Risk Aversion, Unemployment, and Aggregate Risk Sharing with Financial Frictions*

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PRELIMINARY

Abstract

If agents in workhorse business cycle models with financial frictions are allowed to index contracts to observable aggregates, they share aggregate financial risk (almost) perfectly. Thus, borrowing-constrained capital holders' wealth share does not collapse following adverse shocks and the financial accelerator mechanism is eliminated. I revisit this issue in the Bernanke, Gertler, and Gilchrist (1999), henceforth BGG, framework and show that this happens in the standard specification with TFP shocks partly because: *i)* borrowers and lenders are implicitly assumed to have identical, logarithmic utility, and *ii)* the representative lender's human wealth comoves closely with aggregate financial wealth. I then demonstrate that non-state-contingent borrowing rates, as initially imposed by BGG, can arise optimally in light of TFP shocks if *i)* lenders' aversion to consumption fluctuations is increased to plausible degrees, or *ii)* at identical preferences for consumption, lenders face uninsurable idiosyncratic liquidity risk brought about by loss of employment.

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1 Introduction

Recent developments in the literature on macro-financial DSGE models have brought to light the fact that in several conventional frameworks, the relevance of financial frictions in aggregate fluctuations is eliminated if economic agents are allowed to share aggregate risk embedded in

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returns to risky assets optimally.[1] One such workhorse model is that set up by Bernanke et al. (1999), building on earlier work by Bernanke and Gertler (1989) and Carlstrom and Fuerst (1997). It features capital-managing entrepreneurs and households who provide funding to the former. A costly state verification (CSV) friction emanates from idiosyncratic shocks to entrepreneurs' held capital, as formalized by Townsend (1979). The presence of such a financial friction gives rise to a financial accelerator mechanism which generates amplification and added persistence of aggregate shocks, compared to a frictionless real business cycle framework. A crucial nuance in this mechanism is the exogenously imposed constraint that the lenders must, on aggregate, receive a predetermined return. That is, the lenders' return does not respond to realizations of aggregate shocks, observed at the time of repayment, and the borrowing entrepreneurs bear all aggregate risk.

This assumption regarding the predeterminacy of returns has drawn criticism, for example by Chari (2003). There is no explicitly modelled reason why in the presence of a risk-averse lender and a borrower with time-varying investment opportunities the counterparties cannot engage in mutual insurance against aggregate risk. This could be achieved by agreeing on a lender's return which is indexed to observable outcomes to be realized in the macroeconomy. Carlstrom et al. (2016) (henceforth, CFP) formalize this idea in the BGG framework. They show that, in the privately optimal one period contract, the *ex post* return to the lenders is indexed one-for-one to the return on entrepreneurial capital, adjusted for fluctuations in the borrower's and lender's marginal valuations of wealth.

The fact that borrowing entrepreneurs' *ex post* liabilities adjust to capital return shocks significantly dampens financial accelerator dynamics. In the non-state-contingent lender return case employed by BGG, a positive aggregate shock to the return on capital, for example due to increased productivity, leads to a significant increase in entrepreneurs' net worth because of their predetermined liabilities. The relatively higher net worth decreases entrepreneurial leverage, increases their ability to hold assets, boosting asset prices which feed into further net worth and investment increases – the financial accelerator mechanism.

However, if lender returns are indexed to capital returns, and possibly other observables, there is sharing of aggregate financial risk between the borrower and lender. A positive shock leads to an increase in *ex post* entrepreneurial liabilities, a smaller increase in net worth, and a dampened drop, if any, in leverage. This is exactly the mechanism at work in the treatment of CFP. Similar ideas have been presented by Krishnamurthy (2003) in a stylized three period model with borrowing contraints in the spirit of Kiyotaki and Moore (1997), and by Di Tella (forthcoming) in the infinite horizon framework developed by Brunnermeier and Sannikov (2014). A key implication for the dynamics of borrowers'/entrepreneurs' balance sheets in all these frameworks, compared to a contract with predetermined lender returns, is that fluctuations in entrepreneurial leverage should be minimal. Or equivalently, the magnitude of relative fluctuations in entrepreneurial net worth should be close to equal to those in held assets, and not amplified by leverage.

