\bar{r}_t\right)-\lambda_{t+1}\right)\right]+\gamma_t\left(F_N\left(K_t,N_t\right)-\bar{w}_t\right)\tfrac{\partial N_t}{\partial T_t}, \quad (18)$$

3 Analytical Results

A key first step in the quantitative analysis is to compute the optimal policy in the long-run. The second step is then to use the optimal long-run policy as a terminal condition when computing the optimal policy during the transition path. We therefore make the standard assumption that the optimal long-run policy is stationary, which we maintain throughout the paper:

Assumption 1. *For each set of initial conditions (B_0, K_0, μ_0), the economy (including policy and all other variables) converges to a unique steady state.*

Note that this does not assume our main result on the independence of initial conditions. Instead we assume that for each set of initial conditions (B_0, K_0, μ_0), there is a unique solution to the maximization problem of the Ramsey planner. Note that this assumption holds in representative agent economies, where given the initial level of debt B_0 and capital K_0 the steady state is unique, but at the same time the steady state depends on the initial debt level, that is different steady states can be reached for different initial conditions. In contrast, we show independence of initial conditions in our incomplete markets economy. The same steady state is reached independent from where the economy started.

Whereas uniqueness is a generic property of maximization problems (it just rules out more than one global maximimum),[3] the second assumption that the optimal solution converges to a steady state is standard and essential for tractability in incomplete market models but little is known whether this is indeed the optimal outcome in incomplete market models. Straub and Werning (2015) show that in a different model, the capitalist-worker model of Judd (1985), that this is not the case if the intertemporal elasticity of substitution is below one (and the weight on capitalists is zero). For these parameter values Proposition 2 in Straub and Werning (2015) shows that no interior steady states exists, implying that the assumption of convergence to an interior steady state is invalid. Such non-existence of steady-states issues do not arise in our numerical applications as we are always able to find a solution to the FOCs which characterize the steady state.[4]

[3]See Aiyagari (1994b) for a proof that a solution to the optimal taxation problem exists.

[4]Straub and Werning (2015) also consider the representative agent Ramsey taxation problem in Chamley (1986) and find that an exogenous upper bound on capital taxes can be binding forever if the initial level of government debt is close enough to the peak of a "Laffer curve". Again these issues seem not to arise in our incomplete markets model. We also impose an upper bound on capital taxation but find it to be binding only for the first period. Instead the planner finds it optimal to lower labor taxes and issue more bonds which requires a sufficiently high after-tax return on assets for households to be willing to absorb the additional debt.

3.1 Steady State

This assumption on the stationarity of the optimal long-run policy means that we can replace all variables in the above FOC with their steady state values. Then the optimal stationary policy is a solution to:

$$u_c(c) \geq \beta(1+\bar{r}')\mathbb{E}[u_c(c')|e] \text{ with equality if } a' > -\underline{a}, \tag{19}$$

$$u_c(c) + \frac{\partial c}{\partial a'}u_{cc}(c)[\lambda(1+\bar{r}) - \lambda'] = \beta\mathbb{E}\left[(1+\bar{r})u_c(c') + \frac{\partial c'}{\partial a'}u_{cc}(c')(\lambda'(1+\bar{r}) - \lambda'')\right]$$
$$+\beta\gamma(F_K(K,N) - \delta - \bar{r}) \text{ if } a' > -\underline{a}, \text{ otherwise } \lambda' = 0, \tag{20}$$

$$1 = \beta(1 + F_K(K,N) - \delta), \tag{21}$$

$$\gamma A = \gamma(F_N(K,N) - \bar{w})\frac{\partial N}{\partial \bar{r}} + \mathbb{E}\left[u_c(c)\lambda\mu(ds,e) + au_c(c) + \frac{\partial c}{\partial \bar{r}}u_{cc}(c)(\lambda(1+\bar{r}) - \lambda')\right], \tag{22}$$

$$\gamma N = \gamma(F_N(K,N) - \bar{w})\frac{\partial N}{\partial \bar{w}} + \mathbb{E}\left[enu_c(c) + \frac{\partial c}{\partial \bar{w}}u_{cc}(c)(\lambda(1+\bar{r}) - \lambda')\right]. \tag{23}$$

again with the additional condition

$$\gamma = \mathbb{E}\left[u_c(c) + \frac{\partial c}{\partial T}u_{cc}(c)(\lambda(1+\bar{r}) - \lambda')\right] + \gamma(F_N(K,N) - \bar{w})\frac{\partial N}{\partial T}, \tag{24}$$

if transfers T_t are a choice variable of the planner.

3.2 Optimal Long-run Level of Capital

Whereas most of our results are naturally based on numerical simulations, we can still analytically derive the optimal level of capital in the long-run. A key property of the steady state is that the capital level satisfies the modified golden rule (see also Aiyagari (1995) and Acikgoz (2014)). Equation (21) implies:

Theorem 1. *The capital satisfies the modified golden rule:* $\beta(1 + F_K(K,N) - \delta) = 1$.

The modified golden rule states that it is optimal to equalize the return on capital and the rate of time preference, that is resources are efficiently allocated across time. This result is well known from representative agent economies where distributional concerns are by assumption absent. Theorem 1 shows that we obtain the same efficiency result in our incomplete market economy where redistribution might induce a deviation from production efficiency, reminiscent of the production efficiency result in Diamond and Mirrlees (1971).

As is well known agents engage in precautionary savings to smooth consumption in response to idiosyncratic income fluctuations and this smoothing is the better the more assets are available. The planner does not issue more capital to increase the availability of assets though but instead issues more government debt which has the advantage that debt can be used as well as capital for consumption smoothing but does not interfere with efficiency. This reasoning is reflected in the absence of a "precautionary savings" term in the FOC determining the optimal level of capital.

A higher than efficient capital stock could also be used to increase wages which would benefit those whose consumption is primarily financed from labor and not asset income as it is the case in Dávila et al. (2012). In our Ramsey taxation problem the planner can increase the capital stock as well but only by lowering capital income taxes but can use labor taxes to change the after-tax labor income.[5] We show that the planner uses labor taxes to modify the after-tax wage and not a higher capital stock, which is again reflected in the absence of a "wage" term in the FOC determining the optimal level of capital.[6]

These arguments also establish that the availability of government debt and labor taxes are necessary for theorem 1 to be valid. Without these instruments the modified golden rule does not hold. If labor taxes are not available, the planner needs to take into account that a higher capital stock leads to higher wages and if government debt is not available the planner needs to take into account that a higher capital stock improves consumption smoothing.

3.3 Optimal Long-run Level of Debt

As is well known, the steady state Ramsey planner solution depends on initial conditions, i.e. the initial government debt level, when markets are complete (see e.g. Lucas (1990) and Chari and Kehoe (1999)). The next theorem shows that this result is overturned if markets

[5] Dávila et al. (2012) study a different problem, the constrained efficient allocation in a model with exogenous labor, where the planner also maximizes the discounted present value of lifetime utility but decides how much each individual has to save without the need to implement those decisions through a properly designed tax scheme. They find, using a calibration similar to ours, that the optimal level of capital is much higher than the current U.S. level as the rich have to save more such that aggregate capital and thus wages increase.

[6] Lowering debt while keeping the total amount of households' assets constant increases capital but lowers the marginal product of capital (MPK). For a fixed after-tax interest rate $\bar{r}$ (which is necessary to keep total assets $K + B$ constant), a lower MPK is equivalent to lower capital income taxes.

are incomplete.

Theorem 2. *The long-run values of government debt, of the labor income tax rate and of the capital income tax rate are generically independent of the initial level of government bonds (and the initial capital stock).*

To better understand this result, it is important to recognize the key difference between complete and incomplete market models is that households face credit constraints in the incomplete markets world and do not in the complete markets world. If markets are complete and thus in the absence of credit constraints the optimal steady state is linked to the initial steady state through the optimality conditions along the transition path. The optimality conditions allow to compute the solution backwards starting at the optimal steady state. One can infer all period t variables from knowing all variables at period $t+1$. For example from the capital stock in period $t+1$ one infers the interest rate which using the consumption Euler equation yields consumption in period t which in turn allows to infer the level of investment and capital in period t. Credit constraints break this link. Knowing the interest rate and period $t+1$ consumption of households who are credit constraint in period t is not sufficient to infer their period t consumption level. A binding credit constraint prevents us from using the consumption Euler equation as in the complete markets case. As a result there is no deterministic link between the optimal and the initial steady state. Note that, from a computational perspective, this missing link also prevents us from using a simple "backward shooting" algorithm. But, as we explain in Section 5, it is Theorem 2 which renders the computational algorithm tractable as we can first compute the steady state independent from the transition path and in a second step solve for the transition path knowing both the initial and terminal conditions.[7]

The intuition for why there is a unique optimal level of government debt is straightforward. As in Barro (1979) and as the case in complete markets models the planner aims to smooth distortions over time using government debt. But with incomplete markets there is an additional benefit of providing more bonds, better consumption smoothing. The planner

[7] The credit constraints also explain why the optimal steady state wealth distribution is independent from initial conditions. One property of the Aiyagari model is that the credit constraint will be eventually binding for everyone. At the point in time when the credit constraint is binding a household's life is reset and the individual history until this point is wiped out. Eventually everyone's history was eliminated at some point such that the current situation is independent from the initial one, implying that each individual's initial income level will be irrelevant for the long-run income position.

therefore deviates from full distortion smoothing and instead faces a trade-off between consumption and distortion smoothing. As a result the planner lowers labor taxes initially as there are two benefits - lower distortions today and higher debt and thus better consumption smoothing - but just one cost (higher distortions tomorrow). Of course there are limits to how high debt can become as eventually future distortions become too big and outweigh the initial lower distortions and the benefits of higher liquidity. The optimal level of government debt is determined as equalizing the benefits and costs at the margin. As a result in the long-run both labor taxes and government bonds are high what has the additional advantage that risky labor income is replaced with safe capital income.

A more formal intuition, and that is moving us closer to how the proof actually works, is to note that there are not enough independent optimality conditions to determine the long-run steady-state if markets are complete. Government bonds are not net worth since Ricardian equivalence holds in complete market models and therefore agents are willing to hold any amount of bonds in steady state. As a result bonds B appear only in the government budget constraint (the household budget constraint is dropped by Walras' Law) but this is not sufficient to pin down its long-run level. The steady-state government budget constraint just determines pairs of B and labor taxes τ_n which satisfy this constraint but does not determine each separately. In other words, an equation is missing and thus the long-run level of government debt (and also labor taxes) is not determined just from the steady state FOCs but only when initial conditions are taken into account.

We now argue that incomplete market models provide an additional equation - the asset demand equation - which serves to determine the long-run debt level since bonds are net worth in this class of models.[8]

[8]Some intuition can also be gained from a simple reduced-form model where bonds by assumption have a value is one where the representative agent's utility equals

$$\sum_{t=0}^{\infty} \beta^t (u(c_t) + \chi(B_{t+1}))$$

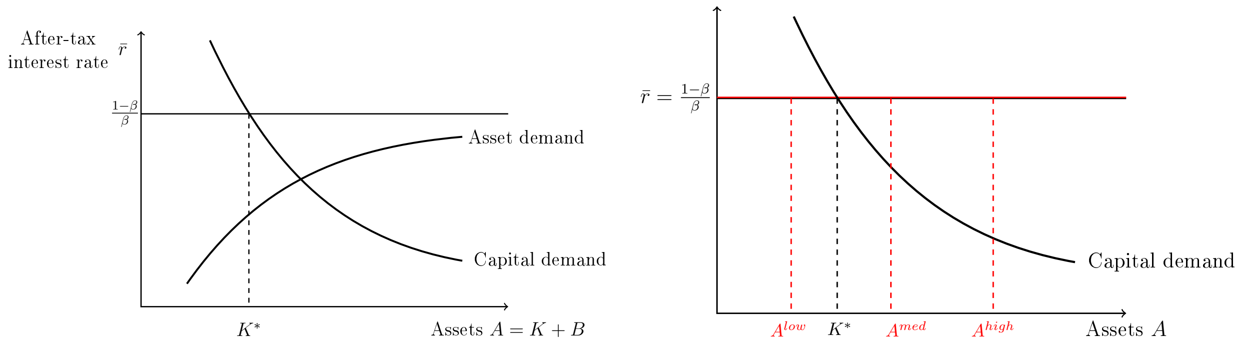
and the household budget constraint is (inelastic labor $n = 1$)

$$B_{t+1} = (1 + \bar{r}_{t+1})B_t - c_t + w_t.$$

In steady state the planner has to respect households demand for bonds function,

$$1 - \frac{\chi'(B)}{u'(c)} = \beta(1 + \bar{r}),$$

which is the additional equation that determines the long-run level of bonds in the Ramsey planner problem.