A descriptive summary of aggregate time-series data on US firms' balance sheets demonstrates

[1] Throughout, I will use the term "aggregate risk" to refer to the stochastic nature of aggregate returns to holding risky assets, specifically, productive capital. And "sharing aggregate risk" refers to how fluctuations in these returns translate into realized returns on agents' financial wealth.

a considerable degree of volatility.[2] For a very simplistic illustration, Figure 1 displays quarterly aggregate non-financial corporate balance sheet data from the Federal Reserve Board Flow of Funds Accounts (FOFA Table B.103).[3] It graphs the HP-filtered cyclical components of corporate net worth and leverage, alongside that of gross value added (GVA) in the nonfarm business sector – all in logs, for the period 1976Q4–2015Q3.[4] As is evident, aggregate non-financial corporate sector leverage exhibits non-negligible countercyclicality over the business cycle. And it is quite striking that since the 1980s, the unconditional second moments of the cyclical components of the balance sheet variables have seemed to drift farther from the implications of privately optimal aggregate risk sharing covered above. Increased volatility in US firms' financial flows and balance sheet variables has been pointed out in earlier work by Jermann and Quadrini (2009) and Fuentes-Albero (2016), respectively. This finding becomes the more intriguing if one were to expect that the rapid development of financial markets and instruments during this period should have made aggregate state dependent borrowing contracts and privately optimal risk sharing more easily implementable. For example, regarding the management of interest rate risks, the market for interest rate swaps emerged in the early 1980s and grew rapidly during the decade (Saunders, 1999). And all this happened during a time of lower volatility in the real economy, well-documented as the Great Moderation and evident in the fluctuations of GVA.

Explaining the observed changes in balance sheet dynamics is beyond the scope of this paper. And this simplistic picture of balance sheets could be obscuring non-finanical firms' activities in holding large amounts of financial assets (Armenter and Hnatskova, 2016). It is nontheless clear that significant fluctuations in the net worth and leverage of non-financial firms are a prevalent phenomenon in the US economy. Also, these fluctuations are synchronized with the business cycle, exactly like the basic financial accelerator mechanism under non-state-contingent debt would predict.[5] And as demonstrated for example in the work by Giroud and Mueller (2017), the health of non-financial firms' balance sheets had significant relevance for real activity during the Great Recession.

In this paper, I pursue the idea that non-aggregate-state-contingent lender returns, and the implied countercyclical fluctuations in entrepreneurial leverage, might be the outcome of privately optimal aggregate risk sharing between households and entrepreneurs in the BGG framework. A slight reformulation of the borrowing entrepreneurs' problem and preferences allows to establish that, to a first order, the conventional BGG-CFP assumption of individual entrepreneurs with linear utility consuming a constant fraction of wealth is equivalent to assuming that the mass

[2]To be more precise in the labeling between model and data, one could think of entrepreneurial wealth in the model as inside equity, and the entrepreneurs' external financing as the sum of outside equity and debt financing. The degree of external finance return indexation to the return on assets then mirrors the relative magnitudes of outside equity and debt. In the model's solution, to a first order, this degree of indexation is constant over time, implying a constant ratio of outside equity to external finance. As long as this ratio implied by the model is less than 1 (i.e. entrepreneurs wealth is fully held as inside equity), one can easily establish a positive relation between $\textit{leverage in model} \equiv \frac{assets}{inside\ equity}$ and $\textit{leverage in data} \equiv \frac{assets}{equity}$.

[3]All of these statements follow also for the non-financial noncorporate business sector.

[4]For this motivating data description, I just focus on the general balance sheet entries of Total Assets (FOFA Table B.103, line 1) and Total Liabilities (FOFA Table B.103, line 25) – both measured at market values. $\textit{Net worth} = \textit{Total Assets} - \textit{Total liabilities}$, $\textit{Leverage} = \textit{Total Assets}/\textit{Net worth}$. GVA measure from NIPA-BEA Table 1.3.5. All variables seasonally adjusted and deflated by the implicit price index for the nonfarm business sector (NIPA-BEA Table 1.3.4).

[5]The correlation between the above graphed GVA and leverage cyclical components is -0.56 in the sample post-1984, a commonly estimated structural break date for the Great Moderation.