(a) Asset Market: Aiyagari model

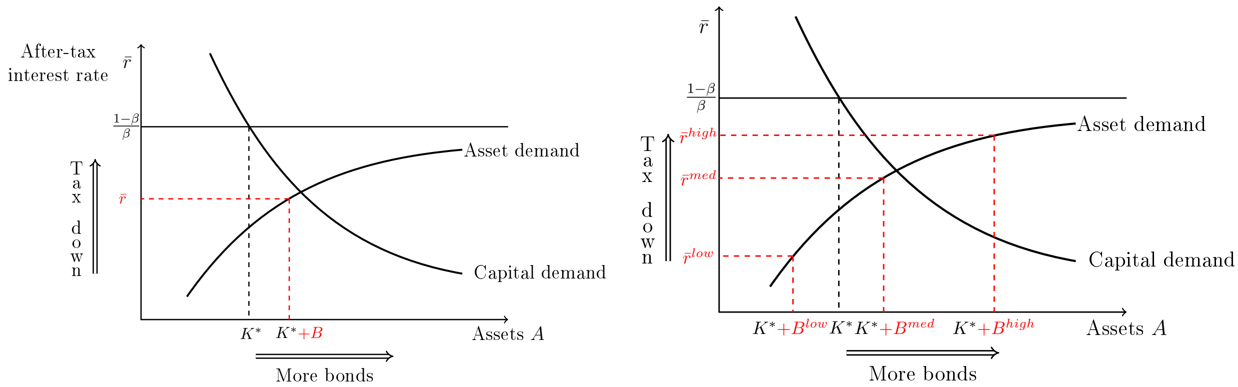
(b) Asset Market: Complete Markets

Figure 1: Asset Markets in (In)complete Markets

As is well known, aggregate households asset demand in the Aiyagari economy is described through a mapping between the after-tax interest rate $\bar{r}$ and assets A as illustrated in Figure 1a. Since the capital stock is at its modified golden rule level K^* where the marginal product of capital equals $1/\beta$ (Theorem 1), total assets $A = K^* + B$ are one-to-one related to the number of bonds. Figure 2a shows that picking a specific capital income tax rate and therefore an after-tax interest rate $\bar{r}$, automatically also chooses a specific amount of bonds B and vice versa. The planner therefore faces a trade-off, illustrated in Figure 2b, between supplying more bonds/liquidity and lower capital income tax rates. Choosing a low level of bonds, B^{low}, allows for a low after tax interest rate $\bar{r}^{low}$, that is a high tax on capital income. Choosing higher levels of bonds, B^{med} or B^{high}, provides more liquidity and thus enhances consumption smoothing but the capital income tax rates has to be lowered as households require higher after-tax interest rates, $\bar{r}^{med}$ or $\bar{r}^{high}$, to be willing to absorb the higher amount of assets $K^* + B$. This $B - \bar{r}$ trade-off provides the additional equation which allows us to determine the long-run level of debt using just the steady state FOCs. This trade-off is absent in complete markets models and therefore the long-run level of bonds is not determined as illustrated in Figure 1b. In a steady state $1 + \bar{r} = 1/\beta$ and Ricardian equivalence implies that the representative agents is willing to hold any amounts of bonds, $A^{low}, A^{med}, A^{high}$.

The formal proof uses ideas and concepts developed by Debreu (1970) to show the generic local uniqueness of competitive equilibria. The same approach can be used here since both in Debreu (1970) and here one has to show that a set of equations is locally invertible and

The intuition in out incomplete markets model is the same with the important difference that bonds have a real value not by assumption but endogenously.



(a) Additional Equation

(b) B-$\bar{r}$ trade-off

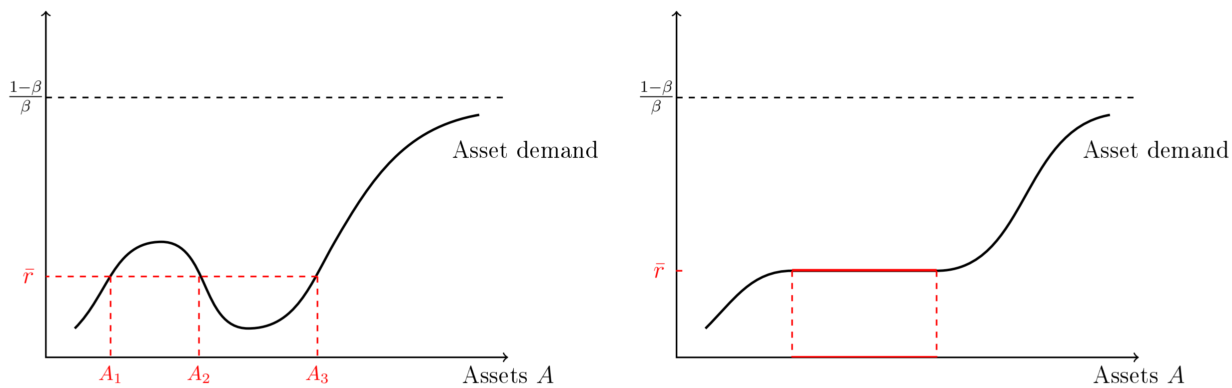
Figure 2: Additional Equation: B-$\bar{r}$ trade-off

thus has a unique local solution. In Debreu (1970) this set of equations is given by the excess demand function and here it is the set of equations characterizing the optimal steady state. Local uniqueness is guaranteed generically, that means it holds for a set of parameters of measure one (here the distribution of idiosyncratic prodcutivity; initial endowments in Debreu (1970)).[9] As in Debreu (1970) local uniqueness implies that there are at most a finite number of solutions to the necessary FOCs of the optimal steady state. Figures 1 and 3 illustrate this reasoning. Figures 1 show the simple case where the asset demand curve is monotonically increasing and therefore each level of assets is associated with a different after tax capital income tax rate $\bar{r}$, that is we obtain only one solution. Figure 3 illustrates that a finite number of solutions is possible, that is multiple levels of assets, A_1, A_2, A_3, are associated with the same $\bar{r}$. What both figures have in common is that all solutions are locally unique, that is can be separated by open sets.[10] Adopting the arguments in Debreu (1970) shows that this is the generic case. Figure 3b shows a non-generic case where a continuum of assets levels A is associated with the same $\bar{r}$ and thus an infinite number of solution would be possible. Following the arguments in Debreu (1970) we show that this is a pathological case and not robust to small perturbations of fundamentals (distribution of productivity shocks).

Since the steady state depends continuously on initial conditions - such as the initial debt level - the finiteness of the number of steady states implies that the steady state does not

[9] The same proof to show local uniqueness can be used to show that the constraint qualification is generically satisfied such that the Karush-Kuhn-Tucker optimality conditions are necessary.

[10] For each solution e there is an open set U_e such that $e \in U_e$ and no other solution is in U_e.



(a) Finite Number of SS

(b) Non-generic Indeterminacy

Figure 3: Generic Local Uniqueness

depend on initial conditions.

4 Quantitative Analysis: Steady State

The quantitative analysis has two main parts. First, we compute the optimal policy in the long-run in this Section. Second, we use the optimal long-run policy as a terminal condition to compute the optimal policy during the transition path in Section 5.

We start by calibrating the model to the U.S. economy and then compute the optimal values for the capital and labor tax rates, the capital stock and the level of debt in the steady state.[11] We also compute the optimal policy for a different Frisch elasticity, a different elasticity of intertemporal substitution, for the income process used in Aiyagari (1995) with much smaller income inequality than in our benchmark and we also allow for permanent productivity differences. We use the same calibrations and solve for the optimal policies when lump-sum transfers are an available instrument, obviously a very effective tool for redistribution

4.1 Calibration

To calibrate the initial state of the benchmark economy to the U.S., we first set the initial values of the following variables according to the literature. Following Trabandt and Uhlig

[11] These are the same policy instruments as used in Acikgoz (2014).

Parameters	Value	Description	Source/Target
		Exogenous Parameters	
σ	2	Coefficient of Risk Aversion	
ϕ	1	Frisch Elasticity	
α	0.36	Capital Share	
δ	0.08	Depreciation Rate	
τ_l	28%	Labor Income Tax Rate	Trabandt and Uhlig (2011)
τ_k	36%	Capital Income Tax Rate	
$\frac{B}{Y}$	62%	Debt to GDP Ratio	Holter et al. (2015)
$\frac{G}{Y}$	7.3%	Gov. Expenditure to GDP Ratio	Prescott (2004)
		Calibrated Parameters	
ρ	0.93	Persistence of Labor Productivity	$a_{90}/a_{50} = 7.55$
σ_u	0.3	Std. Dev. of Labor Productivity	$var(\log y) = 1.3$
β	0.94	Discount Rate	$K/Y = 3$
χ	13.5	Disutility from Labor	$mean(n) = 0.33$

Table 1: Benchmark Calibration

(2011), initial labor income tax rate is set to 28% and capital income tax rate to 36%, as shown in table 1. The Debt-to-GDP ratio is 62% as in Holter et al. (2015), and government expenditure is 7.3% of GDP, same as in Prescott (2004). Then, some parameters in the utility function and production function are set as follows: $\sigma = 2, \phi = 1, \alpha = 0.36$ and $\delta = 0.08$. The values for σ, α and δ are those used in most of the literature. The value of the Frisch elasticity ϕ is set higher than what are considered typical choices in the empirical labor literature but lower than the choice of many macroeconomists. As this parameter is important for the size of labor taxes in standard models, we provides several robustness checks. Anticipating our results of high labor income taxation, this high choice shows that this finding is not due to an inelastic household labor supply.

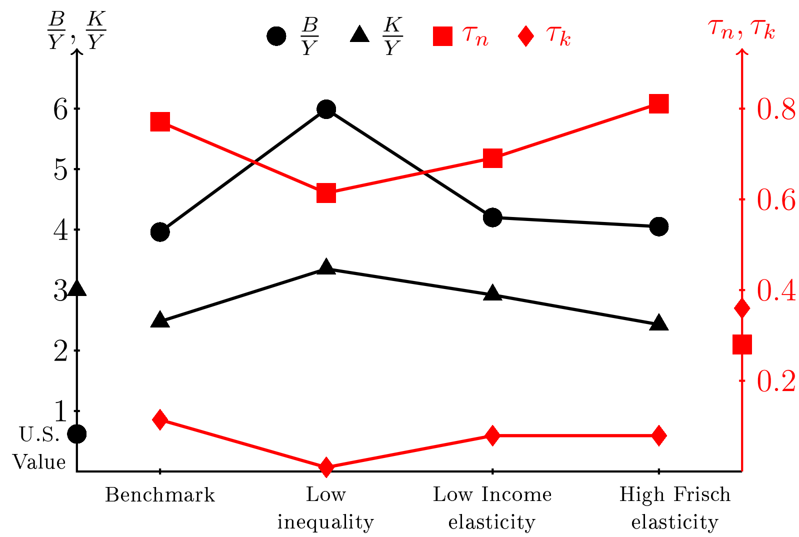
The rest of parameters are set to match related targets in the U.S. economy. The income process is the following AR(1) process: $\log e_t = \rho \log e_{t-1} + u_t$, $u_t \sim N(0, \sigma_u)$ where $\rho = 0.93$ and $\sigma_u = 0.3$, which are calibrated to two targets in the U.S. economy: first, the variance of log labor income - 1.3, and second, the ratio of earning at the 90 percentile over earning at the 50 percentile - 7.55. The time preference is set as $\beta = 0.94$ to match a capital-output ratio of 3. Disutility from labor is $\chi = 13.5$ such that the labor supply on average is 0.33.

4.2 Results

We numerically solve the set of equations that characterize the steady state of the optimal policy problem - equation (19) to (23) - based on the algorithm in Acikgoz (2014). We conduct this experiment for various calibrations: The benchmark calibration shown in section 4.1 and several other parametrizations where we change one parameter at a time. We consider a low Frisch labor supply elasticity of $\phi = 0.5$ instead of 1, a small income effect of $\sigma = 1$ instead of 2 and a low inequality calibration as in Aiyagari (1995) where we set $\rho = 0.6$ and $\sigma_u = 0.2$ instead of 0.93 and 0.3 (We also have to change β and χ to match the benchmark capital output ratio and labor supply). Finally we allow for permanent productivity differences in addition to an stochastic element implying that not all income states can be reached from any other state, e.g. the most productive worker today can never fall below average productivity. While we discuss the results in detail in the next sections Figure 4 provides an overview.

Several robust conclusions emerge across these parametrizations although the precise numbers of course differ as we show in the next sections. In the long-run, the level of government debt is very high relative to the current U.S. level. Tax distortions apparently do not put a tight bound on the welfare maximizing debt level. One reason is that higher labor taxes tax the risky income stream and replace it with riskfree capital income from holding bonds. The high level of debt together with the modified golden rule for capital imply that households require a higher after-tax interest rate and thus the tax on capital income is low across parametrizations. The high level of debt also implies large interest rate payments requiring a quite high tax on labor income, again robustly across all calibrations. The results therefore show that the planner does not use high capital income taxes for redistribution but instead decides to tax the risky labor income at a high rate and provides safe interest rate income from holding a large amount of debt which serves to smooth consumption very well.

The detailed results for the benchmark calibration are considered in section 4.2.1, for a low Frisch elasticity in section 4.2.3, a low income elasticity in section 4.2.4, a low inequality economy in section 4.2.2 and permanent differences in section 4.2.5. We not only consider the case where transfers are exogenous but we also report results when we include transfers as an instrument (the details of the numerical approach are delegated to the appendix).



Note - Results for debt-to-GDP ratio, B/Y, capital to output ratio, K/Y, capital income tax rate, τ_k and labor income tax rate, τ_n. Column (1) are current U.S. values, column (2) is the benchmark, column (3) considers a low Frisch elasticity $\phi = 0.5$, column (4) a low income elasticity $\sigma = 1$, column (5) a low inequality calibration as in Aiyagari (1995) and column (6) considers permanent income differences.

Figure 4: Results Overview

4.2.1 Results: Benchmark Calibration

The findings for the optimal Ramsey policies for using three instruments (labor tax τ_n, capital tax τ_k and transfers T) in a steady state are summarized in column (1) of Table 2 while the corresponding numbers - calibrated to the U.S. economy - are in column (4), as a comparison. In the long-run, optimal labor tax rate is as high as 76.7%, while the capital tax rate is 10.9% - higher than the optimal tax rate of 0 in a complete markets model, though lower than the current capital tax rate in the U.S. The quite sizable tax income is spent on redistribution through lump-sum transfer - 9.16% of GDP - and more importantly, on interest payment of the government debt - the debt level is as high as 5.5 times GDP. The capital satisfies the modified golden rule, so the capital output ratio is 2.48, slightly lower than the current ratio in the U.S. The high labor tax and the large transfer reduce labor supply from 0.33 to 0.21. This policy leads to a larger inequality of labor income but reduces the inequality of wealth.

One important feature of this steady state is a high tax rate on labor and a large amount of redistribution. First, the social planner largely reduces income inequality by setting a high labor tax rate, even though given the high Frisch elasticity, the distortion on the labor

Table 2: Ramsey Solutions of the Benchmark Economy

	Ramsey (τ_k, τ_n, T) (1)	Ramsey (τ_k, τ_n) (2)	Ramsey $(\tau_k, \tau_n, T=0)$ (3)	U.S. (4)
τ_l	76.7%	77.1%	75%	28%
τ_k	10.9%	11.4%	11.5%	36%
$\frac{T}{Y}$	9.16%	18.4%	0	13.4%
$\frac{B}{Y}$	5.5	3.96	6.98	0.62
$\frac{K}{Y}$	2.48	2.48	2.48	3.00
$\int n$	0.21	0.20	0.23	0.33
$N = \int en$	0.269	0.265	0.275	0.325
coeff. var. a	0.69	0.82	0.61	1.5
coeff. var. y	1.69	1.70	1.65	1.25
$var\,(\log y)$	3.09	3.03	3.06	1.3
$var\,(\log(y+T))$	0.43	0.23	3.06	0.49
$var\,(\log(y+T+\bar{r}a))$	0.35	0.29	0.41	0.47

Note - The table contains the optimal Ramsey steady-state policies. (1): Labor tax τ_n, capital tax τ_k and transfers T are available instruments. (2): Labor tax τ_n and capital tax τ_k are available instruments. Transfers T are fixed. (3): Labor tax τ_n and capital tax τ_k are available instruments. Transfers T are set to zero. (4): U.S. economy (calibration target)

supply is quite sizable, a 36% lower labor supply compared to the level in the calibrated U.S. economy. Effective labor N drops by less (17%) as it is low productivity households who reduce their labor supply most such that the inequality of after tax labor income $log(y) = \log(\bar{w}en)$ is higher in the optimal solution. However, the planner spends a large fraction of the tax income as lump-sum transfer, which reduces inequality of after tax and transfer income $\log(y+T)$ from 0.484 to 0.431 leading to an improvement of low-income households' welfare. The results show that the planner also reduces inequality through reducing wealth inequality as the coefficient of variation drops from 1.5 to 0.69, a drop which materializes in lower inequality of income $\log(y+T+\bar{r}a)$ of 0.346 relative to the benchmark level of 0.465.

To better understand the importance of lump-sum transfers in redistribution we now consider the same optimal policy problem with one modification: either we we fix the size of transfers at their current level or do not allow the planner to use lump-sum transfers and set $T = 0$. The findings are reported in columns (2) and (3) of table 2. When $T = 0$ is enforced (column (3)), the optimal labor tax rate is slightly lower, 75% and the capital tax rate is 11.5%. Now that the government pays no transfer, the still high revenue from taxing labor income is spent on the interest payment of huge government debt, which increases to 6.98 times GDP, even larger than the debt level of 5.5 for the case with transfers. Households on average hold high levels of assets and their returns are an important source of income. The asset returns help to insure against the negative labor productivity shocks, especially in this case where asset inequality is low - the coefficient of variance of household assets is only 0.61, compared to 1.5 in the original calibration. This means that even some labor income poor households hold sizeable assets, and receive substantial interest payments on government bonds. Although they do not receive direct transfers from the government anymore, the indirect transfers through a more equal distribution of wealth substitutes for the absence of the (more effective) redistribution through transfers. As a result the inequality of $\log(y+T+\bar{r}a) = 0.407$ is larger than when transfers payments are allowed for but smaller than in the benchmark although the variance of $\log(y) = \log(y+T) = 3.06$ is much higher than in the benchmark since labor supply is again more unequal. If we fix transfers at the current U.S. level of 13.4% of current U.S. GDP, which is higher than the optimal value reported in column (1), labor supply decreases even further but now the optimal level of debt drops to about 4 times GDP (column (2) of 2). Now that transfers already ansure a large amount of redistribution less redistribution and insurance though government debt is needed.

4.2.2 Results: A Low Inequality Economy

The income process in our benchmark calibration implies quite large inequality. If the income inequality is smaller, as in Aiyagari (1995), the motive for redistribution is smaller and the optimal policies are very different. Based on the benchmark economy, we change the parameters of the income process to $\rho = 0.6$ and $\sigma_u = 0.2$. Then we recalibrate the model by changing β to 0.973 and χ to 12.76 to match a capital output ratio of 3 and a average labor supply of 0.33.

The optimal long-run tax rates in this low inequality economy are smaller, compared to the benchmark economy. As shown in table 3, the labor income tax rate is now down to 59% though it is still quite high. The capital income tax rate is close to 0 - only 0.7%. The transfer is set to its lowest possible value, that is, 0. The government debt is even larger than in the benchmark, 11 times GDP.

In this low inequality economy, the planner has, compared to the high inequality benchmark economy, less incentives to reduce the income risks and to redistribute to low-income

	Ramsey (τ_k, τ_n, T) (1)	Ramsey (τ_k, τ_n) (2)	U.S. (3)
τ_l	59.0%	61.4%	28%
τ_k	0.7%	0.9%	36%
$\frac{T}{Y}$	0	15.0%	13.4%
$\frac{B}{Y}$	11.0	5.99	0.62
$\frac{K}{Y}$	3.35	3.35	3.00
$\int n$	0.29	0.28	0.33
$N = \int en$	0.305	0.296	0.352
coeff. var. a	0.49	0.58	0.71
coeff. var. y	0.65	0.60	0.51
$var\,(\log y)$	0.38	0.33	0.24
$var\,(\log(y+T))$	0.38	0.11	0.13
$var\,(\log(y+T+\bar{r}a))$	0.09	0.06	0.1

Table 3: Ramsey Solutions of Low Inequality Economy

households, resulting in a lower labor tax rate and zero transfers. Meanwhile, the planner still makes large payments to households, through the interest payments of government debt, and this helps to insure against the labor productivity shocks, as we discussed in the benchmark case without transfers.

Our findings show how a welfare-maximizing planner should redistribute and make use of two instruments. One instrument is to pay lump-sum transfers, the classic way to redistribute. In an incomplete markets setting, the government has a second option, as the capital-income of households can be increased. This way of redistribution involves issuing government debt but at the same time keeping the steady state capital fixed at the level that satisfies the modified golden rule.

Which instrument should the planner use, transfers or government debt? It turns out that the answer to this question depends on the labor supply elasticity and the amount of inequality. Using transfers is more effective in redistributing from high to low-income households but it comes with a disadvantage as it also reduces labor supply through income effects. As a result transfers are used when inequality is high (and thus the need to redistribute is high). Debt is more effective than transfer when inequality is low as labor supply is reduced much less than when transfer are paid. The use of transfer or debt for redistribution also depends on the labor supply elasticity, where a higher one implies lower transfer and more debt ceteris paribus. We will discuss this in more detail below.

Table 4: Low Labor Supply Elasticity

	Ramsey (τ_k, τ_n, T) (1)	Ramsey (τ_k, τ_n) (2)	U.S. (3)
τ_l	80.0%	81.1%	28%
τ_k	8.1%	7.9%	36%
$\frac{T}{Y}$	4.6%	18.0%	13.4%
$\frac{B}{Y}$	6.11	4.05	0.62
$\frac{K}{Y}$	2.43	2.43	3.00
$\int n$	0.23	0.22	0.33
$N = \int en$	0.28	0.28	0.33
coeff. var. a	0.63	0.79	1.53
coeff. var. y	1.64	1.66	1.31
$var\,(\log y)$	2.20	2.20	1.30
$var\,(\log(y+T))$	0.59	0.18	0.51
$var\,(\log(y+T+\bar{r}a))$	0.37	0.29	0.51

4.2.3 Results: Low Labor Supply Elasticity

Now we consider a lower labor supply elasticity and set the Frisch elasticity to $= 1/\phi = 0.5$. As a result of a recalibration we use now $\beta = 0.936$, $\rho = 0.927$, $\sigma_u = 0.335$ and $\chi = 37.7$, so that the modified golden rule capital stock is close to the one in the benchmark. Results are reported in Table 4.