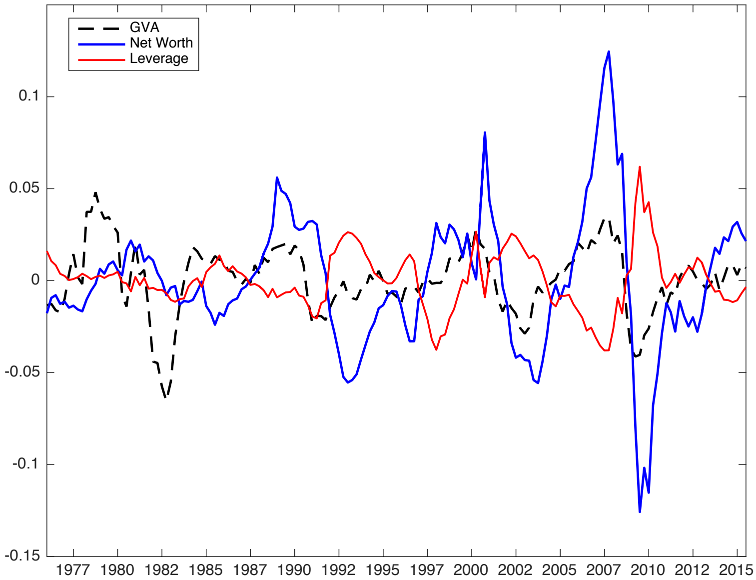

Figure 1: Log-deviations from HP-trend of gross value added in the US non-farm business sector, non-financial corporate net worth and leverage. HP-filter smoothing parameter 1,600. Data: FOFA, NIPA-BEA.

of entrepreneurs have logarithmic utility from consumption and engage in mutual sharing of idiosyncratic project risk as a family, while running individual projects subject to limited liability – effectively constructing a representative entrepreneur.[6]

This establishes that by assuming logarithmic utility from consumption for the representative lender, as BGG and CFP do, one is effectively studying a risk-sharing problem between two agents with identical preferences for consumption. While being a valid theoretical benchmark, it also demonstrates that if one were to instead set up the conventional BGG specification with households that do not have logarithmic utility, the high degree of sharing aggregate financial risk found by CFP might not necessarily follow. Relatively more aggregate risk would trivially be taken on by agents with lower aversion to fluctuations in consumption. To shed light on this issue, I consider households with conventional Epstein and Zin (1989) preferences and compute under which values of risk aversion and intertemporal elasticity of substitution does the implied optimal one-period financial contract yield non-contingent borrowing rates in response to persistent yet stationary total factor productivity shocks. Under the calibration employed by CFP, this happens with a household risk aversion parameter of 13.2 and intertemporal elasticity parameter of 1.0, or CRRA utility with risk aversion of 5.92, for example.[7] Under a close to unit root TFP process, these numbers can be significantly smaller.

In addition to the agents' preferences affecting their optimal sharing of aggregate financial risk, also their exposure to aggregate risk through other sources of wealth matter. In the specification

[6] More precisely, the equivalence holds in the limit as the linear-utility entrepreneurs become infinitely patient, i.e. their discount factor approaches 1, while their savings (survival) rate is strictly below the discount factor of the households.

[7] To be precise, one must be careful with definitions of risk aversion when agents can vary labor supply in response to shocks to wealth, as is the case in the BGG model. See Swanson (2015) and Section 3.1 for more.

used by CFP, entrepreneurs' total wealth equals their financial wealth while the representative household is also endowed with human wealth. With logarithmic utility requiring optimal consumption to be a constant fraction of one's total wealth, optimal risk sharing between households and entrepreneurs effectively requires sharing financial returns in a way that works to neutralize fluctuations in human wealth. That is, for any positive shock to households' human wealth, they should cede more of their financial returns to the entrepreneurs. With relative fluctuations in aggregate financial wealth and human wealth comoving closely in response to TFP shocks, the agents end up sharing realized financial returns close to equally.

Given that idiosyncratic shocks to human wealth are naturally less diversifiable than the idiosyncratic risk embedded in owning individual assets or financing entrepreneurial projects, I also consider households' countercyclical uninsurable idiosyncratic risk as an effective source of their added risk aversion. I do so by introducing uninsurable household liquidity risk in the model, with the risk emanating from temporary spells of unemployment. In the asset pricing literature, the introduction of countercyclical idiosyncratic risk is a well-known way of increasing agents' effective risk aversion towards aggregate fluctuations.[8] This risk will separate unemployed households' consumption from aggregate human wealth and generate a force towards less financial risk sharing in the model, even when borrowers and lenders have identical expected utility preferences over consumption.