A lower elasticity of labor supply implies that labor supply is less sensitive to an increase in labor taxes, rendering a labor tax of 80% optimal. At the same time although the labor tax rate is higher than in the benchmark economy, now in the optimal steady state, the labor supply is in fact slightly higher: 0.23, compared to 0.19 in the benchmark, as one would expect if the labor supply elasticity is reduced.

4.2.4 Results: Small Income Effect

How labor supply reacts to the labor tax, depends not only on the Frisch elasticity which governs the substitution effect, but also on the coefficient of relative risk aversion, which determines the income effect. Next, we consider the case with a smaller income effect by setting the coefficient of relative risk aversion $\sigma = 1$. As a result of a recalibration we use now $\beta = 0.958$, $\rho = 0.900$, $\sigma_u = 0.287$ and $\chi = 45.6$, so that the modified golden rule capital stock is now larger than in the benchmark. Results are reported in Table 5.

Table 5: Small Income Effect

	Ramsey (τ_k, τ_n, T) (1)	Ramsey (τ_k, τ_n) (2)	U.S. (3)
τ_l	68.5%	69.1%	28%
τ_k	6.38%	7.9%	36%
$\frac{T}{Y}$	0	18.5%	13.4%
$\frac{B}{Y}$	8.61	4.20	0.62
$\frac{K}{Y}$	2.92	2.92	3.00
$\int n$	0.23	0.22	0.33
$N = \int en$	0.32	0.31	0.42
coeff. var. a	0.69	0.95	1.52
coeff. var. y	1.61	1.62	1.38
$var\,(\log y)$	2.01	1.94	1.30
$var\,(\log(y+T))$	2.01	0.27	0.51
$var\,(\log(y+T+\bar{r}a))$	0.53	0.31	0.49

A larger income effect increases labor supply, because a higher labor tax makes the households poorer and as a result increases their labor supply, holding other things constant. The smaller income effect in this experiment implies that labor supply is more responsive to labor taxes, rendering a lower labor tax rate of 68.5% optimal. The smaller income effect also decreases the negative income effects on labor supply of paying higher transfers, which suggest that a higher level of transfers than in the benchmark might be optimal. However such an argument overlooks that a smaller income effect also reduces the welfare gains from redistribution since households are less averse to consumption fluctuations in this case, which suggest a smaller level of transfers than in the benchmark. Our results show that the latter effect dominates the first one and transfers are now zero in the optimal steady state. But we can conclude that our finding of a high tax on labor income is robust with respect to what we assume for the elasticity of labor supply and the income effect.

4.2.5 Results: Permanent Income Differences

4.2.6 Understanding High Labor Taxes

The optimal high redistribution policy involves high labor tax rates and transfer payments both with adverse consequences for aggregate employment. To understand why this policy is optimal we now discuss how the policy changes affect aggregate steady state labor supply,

$$N = \int_{E\times A} en(a,e,s(a,e;\tau_n,T);\tau_n,T)d\mu(a,e), \tag{25}$$

where $n(a,e,s(a,e;\tau_n,T);\tau_n,T)$ is individual labor supply of a household with asset level a and labor productivity e, who saves $s(a,e;\tau_n,T)$ and faces the labor tax rate τ_n and the transfer T. The change in aggregate supply between the optimal steady state and the initial calibrated steady state can then be decomposed into the contribution due to the changes in labor taxes, in transfers, in prices, in savings and in the joint distribution of assets and shocks:

$$\begin{aligned}
& N^* - N \qquad (26)\\
= & \int_{E\times A} en^*(a,e,s^*(\cdot);\tau_n^*,T^*)d\mu^*(a,e) - \int_{E\times A} en(a,e,s(\cdot);\tau_n,T)d\mu(a,e)\\
= & \underbrace{\int_{E\times A} en(a,e,s(a,e;\tau_n,T);\boldsymbol{\tau_n^*},T)d\mu(a,e) - \int_{E\times A} en(a,e,s(a,e;\boldsymbol{\tau_n},T);\tau_n,T)d\mu(a,e)}_{\text{Labor tax change} = -0.0097}\\
+ & \underbrace{\int_{E\times A} en(a,e,s(a,e;\tau_n,T);\tau_n^*,\boldsymbol{T^*})d\mu(a,e) - \int_{E\times A} en(a,e,s(a,e;\tau_n,\boldsymbol{T});\tau_n^*,T)d\mu(a,e)}_{\text{Change in Transfer} = 0.0379}\\
+ & \underbrace{\int_{E\times A} e\boldsymbol{n^*}(a,e,s(a,e;\tau_n,T);\tau_n^*,T^*)d\mu(a,e) - \int_{E\times A} e\boldsymbol{n}(a,e,s(a,e;\tau_n,T);\tau_n^*,T^*)d\mu(a,e)}_{\text{Change in Prices} = -0.0207}\\
+ & \underbrace{\int_{E\times A} en^*(a,e,\boldsymbol{s^*(a,e;\tau_n^*,T^*)};\tau_n^*,T^*)d\mu(a,e) - \int_{E\times A} en^*(a,e,\boldsymbol{s(a,e;\tau_n,T)};\tau_n^*,T^*)d\mu(a,e)}_{\text{Change in Savings} = 0.0302}\\
+ & \underbrace{\int_{E\times A} en^*(a,e,s^*(a,e;\tau_n^*,T^*)d\boldsymbol{\mu^*}(a,e) - \int_{E\times A} en^*(a,e,s^*(a,e;\tau_n^*,T^*);\tau_n^*,T^*)d\boldsymbol{\mu}(a,e)}_{\text{Change in Distribution} = -0.0895}
\end{aligned}$$

where optimal values have a superscript $*$.

The labor tax change keeps the labor supply function n and the saving function s as well as prices and the distribution fixed and is thus a combination of the substitution and the income effect, which depends on household's wealth. Keeping the saving function and not consumption constant allows us to also compute the effects of paying transfers which increase consumption and thus lower labor supply. The next step considers the change in prices as this induces a different labor supply function n^* instead of n. Households also adjust their savings behavior in response to the policy change leading to an adjustment in labor supply. Finally households labor supply also depends on the assets a it holds and the productivity shock e so that a change in the joint distribution μ over assets and productivity leads to a different aggregate labor supply.

Next we investigate how the optimal steady state policies and labor supply depend on the labor supply elasticity and income effects. In the benchmark both the labor supply elasticity and the income elasticity are quite high and we redo our analysis now for lower values for these elasticities to see whether labor is still taxed at a high rate and capital income is taxed at a quite low rate.

4.3 Comparison with the Literature

Dyrda and Pedroni (2015) also compute the optimal transition path in an incomplete markets economy, using however a quite different approach. In particular they do not characterize the optimal steady policies first before computing the transition but instead compute both jointly.[12] They find that the optimal capital tax rate in the long-run is as high as 45% and the labor tax rate is at a relatively low level, 13%.At the steady state the lump-sum transfer is 3.5% of GDP and debt is -125%. As a comparison, we use their utility function and income process to calibrate our model to match the targets used in their calibration.[13] These targets are actually not so different from what we use in our paper, except that the government expenditure in their calibration is higher than ours: 15% of GDP v.s. 7.3%. The targets and parameters are shown in table **??**. Then we use our approach to solve for the optimal steady state. TBC

[12]They approximate the optimal policies, including tax rates and transfers, over time, using splines with nodes, and then search for the splines and the associated policies that lead to the highest welfare for the Ramsey planner.

[13]The only difference is that in our model, the capital tax also applies to households with negative asset, i.e., these households receive tax deduction for interest paid on debt. In their model, the capital tax only applies to households with positive asset. Given that the borrowing limit in their paper is not so different from 0, we find that this difference does not change the calibrated equilibrium.

5 Quantitative Analysis: Transition

A main objective of this paper is, as emphasized a couple of times before, to compute the path of tax rates and government debt which maximizes welfare at date 0. This is a huge computational challenge: Several hundred or thousands of variables have to be chosen in a highly nonlinear optimization problem. However, our previous result on the optimal steady state turn this non-manageable optimization problem into a manageable one. We have shown that the optimal steady state is independent of initial conditions. From a computational point of view this means that we know the optimal long-run policies and allocations without having to compute the transition. Instead we know the initial conditions (economy calibrated to U.S. economy) and we know the terminal condition, the optimal steady state characterized above. The (still huge) computational problem is then to find the optimal policy path which satisfies all necessary first-order conditions along the transition and at the same time satisfies the initial and terminal conditions. This problem is still a challenge as it involves solving hundreds or thousands of nonlinear equations but is significantly easier (and therefore tractable) than the original problem, which has to find the optimal transition and the optimal terminal point at the same time. Given the large number of variables involved there is no way to check the global validity of a candidate solution, a check which is not necessary in the current approach.

Knowing the optimal path of policies allows us to compute the welfare gains of switching to the optimal policy and helps to understand the properties of the optimal steady states policies better as those obviously depend on the transition.

5.1 Computational Algorithm

Appendix III.2 outlines the details of computing an optimal transition, starting from the model calibrated to the U.S. economy and going to the optimal long run steady state.

5.2 Calibration of Initial Steady State

In order to facilitate the computation of the optimal transition path we recalibrate the model with 10-year periods. We make some changes that make the economy similar but not exactly identical to the benchmark economy in Section 4. Most notably we assume that the variance of the innovations to labor productivity, σ_u, is 10 times the variance on a one-yearly basis

Parameters	Value	Description	Source/Target
		Exogenous Parameters	
σ	2	Coefficient of Risk Aversion	
ϕ	1	Frisch Elasticity	
α	0.36	Capital Share	
δ	0.57	Depreciation Rate	
τ_l	28%	Labor Income Tax Rate	Trabandt and Uhlig (2011)
τ_k	36%	Capital Income Tax Rate	
$\frac{B}{Y}$	62%	Debt to GDP Ratio	Holter et al. (2015)
$\frac{G}{Y}$	7.3%	Gov. Expenditure to GDP Ratio	Prescott (2004)
		Calibrated Parameters	
ρ	0.39	Persistence of Labor Productivity	$a_{90}/a_{50} = 7.55$
σ_u	0.94	Std. Dev. of Labor Productivity	$var(\log y) = 1.3$
β	0.50	Discount Rate	$K/Y = 3$
χ	37.0	Disutility from Labor	$mean(n) = 0.33$

Table 6: Benchmark Calibration with 10-year Periods

an we then adjust the parameter governing the persistence of labor productivity, ρ, to still hit the calibration target $a_{90}/a_{50} = 7.55$. We also set $K/Y = 0.3$.

5.3 Results Transition

For now we focus on a transition path with optimal choice of debt, capital tax and labor income tax. Transfers are kept at the calibrated benchmark level. Figure 5 plots the optimal transition path of capital taxes, τ_k, and labor income taxes, τ_l, and Figure 6 plots the optimal path of the capital stock, K, and government debt, B. It is optimal to subsidize labor in the first few periods and to finance this by increasing debt and taxing capital high. The labor income tax, τ_l, starts out at -8.6% and gradually increases to a level of about 80% after 100 years (the long run steady state level of τ_l is 80.3% with the 10-year calibration). The capital tax starts out at 99.8% and gradually decreases to a level of about 10% after 100 years (the long run steady state level of τ_k is 11.6% with the 10-year calibration). Debt is rapidly increasing in the first few periods and reaches a peak after 50 years before it starts decreasing towards the steady state level. At the peak, debt relative to GDP in the initial steady state, B/Y_0, is about 0.29 (2.9 in yearly terms), whereas the long run level is about 0.18 (1.8 in yearly terms). The capital stock decreases smoothly towards the long run level for the first 70 years and then stays relatively constant.

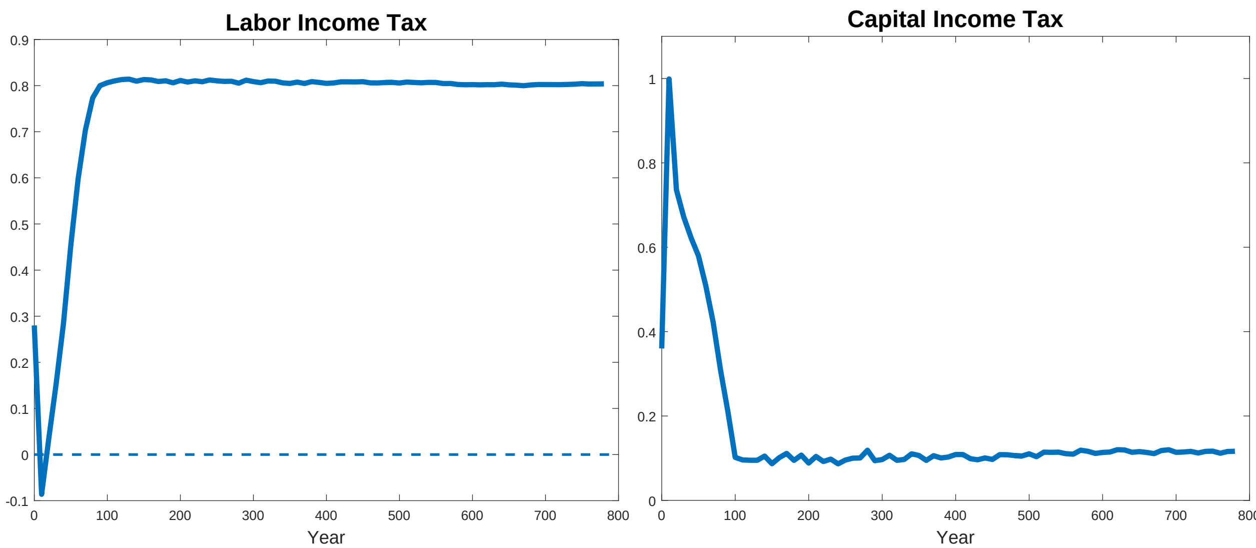


Figure 5: Optimal Transition Path for labor tax τ_l (left) and capital tax τ_k (right)

We solve an 800 year transition path even if there is not that much going on in terms of the optimal policies after the first 100 years. It does, however, take 500-600 years before the distribution of agents in the economy converges exactly to the long run steady state. To see this one can for instance look at the average value of the Lagrange multipliers, λ, which we plot in Figure 7. It converges to it's steady state level after about 550 years.

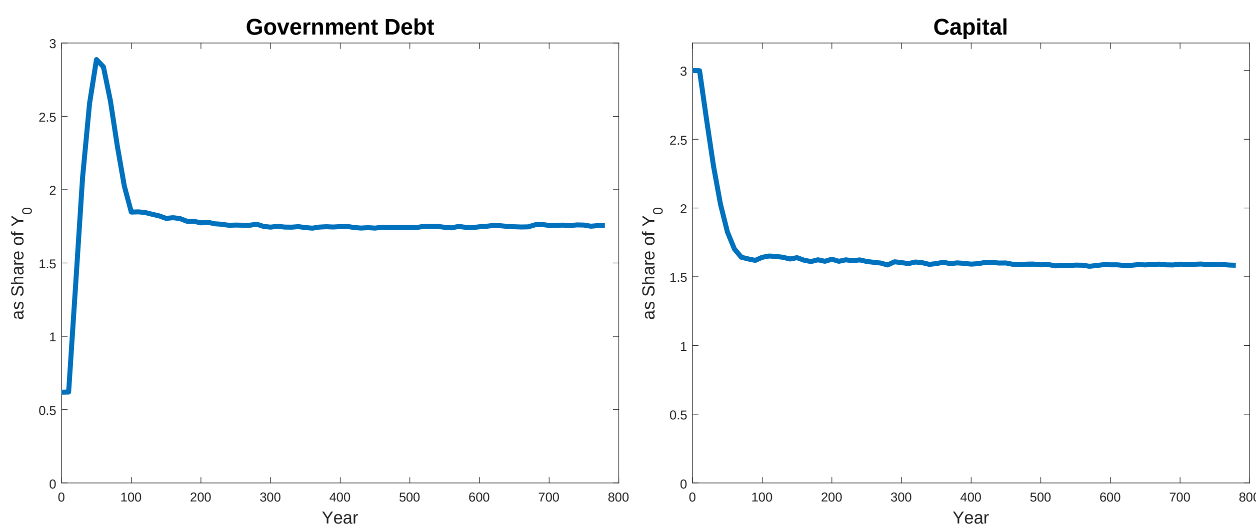


Figure 6: Optimal Transition Path for B (left) and K (right). Y_0 is 10-year output.

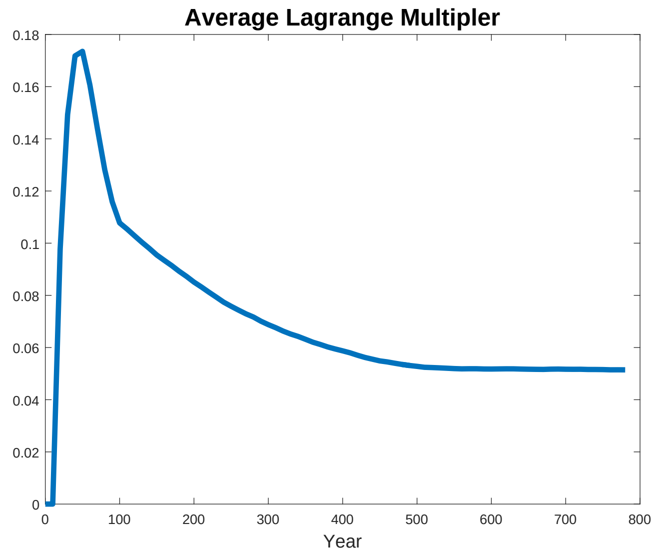


Figure 7: Optimal Transition Path for Average λ

5.4 Welfare Gains

To quantify the lifetime welfare gain to agents in period 0 of the transition relative to the initial steady state, we ask the question of by how many % we need to increase consumption in every state of the initial steady state to make the expected value function equal to to the expected value function in the first period of the transition. In other words we solve for a constant, φ, such that:

$$E[V_{SS}(a,e) \ \mid c(a,e) = \varphi c(a,e)] = E[V_0(a,e)] \tag{27}$$

The lifetime welfare gain achieved from moving to the optimal policy taking into account the full transition period is equivalent to increasing the consumption in all states of the initial steady state by 4.2%.

6 Conclusion

In incomplete markets model of the Bewley-Imrohoroglu-Huggett-Aiyagari type, inequality is to a large degree purely due to luck. This calls for a large amount of redistribution in an optimal welfare maximizing policy. Several classic instruments are available for redistribution: labor income taxation, capital income taxation and paying transfers. However, a large amount of redistribution could also inflict large efficiency losses limiting how much redistribution is desirable. Whereas all of these instruments can reduce inequality, it is therefore unclear how much redistribution should come from which instrument. Each of these instruments comes with efficiency losses in terms of distorting labor supply and/or capital accumulation. Furthermore, if markets are incomplete, the planner can also reduce the inequality of wealth through issuing more debt such that low-labor income households can also rely on their asset income for consumption purchases.

This paper suggests the following conclusions how to redistribute in a welfare maximizing way: The optimal policy to provide insurance is to heavily tax labor in the long-run, and redistribute through transfers and government bonds. In particular redistribution is through high labor taxation but capital income is taxed at a low rate only, a conclusion that holds in high and low inequality economies and is also robust to changing parameters such as the labor supply elasticity.

The choice whether to use transfers or higher debt, however, depends on the properties of the economy. Paying transfers is an effective tool to redistribute when the income inequality is high, but not if inequality is low since then the disincentive effects on labor supply outweigh the gains from redistribution. In the latter case instead, a large amount of government debt is used to lower wealth inequality and thus makes consumption more equal across households. Given this role and the smaller disincentives effects than paying transfers, the debt level is generally high, not only in low inequality economies but also in the high inequality economies since redistributing through transfers only would inflict too large disincentive effects on the economy.

Results during the transition to the long-run optimum are quite different though. During the transition debt is accumulated and this increase in government revenue is used to lower labor taxes below its current U.S. level. Only when the long-run steady state is approached and the amount of debt and associated interest rate payments are high, it becomes necessary to increase labor taxes to balance the budget. At that time, capital taxes have already converged to a low level after initial periods of high high taxation, a well known result as capital is supplied quite inelastically in the short-run.

We prove two theoretical results which enable this quantitative analysis. We show theoretically that the optimal capital stock is at the modified golden rule and that the long-run optimal steady state is independent of initial conditions. In particular, there is a unique long-run optimal level of government debt independent of the initial level of debt in our incomplete markets model, a result not valid in complete market models.

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APPENDICES

I Appendix: Derivations

In this section, we provide the derivations of key equations for the household problem, the Ramsey planner's problem and the steady state.

I.1 Households' Problem

A household's labor supply can be expressed as a function of effective wage and consumption, using the F.O.C of n_t:

$$\begin{aligned} u_c\left(c_t, n_t\right) e_t \bar{w}_t + u_n\left(c_t, n_t\right) = 0 \Rightarrow \\ \frac{-u_n\left(c_t, n_t\right)}{u_c\left(c_t, n_t\right)} &= e_t \bar{w}_t \Rightarrow \\ \frac{\chi n_t^{\frac{1}{\phi}}}{c_t^{-\sigma}} &= e_t \bar{w}_t \Rightarrow \\ n_t &= \left(\chi^{-1} e_t \bar{w}_t c_t^{-\sigma}\right)^{\phi}, \end{aligned}$$

and labor income can be also expressed as a function of wage and consumption, as follows:

$$y_t = \left(\chi^{-1} e_t^{1+\frac{1}{\phi}} \bar{w}_t^{1+\frac{1}{\phi}} c_t^{-\sigma}\right)^{\phi}.$$

Moreover,

$$e_t w_t u_{ct} + u_{nt} = 0$$

will be a useful expression to simplify expressions later. Given the expressions of n and y, using the F.O.C w.r.t. a_{t+1} and Kuhn-Tucker condition for the borrowing constraint, a household's policy functions solve the following system of necessary conditions:

$$\begin{aligned}
u'(c_t) &\geq \beta(1+\bar{r}_{t+1})\mathbb{E}\left[u'(c_{t+1})\right], \\
0 &= (a_{t+1} + \underline{a})\left(u'(c_t) - \beta(1+\bar{r}_{t+1})\mathbb{E}u'(c_{t+1})\right), \\
c_t + a_{t+1} &\leq a_t(1+\bar{r}_t) + y_t + T_t \\
a_{t+1} + \underline{a} &\geq 0.
\end{aligned}$$