To study the relevance of idiosyncratic lender risk, I employ liquidity (cash-in-advance) constraints as a convenient tool to arrive at limited household heterogeneity in equilibrium. Such an approach yields analytical tractability which allows for simple solution methods and a well-defined privately optimal financial contract between the households and entrepreneurs. There has been a considerable amount of recent work on similar models of transitory idiosyncratic risk with tractable (finite) wealth distributions, for example by Challe and Ragot (2015), Le Grand and Ragot (2016) and Challe et al. (2017). However, the way in which tractability in the wealth distribution of heterogeneous agents is achieved in these treatments relies on their relatively low levels of wealth. This way, individuals hit by adverse transitory income shocks deplete their wealth and become identical to those who had been unfortunate in previous periods. Yet in a setting in which the heterogeneous agents under consideration own a nontrivial part of the economy's productive capital – it is common to calibrate the household wealth share to 1/2 in the BGG setting – it is unnatural to expect that any optimally behaving agent facing zero income would deplete their individual wealth in a short period of time.

For analyzing idiosyncratic risk exposure of wealthy households in a tractable manner, it is therefore natural to consider issues of wealth illiquidity and temporal division of consumption and savings decisions. A suitable structure to introduce such phenomena in an economic environment is that presented by Lagos and Wright (2005). While they use bilateral trade and search and matching to study the essentiality of money, I employ a setup in which agents acquire consumption goods in a centralized market, yet are subject to liquidity constraints. The elimination of heterogeneity in households' portfolios is achieved by linear disutility from the production of a good that constitutes a small part of output in the economy. Such a setup provides a convenient

[8] The work of Mankiw (1986), Constantinides and Duffie (1996), Krusell and Smith (1997), Storesletten et al. (2007), Schmidt (2016), Constantinides and Ghosh (2017) are a few prominent examples.

way to study idiosyncratic risk in the business cycle with minimal departures from the neoclassical growth model, and bypasses standard criticisms of frameworks that exhibit linear utility, such as infinite Frisch elasticity of labor supply or trivial consumption dynamics.

The rest of the paper is organized as follows. Section 2 describes the environment with a representative household and a family of entrepreneurs, defines the competitive equilibrium and discusses some properties. In Section 3, I calibrate the model and analyze optimal risk sharing for various household preferences. Section 4 extends the framework to allow for household heterogeneity and liquidity risk and defines the corresponding competitive equilibrium. In Section 5, I calibrate the idiosyncratic risk features of this framework and analyze the implications for aggregate risk sharing and aggregate dynamics. Section 6 concludes and discusses further extensions of the model.

2 The Benchmark Representative Household Model

2.1 The Environment

For comparability with earlier work in the literature, the framework of the model environment closely follows the treatment of BGG and CFP. Time in the model is discrete and infinite. The model features two central types of agents, called households and entrepreneurs – a unit mass of each. In addition, there are new capital producers, a representative financial intermediary and a representative numeraire producer, all discussed below.

The representative household is infinitely-lived, has time discount factor β and labor-augmented Epstein-Zin preferences. It consumes the final good and sells labor in competitive markets. The household saves in period t by depositing savings in a financial intermediary. These deposits yield gross real returns R^d_{t+1} in $t+1$. The returns are not predetermined in t and are realized at $t+1$, possibly depending on aggregate shocks.

As in BGG and CFP, the representative financial intermediary accepts deposits from households and extends loans, between t and $t+1$, to the continuum of entrepreneurs. The intermediary is effectively a pass-through entity that diversifies all idiosyncratic risk arising from lending to entrepreneurs hit with individual shocks. Yet aggregate risk on each extended loan, and on the whole loan portfolio remains. As CFP, I assume that there is free-entry into the financial intermediation market and gross returns to the depositors cannot be negative. This implies that in equilibrium, the gross real returns on households' deposits, R^d_{t+1} will equal the returns on the intermediary's loan portfolio.[9]

Entrepreneurs, indexed by $j \in [0, 1]$, have expected utility preferences with strictly concave momentary utility over consumption and a discount factor $\beta_e \in (0, 1)$ between periods. The entrepreneurs belong to a representative family that pools their dividends and optimally provides equal consumption to each of them in a period.[10] Entrepreneurs are also assumed to be the only

[9] Appendix D provides a more rigorous foundation for these statements and the financial intermediary's optimal behavior when households own the intermediary, for the general case of the model as introduced in Section 4.

[10] Equivalently, one can think instead of there being a representative entrepreneur who owns a continuum of firms,

agents who can hold capital between periods t and $t+1$. At the end of each period, they purchase physical capital, financed by their accumulated entrepreneurial wealth, referred to as *net worth*, and external financing provided by the financial intermediary. At the beginning of period $t+1$ each entrepreneur's capital holdings K_{t+1}^{j} are scaled by an idiosyncratic shock ω_{t+1}^{j} which is observed by the entrepreneur, but by the lender only if a monitoring cost is incurred. This idiosyncratic shock is i.i.d across time and entrepreneurs and independent of any aggregate realizations, with density $f(\omega)$, cumulative distribution $F(\omega)$ and a mean of one. Let R_{t+1}^{k} denote the aggregate return to a unit of capital, meaning the average return in the cross-section of entrepreneurs. R_{t+1}^{k} is perfectly observed by all agents. Then, the total return to a unit of numeraire invested in entrepreneur j's capital project at time t is $\omega_{t+1}^{j}R_{t+1}^{k}$. To be more precise:

$$R_{t+1}^{k} \equiv \frac{r_{t+1} + (1-\delta)Q_{t+1}}{Q_t} \tag{1}$$

where r_{t+1} is the rental rate on capital, δ the depreciation rate and Q_{t+1} the relative price of capital in $t+1$.

As is conventional in this line of models starting with Carlstrom and Fuerst (1997), I assume that there is enough inter-period anonymity in financial markets that only one-period contracts between the entrepreneurs and the intermediary are feasible. More specifically, I assume that once agents enter $t+1$ and entrepreneurial capital shocks are realized, the entrepreneurs pay back the intermediaries in the form of *capital*, as dictated by the previously signed contract, and the latter then receive the proceeds from the ownership of their share of the capital. Returns to capital are derived from capital gains in the price of capital when selling it and renting the capital out to a representative numeraire good producer, as evident in the definition of R_{t+1}^{k} above. This assumption is innocuous in the current representative household framework and one could equivalently allow the entrepreneurs to operate the capital, sell it and divide the proceeds with the lender. It becomes relevant in the heterogeneous household framework, as explained in Section 4.1.1.

As in BGG, I assume that monitoring costs are a proportion μ of the realized gross payoff to a given entrepreneur's capital: $\mu R_{t+1}^{k} Q_t K_{t+1}^{j}$. Also, entrepreneurs have limited liability in that each individual entrepreneur's project cannot make payments in the form of capital in excess of the proceeds $\omega_{t+1}^{j} K_{t+1}^{j}$. That is, even though the entrepreneur is part of a larger family, equity injections or dividend payments to the family can only be made after the payments with the lender have been settled. This assumption renders each individual contracting problem identical to that in BGG and CFP. The entrepreneurs are assumed to liquidate all their capital and all capital must be repurchased. This assumption dates back to BGG who make it to ensure that agency problems affect the entire capital stock and not just the marginal investment. Finally, as is a common assumption in the literature to prevent the entrepreneurs "growing out" of their financial constraints in the long run and become self-financing, I assume that $\beta_e < \beta$.

There is a representative final good producer who runs a Cobb-Douglas production function

each run by a manager $j \in [0,1]$ who maximizes shareholder value by paying dividends and making investment and financial contracting decisions. Although this would be a more conventional framing in light of the literature on heterogeneous firms, I choose to follow the narrative of a continuum of entrepreneurs jointly sharing idiosyncratic risk in order to stay in line with the account of BGG and CFP.

in aggregate labor L_t and capital K_t producing: $A_t K_t^{\alpha} L_t^{1-\alpha}$. It rents capital from entrepreneurs and financial intermediaries, for rental rate r_t, and labor from the household for wage rate W_t, both in competitive markets. A_t is a TFP shock that follows a stationary AR(1) process in logs. It is the only source of aggregate uncertainty in the model and its realization is publicly observed at the beginning of time t.

The household also own competitive new capital producers who produce new capital subject to adjustment costs and sell it to entrepreneurs. Following CFP, they take $I_t \vartheta\left(\frac{I_t}{I_{ss}}\right)$ units of the final good and transform these into I_t investment goods, i.e. gross capital investment. ϑ is convex and I_{ss} is the steady state level of gross investment. These investment goods are sold at price Q_t. I make the standard assumptions that $\vartheta(1) = 1$, $\vartheta'(1) = 0$ and $\vartheta''(1) = \phi_Q$. This normalizes the capital price in steady state to 1 and guarantees that at steady state, the elasticity of the capital price to I_t is ϕ_Q, a key calibration target. New capital producers earn possibly non-zero profits in equilibrium, paid to households, whereas steady state profits are zero.

2.2 Equilibrium

In this section, I present the agents' problems and derive their equilibrium optimality conditions.