I.2 Planner's Problem

Given the planner's problem described in the main text, here we derive the Lagrangian equation (12). First, denote the history of a household's labor productivity from period 0 to t as $h^t = \{h^{t-1}, e_t\}$ where $h^0 = \{e_0\}$. Let θ_{t+1}, η_{t+1} and γ_t represent the present value Lagrangian multipliers for (7), (8) and (2) respectively. Then the Lagrangian can be expressed as

$$\begin{aligned}
L = &\sum_{t=0}^{\infty} \beta^t \sum_{h^t} \Pi\left(h^t\right) \Bigg(u\left(c_t\left(h^t\right)\right) \\
&+\theta_{t+1}\left(h^t\right)\left(u'\left(c_t\left(h^t\right)\right) - \beta(1+\bar{r}_{t+1})\sum_{h^{t+1}} \Pi\left(h^{t+1}|h^t\right) u'\left(c_{t+1}\left(h^{t+1}\right)\right)\right) \\
&-\eta_{t+1}\left(h^t\right)\left(a_{t+1}\left(h^t\right) + \underline{a}\right)\left(u'\left(c_t\left(h^t\right)\right) - \beta(1+\bar{r}_{t+1})\sum_{h^{t+1}} \Pi\left(h^{t+1}|h^t\right) u'\left(c_{t+1}\left(h^{t+1}\right)\right)\right)\Bigg) \\
&+\sum_{t=0}^{\infty} \beta^t \gamma_t \left(F(K_t, N_t) + (1-\delta)K_t + B_{t+1} - \left(G_t + T_t + (1+\bar{r}_t)B_t + \bar{r}_t K_t + \bar{w}_t N_t\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t=0}^{\infty} \beta^t \sum_{h^t} \Pi\left(h^t\right) \Bigg(u\left(c_t\left(h^t\right)\right) + u'\left(c_t\left(h^t\right)\right) \\
&\quad \left(\theta_{t+1}\left(h^t\right) - \theta_t\left(h^{t-1}\right)\left(1+\bar{r}_t\right) - \eta_{t+1}\left(h^t\right)\left(a_{t+1}\left(h^t\right) + \underline{a}\right) + \eta_t\left(h^{t-1}\right)\left(a_t\left(h^{t-1}\right)+\underline{a}\right)\left(1+\bar{r}_t\right)\right)\Bigg) \\
&+ \sum_{t=0}^{\infty} \beta^t \gamma_t \left(F\left(K_t, N_t\right) + (1-\delta) K_t + B_{t+1} - \left(G_t + T_t + \left(1+\bar{r}_t\right) B_t + \bar{r}_t K_t + \bar{w}_t N_t\right)\right).
\end{aligned}
$$

Define $\lambda_{t+1} \equiv \eta_{t+1}\left(a_{t+1} + \underline{a}\right) - \theta_{t+1}$, and the Lagrangian can be further simplified as

$$
\begin{aligned}
L =& \sum_{t=0}^{\infty} \beta^t \sum_{h^t} \Pi\left(h^t\right)\left(u\left(c_t\left(h^t\right)\right) + u'\left(c_t\left(h^t\right)\right)\left(\lambda_t\left(h^{t-1}\right)\left(1+\bar{r}_t\right) - \lambda_{t+1}\left(h^t\right)\right)\right) \\
&+ \sum_{t=0}^{\infty} \beta^t \gamma_t \left(F\left(K_t, N_t\right) + (1-\delta) K_t + B_{t+1} - \left(G_t + T_t + \left(1+\bar{r}_t\right) B_t + \bar{r}_t K_t + \bar{w}_t N_t\right)\right),
\end{aligned}
$$

subject to 3, 4, 9, 11 and 10, starting from initial conditions $a_0\left(h^{-1}\right) = a_0, B_0$ and $\lambda_0\left(h^{-1}\right) = 0$.

The first order conditions can be obtained from the Lagrangian, by taking derivatives w.r.t. to the unknowns $\lambda_{t+1}, a_{t+1}, B_{t+1}, T_t, \bar{r}_t, \bar{w}_t$. This gives us the set of FOCs in the main text, i.e., equation 13 to 17. The FOCs, together with the constraints, i.e., equation 3, 4, 11 and 10, characterize the necessary conditions for the interior solution of the planner's problem.

In these FOCs, partial derivatives including $\frac{\partial N_t}{\partial T_t}, \frac{\partial N_t}{\partial \bar{r}_t}$ and so on can be expressed as

$$
\begin{aligned}
\frac{\partial N_t}{\partial T_t} &= \int \frac{\partial n_t}{\partial c_t}\frac{\partial c_t}{\partial T_t} d\mu_t, \\
\frac{\partial N_t}{\partial \bar{r}_t} &= \int \frac{\partial n_t}{\partial c_t}\frac{\partial c_t}{\partial \bar{r}_t} d\mu_t, \\
\frac{\partial N_t}{\partial \bar{w}_t} &= \int \left(\frac{\partial n_t}{\partial \bar{w}_t} + \frac{\partial n_t}{\partial c_t}\frac{\partial c_t}{\partial \bar{w}_t}\right) d\mu_t.
\end{aligned}
$$

Moreover, expressions for $\frac{\partial n_t}{\partial c_t}, \frac{\partial c_t}{\partial T_t}$, and similar partial derivatives are easy to obtain given equation (5), $n_t = \left(\chi^{-1} e_t \bar{w}_t c_t^{-\sigma}\right)^{\phi}$, and (6), $y_t = e_t n_t \bar{w}_t = \left(e_t \bar{w}_t\right)^{1+\phi}\left(\chi^{-1} c_t^{-\sigma}\right)^{\phi}$, which describe how n_t and y_t depend on c_t and $\bar{w}_t$. Using also the household budget constraint, equation (3), we obtain the partial derivatives:

$$\begin{aligned}
\frac{\partial n_t}{\partial c_t} &= -\sigma\phi\left(\chi^{-1}\bar{w}_t e_t\right)^{\phi} c_t^{-\sigma\phi-1} = -\sigma\phi\frac{n_t}{c_t}, \\
\frac{\partial n_t}{\partial \bar{w}_t} &= \phi\frac{n_t}{\bar{w}_t}, \\
\frac{\partial c_t}{\partial T_t} &= \frac{\partial y_t}{\partial c_t}\frac{\partial c_t}{\partial T_t} + 1 \quad \Rightarrow \quad \frac{\partial c_t}{\partial T_t} = \frac{1}{1-\frac{\partial y_t}{\partial c_t}} = \frac{1}{1+\sigma\phi\frac{y_t}{c_t}}, \\
\frac{\partial c_t}{\partial \bar{w}_t} &= \frac{\partial y_t}{\partial c_t}\frac{\partial c_t}{\partial \bar{w}_t} + \frac{\partial y_t}{\partial \bar{w}_t} \quad \Rightarrow \quad \frac{\partial c_t}{\partial \bar{w}_t} = \frac{\frac{\partial y_t}{\partial \bar{w}_t}}{1-\frac{\partial y_t}{\partial c_t}} = \frac{e_t n_t}{1+\sigma\phi\frac{y_t}{c_t}},
\end{aligned}$$

I.3 Steady State

Given the assumption that variables are stable at the steady state, we can obtain the FOCs at the steady state by simply replacing variables in the FOCs of the transition dynamics at their steady state values. For example, $\bar{r}_t, \bar{r}_{t+1}$ can be replaced by the steady state value $\bar{r}$. Same for $\bar{w}_t, B_t$, and all the aggregate variables. Notice that households' choice variables a_{t+1}, λ_{t+1} are different, because they are not constant variables but depend on the state of the household. Following Straub and Werning (2015), we focus on the recursive formulation of the problem, that is to say, current period variables a and λ are the state variables which summarize the history and decide next period choice variables a' and λ', together with current period productivity shock e. We can then replace a_t and λ_t with a and λ, and replace a_{t+1} and λ_{t+1} with a' and λ'. Now the steady state solution is characterized by a set of FOCs, as equation 19 to 23, together with following constraints:

$$c + a' = a\left(1+\bar{r}\right) + y\left(e, \bar{w}\right) + T, \tag{A1}$$

$$G + \left(1+\bar{r}\right)B + \bar{r}K + \bar{w}N + T \leq F\left(K, N\right) + B, \tag{A2}$$

$$K = A - B, \tag{A3}$$

$$A = \int a d\mu, \tag{A4}$$

$$N = \int e n d\mu. \tag{A5}$$

II Proofs of Section 3

Proof of Theorem 1

Equation $\gamma = \beta \left(1 + F_K \left(K', N'\right)\right) \gamma'$ implies in the steady state since $\gamma = \gamma'$ that $1 = \beta \left(1 + F_K \left(K, N\right)\right)$, which is the modified golden rule.

Proof of Theorem 2

The idea of the proof is as follows. We first show that the Ramsey problem is generically regular (building on Debreu (1970) and Dierker and Dierker (1972)) which implies that a stationary solution to the the Ramsey problem is locally unique. We then show that the steady state depends continuously on initial conditions such as the initial debt level. Together with the local uniqueness this implies that the steady state does not depend on initial conditions. The proof for now assumes that labor supply is exogenous, $n = N = 1$, and we explain later that the arguments generalize in a straightforward way to the case with endogenous labor supply.

We first show local uniqueness and divide this proof into several steps. As a first step we show that the steady state, which is characterized as a solution to

$$
\begin{aligned}
u_c(c) &\geq \beta(1+\bar{r}')\mathbb{E}[u_c(c')|e] \text{ with equality if } a' > -\underline{a}, && \text{(A6)}\\
u_c - \frac{\partial c}{\partial a'}u_{cc}(\lambda(1+\bar{r})-\lambda') &= \beta\mathbb{E}_t\left[(1+\bar{r}')u_c' + \frac{\partial c'}{\partial a'}u_{cc}'(\lambda'(1+\bar{r}')-\lambda'')\right]\\
&+ \beta\gamma(F_K(K',N')-\delta-\bar{r}'),\\
&\quad \text{if } a' > -\underline{a}, \text{ otherwise } \lambda' = 0. && \text{(A7)}\\
1 &= \beta(1+F_K(K,N)-\delta), && \text{(A8)}\\
\gamma A &= \mathbb{E}[u_c(c)\lambda + au_c(c) + au_{cc}(c)(\lambda(1+\bar{r})-\lambda')], && \text{(A9)}\\
\gamma N &= \mathbb{E}[enu_c(c) + enu_{cc}(c)(\lambda(1+\bar{r})-\lambda')]. && \text{(A10)}\\
c + a' &= a(1+\bar{r}) + y(e,\bar{w}) + T, && \text{(A11)}\\
G + (1+\bar{r})B + \bar{r}K + \bar{w}N &\leq F(K,N) + B, && \text{(A12)}\\
K &= A - B, && \text{(A13)}\\
A &= \int a d\mu, && \text{(A14)}\\
N &= \int en d\mu. && \text{(A15)}
\end{aligned}
$$

can be characterized as the solution to two equations $z^{AM}(\bar{r},\bar{w}) = 0$ and $z^{LM}(\bar{r},\bar{w}) = 0$ in the unknowns $\bar{r}$ and $\bar{w}$, the "excess demand" functions in the Asset Market and the Labor Market. Regularity of the steady state then means that these two functions are locally invertible, what we establish in Step 2 below.

Step 1: Characterization of steady state ("excess demand")

To express the steady state as a solution to two equations we first show the existence of steady-state Lagrange Multipliers q.

i) Proof of Existence of steady-state Lagrange Multipliers q

Here we prove the existence and uniqueness of a linear-affine function $q'(q,a,e) = \alpha_0(a,e)\,q + \alpha_1(a,e)$, which solves the steady state equation (A7), which after using (A6), division by γ and defining $q = \lambda/\gamma$ equals

If $a' = -\underline{a}$, $\quad q' = 0$.

If $a' > -\underline{a}$:

$$\begin{aligned}
\text{If } a' > -\underline{a}: \qquad & -\frac{\partial c}{\partial a'} u_{cc}(c)\left[q(1+\bar{r}) - q'\right] \\
&= \beta \mathbb{E}\left[\frac{\partial c'}{\partial a'} u_{cc}(c')\left[q'(1+\bar{r}) - q''\right] | e\right] + 1 - \beta(1+\bar{r}) \\
&= \beta \mathbb{E}\left[-(1+\bar{r})\frac{\partial c'}{\partial a''} u_{cc}(c')\left[q'(1+\bar{r}) - q''\right] | e\right] + 1 - \beta(1+\bar{r}),
\end{aligned} \tag{A16}$$

where we used that $\frac{\partial c'}{\partial a'} = -(1+\bar{r})\frac{\partial c'}{\partial a''}$.

We establish our results for given interest rate $\bar{r}$ and wage $\bar{w}$, individual saving decisions $a'(a,e)$ and individual consumption decisions $c(a,e)$. Introduce the notation $v := \frac{\partial c}{\partial a'} u_{cc}(c(a,e)) > 0$ and ditto notation for $v' := \frac{\partial c'}{\partial a''} u_{cc}(c(a',e')) > 0$. Rewrite the affine $q'(q,a,e) = \alpha_0(a,e)\,q + \alpha_1(a,e)$ as

$$q'(q,a,e) = \Big[(1+\bar{r})q + H(a,e)/v(a,e)\Big] \cdot K(a,e), \tag{A17}$$

where H, K are nonnegative, $K(a,e) = 0$ for those (a,e) such that $a'(a,e) = -\underline{a}$, so that

$$\alpha_0(a,e) = (1+\bar{r}) \cdot K(a,e) \quad \text{and} \tag{A18}$$

$$\alpha_1(a,e) = K(a,e)H(a,e)/v(a,e). \tag{A19}$$

Similarly

$$q''(q',a',e') = \Big[(1+\bar{r})q' + H'/v'\Big] \cdot K' \tag{A20}$$

for H', K' all nonnegative, $K'(a',e') = 0$ for those (a',e') such that $a''(a',e') = -\underline{a}$.

Insert this into (A16)

$$(1+\bar r)(K-1)vq + KH - 1 + (1+\bar r)\beta = (1+\bar r)\beta\mathbb{E}\left[(1+\bar r)(K'-1)v'q' + H'K'|e\right] \quad \text{(A21)}$$
$$= (1+\bar r)\beta\mathbb{E}\left[(1+\bar r)^2(K'-1)v'Kq + (1+\bar r)(K'-1)HKv'/v + H'K'|e\right] \quad \text{(A22)}$$

Gather the "q" terms:

$$(1+\bar r)(K-1)v = (1+\bar r)^3\beta\mathbb{E}\left[(K'-1)v'|e\right]K \quad \text{(A23)}$$

and solve out for K, taking into account where it must be zero:

$$K = \frac{1_{\{(a,e);a'>-\underline{a}\}}}{1+(1+\bar r)^2\beta\mathbb{E}\Big[(1-K')v'/v\Big|e\Big]}. \quad \text{(A24)}$$

We define an iteration of functions which converge to the solution. As initialization we set $K_0(a,e)\equiv 0$ and define inductively

$$K_{n+1}(a,e) = \frac{1_{\{(a,e);a'>-\underline{a}\}}}{1+(1+\bar r)^2\beta\mathbb{E}\Big[\big\{1-K_n(a'(a,e),e')\big\}v'(a',e')/v(a,e)\Big|e\Big]} \quad \text{(A25)}$$

By induction, it follows that $1\geq K_{m+1}\geq K_m\geq \dots K_0 = 0$. This is obviously true for $n=0$. For $m+1$ it follows from $K_{m+1}\geq K_m$ that

$$\begin{aligned} K_{m+2}(a,e) &= \frac{1_{\{(a,e);a'>-\underline{a}\}}}{1+(1+\bar r)^2\beta\mathbb{E}\Big[\big\{1-K_{m+1}(a'(a,e),e')\big\}v'(a',e')/v(a,e)\Big|e\Big]} && \text{(A26)}\\ &\geq \frac{1_{\{(a,e);a'>-\underline{a}\}}}{1+(1+\bar r)^2\beta\mathbb{E}\Big[\big\{1-K_m(a'(a,e),e')\big\}v'(a',e')/v(a,e)\Big|e\Big]} && \text{(A27)}\\ &= K_{m+1}(a,e). && \text{(A28)}\end{aligned}$$

We therefore obtain a well-defined measurable function K defined by the pointwise $K(a,e) := \sup_m K_m(a,e)$ ($\in [0,1]$ and 0 when $a' = -\underline{a}$).
That was the "q" terms. For the constant term:

$$KH - 1 + (1+\bar r)\beta = (1+\bar r)\beta\mathbb{E}\Big[(1+\bar r)(K'-1)HKv'/v + H'K'\Big|e\Big], \quad \text{i.e.} \quad \text{(A29)}$$

$$H \cdot \underbrace{\left\{1 + (1+\bar{r})^2 \beta \mathbb{E}\left[(1-K')v'/v \middle| e\right]\right\} \cdot K}_{=1 \text{ on } \{(a,e);a'>-\underline{a}\} \text{ by (A24)}} = 1 - (1+\bar{r})\beta + (1+\bar{r})\beta \mathbb{E}\left[H'K' \middle| e\right] \tag{A30}$$

As we can safely put $H = 0$ on $\{(a,e); a' = -\underline{a}\}$, we can iterate from $H_0(a,e) \equiv 0$ the relation

$$H_{m+1}(a,e) = \left\{1 - (1+\bar{r})\beta + (1+\bar{r})\beta \mathbb{E}\left[H_m(a'(a,e),e')K'(a'(a,e),e') \middle| e\right]\right\} \cdot 1_{\{(a,e);a'>-\underline{a}\}} \tag{A31}$$

Now the condition $(1+\bar{r})\beta \sup_e \mathbb{E}K'(a'(a,e),e') < 1$ (recall that $K' \in [0,1]$ and $(1+\bar{r})\beta < 1$ – and except in the trivial case, zero when the credit constraint is binding) is sufficient for a contraction and unique solution H; if we start at 0, then we have bounded monotonicity $1 \geq H_{m+1} \geq H_m \geq 0$, and thus H defined by $H(a,e) := \sup_m H_m(a,e) \in [0,1]$ does the job. We have therefore established the existence a solution $q'(q,a,e) = \alpha_0(a,e)\,q + \alpha_1(a,e) = \Big[(1+\bar{r})q + H(a,e)/v(a,e)\Big] \cdot K(a,e)$.

ii) "Excess Demand" Functions

For a given $\bar{w}$, equations (A6) and (A11) describe households consumption and savings behavior as a function of $\bar{r}$, resulting in an aggregate asset supply function $S(\bar{r},\bar{w})$.[14]

Asset demand D, the sum of capital and bonds, follows from the government budget constraint (A12) using (A13) and (A15)

$$D(\bar{r},\bar{w}) := A = K + B = \frac{F(K,N) - \bar{w}N - G}{\bar{r}},$$

which, since we already established that capital K satisfies the modified golden rule (equation (A8)), is actually just describing how many government bonds are demanded. We therefore define

$$z^{AM}(\bar{r},\bar{w}) = D(\bar{r},\bar{w}) - S(\bar{r},\bar{w}).$$

A solution $\bar{r}$ (for given $\bar{w}$) to $z^{AM}(\bar{r},\bar{w}) = 0$ fully characterizes a stationary Aiyagari economy (and solves equation (A14)).

To derive the second equation $z^{LM}(\bar{r},\bar{w})$ we use the remaining equations (A9) and (A10).

[14] While it is conceivable that aggregate asset supply is not unique given $\bar{r}$ and $\bar{w}$, this is not a concern here since we impose the standard assumption that the planner picks the unique welfare maximizing allocation.

After division by γ equation (A9) reads

$$A = \mathbb{E}\left[u_c(c)\,q + a\frac{u_c(c)}{\gamma} + au_{cc}(c)\left(q\left(1+\bar{r}\right) - q'\right)\right],$$

where q' depends on $\bar{r}$, $\bar{w}$ and other parameters. Solving this equation for γ yields a function $\tilde{\gamma}(\cdot)$:

$$\tilde{\gamma}(\cdot) = \frac{\mathbb{E}\left[\frac{a}{A}u_c(c)\right]}{1 - \mathbb{E}\left[\frac{u_c(c)q}{A} + au_{cc}(c)\left(q\left(1+\bar{r}\right) - q'\right)\right]}$$

Plugging this function into (A10) (and noting $n = 1$) yields

$$\int end\mu = \mathbb{E}\left[e[\frac{u_c(c)}{\tilde{\gamma}} + u_{cc}(c)\left(q\left(1+\bar{r}\right) - q'\right)]\right].$$

We therefore define

$$z^{LM}(\bar{r}, \bar{w}) := \mathbb{E}\left[e[\frac{u_c(c)}{\tilde{\gamma}} + u_{cc}(c)\left(q\left(1+\bar{r}\right) - q'\right)]\right].$$

The optimal steady state then satisfies

$$z^{LM}(\bar{r}, \bar{w}) = \int end\mu.$$

Step 2: Local Invertibility

We first show that the interest rate $\bar{r}$ can generically (in the sense of Debreu (1970)) be expressed locally as a function of $\bar{w}$ (and other parameters). After that we show that $\bar{w}$ is also generically locally invertible.

i) Interest Rate

Acemoglu and Jensen (2015) show that a tightening of the borrowing limit leads to an increase in the supply of assets for given $\bar{r}$ and $\bar{w}$ but will not change the modified golden rule level of capital.[15] The transversality theorem (see e.g. Dierker and Dierker (1972), Shannon

[15] Acemoglu and Jensen (2015) call such an experiment a positive shock. Their objective is more demanding then just showing an increase in the supply function. They characterize the response of the equilibrium output per capita which has to take into account the endogeneity of prices.

(2006)) implies then that

$$\frac{\partial z^{AM}(\bar{r},\bar{w})}{\partial \bar{r}} \neq 0,$$

which implies that $\bar{r}$ is locally invertible and is thus a function of $\bar{w}$ and can be written as $\bar{r}(\bar{w})$.

ii) Wage The after tax wage $\bar{w}$ is determined as the solution to

$$z^{LM}(\bar{r}(\bar{w}),\bar{w},\mu) = \sum_{e\in E} e\mu(e),$$

where we have plugged in $\bar{r}(\bar{w})$ and use, consistent with the numerical implementation, a more convenient discrete space $E = \{e_1 < e_2, \ldots < e_N\}$.

We now follow Debreu (1970) and apply Sard's theorem to the function $F : \mathbb{R}^{N+1} \to \mathbb{R}^N$,

$$F(\bar{w},\{\mu(e_i)\}_{i=1}^N) = (\mu(e_1),\mu(e_2),\ldots,\mu(e_{N-1}),\frac{z^{LM}(\bar{r}(\bar{w}),\bar{w},\mu) - \sum_{i=1}^{N-1} e_i\mu(e_i)}{e_N}).$$

The optimal solution is characterized as $F(\bar{w},\{\mu(e_i)\}_{i=1}^N) = (\mu(e_1),\ldots,\mu(e_N))$.

Following Debreu (1970) we use now Sard' theorem which implies that the set of critical values has measure zero.[16]

Both the distribution μ and the after-tax wage $\bar{w}$ live on a compact space K, $(\bar{w},\{\mu(e_i)\}_{i=1}^N) \in K$. This is obvious for for $\mu(e_i) \in [0,1]$ and for $\bar{w}$ follows from Aiyagari (1994b) who ensures that no-one is willing to work in the market at a wage of 0 and the marginal productivity of labor is bounded since capital and hours (time) are.

The inverse image $F^{-1}(\mu(e_1),\ldots,\mu(e_N))$ of a regular value $((\mu(e_1),\ldots,\mu(e_N))$ is compact since F is continuous and K is compact. Consider now $e := (\bar{w},\{\mu(e_i)\}_{i=1}^N) \in F^{-1}(\mu(e_1),\ldots,\mu(e_N))$, for a regular $((\mu(e_1),\ldots,\mu(e_N))$ implying that the Jacobian does not vanish. The inverse function theorem implies that for each such e there is an open neighborhood U_e of e such that $F^{-1}(\mu(e_1),\ldots,\mu(e_N)) \cap U_e = \{e\}$. Since $F^{-1}(\mu(e_1),\ldots,\mu(e_N))$ is compact it can be covered by finite number of open sets U_e and therefore is finite.

This implies that the set of $\mu(e_1),\ldots,\mu(e_N)$ for which an infinite number of steady states exists consists of critical values only and has therefore measure zero (and so does its closure).

[16]For a continuously differentiable function $F : U \subset \mathbb{R}^m \to \mathbb{R}^n$, a point $e \in U$ is a critical point if the Jacobian matrix of F at e has rank smaller than n. A point $\mu \in \mathbb{R}^n$ is a critical value if there is a critical point $e \in U$ such that $F(e) = \mu$. A point $\mu \in \mathbb{R}^n$ is a regular value if it is not a critical value.

Vice versa, the number of steady state solution is generically, that is on a measure one, finite.

<u>Step 3: Continuity w.r.t. Initial Conditions</u>

Berge's maximum theorem[17] implies that the optimal policy path, and thus in particular the steady state, depends continuously on the initial level of government debt. That is the function $\zeta : \mathbb{R} \rightarrow \mathbb{R}^n$ mapping the initial government debt level into the steady state policies (tax rates and debt level) is continuous.

<u>Step 4: Independence of Initial Conditions</u>

We have shown in Step 2 that the set of solutions to the first order conditions is finite. These first-order conditions do not depend on the initial level of government debt. That is the finite set of solutions does not depend on the initial level of debt. The first order conditions are necessary conditions for an optimum, implying that every optimal policy has to be one of the finite solutions to the first order conditions. What remains to be shown is that each initial debt levels always yields the same solution to the first order conditions, that is that there is no selection of these solutions based on initial conditions. Using our results above, this is straightforward.

A continuous function mapping into a discrete set is constant, implying that ζ maps every initial debt level to the same steady state policy.

<u>Remarks:</u>

<u>Elastic Labor supply</u>

The same arguments hold when labor supply is elastic. We then define

$$z^{LM}(\bar{r}, \bar{w}) := \mathbb{E}\left[en[\frac{u_c(c)}{\tilde{\gamma}} + u_{cc}(c)\left(q(1+\bar{r}) - q'\right)]\right] + \left(F_N(K,N) - \bar{w}\right)\frac{\partial N}{\partial \bar{w}}$$

[17]See Theorem 17.31 in Aliprantis and Border (2006) for the infinite-dimensional version. Here we use the same topology, the product topology, as in Appendix A of Aiyagari (1994a). The equations describing the constraints of the Ramsey planner problem are continuous and the constrained set is by Tychonoff's theorem compact. The maximand of the Ramsey problem is continuous as well. These properties imply that a solution to the optimal tax problem exists, as shown in Aiyagari (1994a), and allow us to apply Berge's maximum theorem. Finally, note that a function is continuous in the product topology iff it all its projections are continuous, implying that the usual real analysis ϵ/δ characterization of continuity holds for all t and in particular for arbitrarily large t.

and the optimal steady state then satisfies

$$z^{LM}(\bar{r},\bar{w}) = \sum_{e} \pi_e e n\left(e,\bar{w}\right).$$

Additional Policy Instrument: Transfers

The same arguments hold when the government can use lump-sum transfers as an additional instrument. We then define a function

$$z^{T}(\bar{r},\bar{w},T) := \mathbb{E}\left[\frac{u_c\left(c\right)}{\tilde{\gamma}} + u_{cc}\left(c\right)\left(q\left(1+\bar{r}\right)-q'\right)\right)\right] + \left(F_N\left(K,N\right)-\bar{w}\right)\frac{\partial N}{\partial T} - 1,$$

so that the first-order condition reads as

$$z^{T}(\bar{r},\bar{w},T) = 0.$$

The arguments as above made for F now apply to the function $\tilde{F}$:

$$\begin{aligned}&\tilde{F}(\bar{w},T,\{\mu(e_i)\}_{i=1}^{N})\\ =&(\mu(e_1),\mu(e_2),\ldots,\mu(e_{N-1}) + z^{T}(\bar{r}(\bar{w},T),\bar{w},T), \frac{z^{LM}(\bar{r}(\bar{w},T),\bar{w},T,\mu) - \sum_{i=1}^{N-1} e_i\mu(e_i)}{e_N}).\end{aligned}$$

The optimal solution is characterized as $\tilde{F}(\bar{w},T,\{\mu(e_i)\}_{i=1}^{N}) = (\mu(e_1),\ldots,\mu(e_N))$.

III Computational Algorithms

III.1 Steady State

To numerically compute the steady state, we first need to introduce the steady state distribution of state variables (a, λ, e), represented by a density function $p(a, \lambda, e)$. Moreover, we denote the density function of (a, e) as $m(a, e)$. Now the steady state equations involving expectation and integration can be explicitly expressed using p and m. Equation 24, 22, 23, A4 and A5 are now:

$$\begin{aligned}
\gamma &= \sum_e \int \int \left(u_c(c) + \frac{\partial c}{\partial T} u_{cc}(c) \left[\lambda (1 + \bar{r}) - \lambda' \right] \right) p(a, \lambda, e)\, da d\lambda \\
&\quad + \gamma \left(F_N(K, N) - \bar{w} \right) \frac{\partial N}{\partial T}, \qquad \text{(A32)} \\
\gamma A &= \sum_e \int \int u_c(c) \lambda p(a, \lambda, e)\, da d\lambda \\
&\quad + \gamma \left(F_N(K, N) - \bar{w} \right) \frac{\partial N}{\partial \bar{r}} \\
&\quad + \sum_e \int \int a \left(u_c(c) + u_{cc}(c) \left[\lambda (1 + \bar{r}) - \lambda' \right] \right) p(a, \lambda, e)\, da d\lambda, \qquad \text{(A33)} \\
\gamma N &= \gamma \left(F_N(K, N) - \bar{w} \right) \frac{\partial N}{\partial \bar{w}} \\
&\quad + \sum_e \int \int en(e, \bar{w}) (u'(c) + u''(c) (\lambda (1 + \bar{r}) - \lambda')) p(a, \lambda, e)\, da d\lambda, \qquad \text{(A34)} \\
N &= \sum_e \pi_e en(e, \bar{w}), \qquad \text{(A35)} \\
A &= \sum_e \int am(a, e)\, da. \qquad \text{(A36)}
\end{aligned}$$

Moreover, the density functions satisfy

$$p(a', \lambda', e') = \sum_e \pi_{ee'} \int I \left[g_{a'}(a, e) = a', g_{\lambda'}(a, \lambda, e) = \lambda' \right] p(a, \lambda, e)\, da d\lambda, \qquad \text{(A37)}$$

$$m(a', e') = \sum_e \pi_{ee'} \int I \left[g_{a'}(a, e) = a' \right] m(a, e)\, da. \qquad \text{(A38)}$$

Using the steady state equations, i.e., equation 19 to 21, A1, A2, and A32 to A38, we compute the steady state variables according to the following steps, with some exceptions similar to Acikgoz (2014)[18].

1. Guess T.
2. Guess $\bar{w}$. Solve for $\bar{r}(\bar{w})$ following Aiyagari (1995):
   (a) Solve for K from (21).
   (b) Guess $\bar{r}$ and solve the household's problem: solve for $c(a,e), a'(a,e)$ from (19) and (A1), keeping in mind that $n = \left(\chi^{-1}\bar{w}ec^{-\sigma}\right)^{\frac{1}{\phi}}, y = \left(\chi^{-1}\bar{w}^{1+\phi}e^{1+\phi}c^{-\sigma}\right)^{\frac{1}{\phi}}$.
   (c) Compute N from (A35).
   (d) Solve for $m(a,e)$ or equivalently $m(.,e)$ from (A38).
   (e) Solve for A from (A36).
   (f) Solve for B from (A3).
   (g) Verify $\bar{r}$ using (A2). If the equation is not satisfied, update $\bar{r}$.
3. Define $q \equiv \frac{\lambda}{\gamma}$, and solve for $q'(a,q,e)$ by iterating on $q'(a,q,e)$ using (20) until q' converges. Guess $q'(a,q,e) = g^0_{q'}(a,q,e)$, and then use equation (20) to find the new $q'(a,q,e) = g^1_{q'}(a,q,e)$ as follows:

$$q' = \frac{-\frac{\partial c}{\partial a'}u_{cc}(c)\,q\,(1+\bar{r}) + \beta\mathbb{E}\left[\frac{\partial c'}{\partial a'}u_{cc}(c')\,q''|e\right] - 1 + \beta(1+\bar{r})}{-\frac{\partial c}{\partial a'}u_{cc}(c) + \beta\mathbb{E}\left[\frac{\partial c'}{\partial a'}u_{cc}(c')(1+\bar{r})\right]}$$

   where $\mathbb{E}\left[\frac{\partial c'}{\partial a'}u_{cc}(c')\,q''|e\right]$ can be computed using $g^0_{q'}(a,q,e)$, and the new q' gives us the new policy function, denoted as $g^1_{q'}(a,q,e)$. Keep updating until $g^i_{q'}(a,q,e)$ converges to $g_{q'}(a,q,e)$. It can be proven that the above functional equation is a contraction mapping.
4. Solve for γ from (22)
5. Check whether (23) is satisfied. If so, stop. Otherwise update $\bar{w}$.
6. Check whether (A32) is satisfied. If so, stop. Otherwise update T.

[18]The utility function in Acikgoz (2014) is of a form that yields no income effect on labor supply. We use an alternative method for finding $q'(a,q,e)$ which will work also when the economy is not in steady state. See point 3.

III.2 Transition

Below we outline the algorithm for computing the transition from the model calibrated to the U.S. economy to the optimal long run steady state.:

1. Choose a number of transition periods, J.

2. Compute the optimal long run steady state as outlined in III.1 and obtain $a_{t+1}(a_t, e_t)$, $c_t(a_t, e_t)$ at time J.

3. Compute the steady state for the economy calibrated to the U.S. and obtain $m_0(a_0, e_0)$ A_0, B_0, K_0.

4. Guess $\{\bar{w}_t, \bar{r}_t, T_t\}_{t=0}^{J}$.

5. Solve the households' problem by backward induction and obtain $a_{t+1}(a_t, e_t)$, $c_t(a_t, e_t)$[19].

6. Compute distribution of asset and productivity $m_t(a_t, e_t)$, using simulation starting from $m_0(a_0, e_0)$.

7. Compute A_t and N_t from (9) and (10).

8. Compute K_t and B_{t+1} going backwards using (11) and (2), namely,

$$
\begin{aligned}
K_t &= A_t - B_t \\
B_{t+1} &= F(K_t, N_t) - G_t + (1 + \bar{r}_t) B_t + \bar{r}_t K_t + \bar{w}_t N_t.
\end{aligned}
$$

9. Compute γ_t backward using

$$
\gamma_t = \beta (1 + F_K (K_{t+1}, N_{t+1})) \gamma_{t+1}
$$

10. Solve for $\lambda_{t+1}(a_t, \lambda_t, e_t)$ from 14.

11. Compute p_t forward by simulations using p_0 and the policy functions: $a_{t+1}(a_t, e_t)$, $c_t(a_t, e_t)$ and $\lambda_{t+1}(a_t, \lambda_t, e_t)$.

[19] We do this, using an endogenous rid approach.

12. Check the errors implied by the guessed $\{\bar{w}_t, \bar{r}_t, T_t\}_{t=0}^{J}$. This means check the equations 18 16 and 17. If they are not satisfied, update the guess for $\{\bar{w}_t, \bar{r}_t, T_t\}_{t=0}^{J}$. In practice we do this by a simplex minimization routine, which minimizes the sum of the squared errors in the equations.