2.2.1 Households

The representative household maximizes its utility function over streams of consumption C_t and labor hours worked L_t:

$$\mathcal{V}_t(D_t) = \left\{(1-\beta)u(C_t, L_t) + \beta \mathbb{E}_t \left[\mathcal{V}_{t+1}(D_{t+1})^{1-\xi}\right]^{\frac{1-\frac{1}{\psi}}{1-\xi}}\right\}^{\frac{1}{1-\frac{1}{\psi}}} \tag{2}$$

subject to the borrowing constraint

$$C_t + D_{t+1} \leq W_t L_t + R_t^d D_t + \Pi_t \tag{3}$$

where D_{t+1} denotes the household's choice of deposits saved in the intermediary, and Π_t^I are profits of new capital producers. Although I consider recursive equilibria, for brevity of notation, I assume that the aggregate state is encompassed by allowing for an aggregate state contingent value function $\mathcal{V}_t$.

Following Uhlig (2010), I assume $u(C, L) = [C\Phi(L)]^{1-\frac{1}{\psi}}$, where Φ is positive, thrice differentiable, decreasing and concave. These preferences are consistent with long run growth and give flexibility in calibrating the elasticity of labor supply. Given this, the household's first order necessary conditions for labor supply and deposits are then:

$$C_t \left[-\frac{\Phi'(L_t)}{\Phi(L_t)} \right] = W_t \tag{4}$$

$$1 = \mathbb{E}_t \left[M_{t+1} R^d_{t+1} \right] \tag{5}$$

$$\text{with } M_{t+1} \equiv \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \left(\frac{\mathcal{V}_{t+1}}{\mathbb{E}_t \left[\mathcal{V}_{t+1}^{1-\xi} \right]^{\frac{1}{1-\xi}}} \right)^{\frac{1}{\psi}-\xi} \left(\frac{\Phi(L_{t+1})}{\Phi(L_t)} \right)^{1-\frac{1}{\psi}} \tag{6}$$

2.2.2 Final Goods and New Capital Producers

The representative numeraire producer's optimization yields the demand for labor and capital:

$$W_t = (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha} \tag{7}$$

$$r_t = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} \tag{8}$$

New capital producers' profits are given by:

$$\Pi^I_t = Q_t I_t - I_t \vartheta \left(\frac{I_t}{I_{ss}} \right) \tag{9}$$

Their optimization with respect to I_t yields that the equilibrium capital price follows:

$$Q_t = \vartheta \left(\frac{I_t}{I_{ss}} \right) + \frac{I_t}{I_{ss}} \vartheta' \left(\frac{I_t}{I_{ss}} \right) \tag{10}$$

The law of motion for aggregate capital is:

$$K_{t+1} = I_t + (1-\delta) K_t \tag{11}$$

2.2.3 Entrepreneurs and the Loan Contract

Let us denote entrepreneur j's accumulated wealth after paying the lender yet before paying dividends in t by E^j_t, for entrepreneurial equity. And let the entrepreneurial net worth N^j_t be the entrepreneur's wealth after paying dividends. This net worth is accumulated by purchasing capital K^j_t in $t-1$, paying back the contracted upon share to the lender in t, earning rental returns and capital gains on the remainder, and paying dividends to the entrepreneurial family.

Because of the imperfect obsevability of entrepreneur j's idiosyncratic capital shock ω^j_{t+1}, the costly state verification problem arises. Entrepreneur j's investment of K^j_{t+1} units of capital yields $\omega^j_{t+1} K^j_{t+1}$ units in $t+1$ which generates an income flow of $\omega^j_{t+1} \left[r_{t+1} + (1-\delta) Q_{t+1} \right] K^j_{t+1} = \omega^j_{t+1} R^k_{t+1} Q_t K^j_{t+1}$. Following Townsend (1979) and Williamson (1986), one can show that if payoffs are linear in the project outcome $\omega^j_{t+1} K^j_{t+1}$, and there is no random monitoring, the optimal contract is risky debt.[11] Since idiosyncratic entrepreneurial risk is fully diversified in the financial

[11] The proof is exactly as for the conventional CSV problem without aggregate uncertainty, only applied for each realization of the aggregate state separately.

intermediary's portfolio, this is true on the lender's side. As for entrepreneur j, following a similar formalization as CFP, below it will be clear that if risky debt is the optimal contract, then the entrepreneur's value function is linear, closing the logical circle.

By risky debt we mean that monitoring only occurs for low realizations of ω_{t+1}^j. More specifically, in the absence of aggregate uncertainty, i.e. when r_{t+1} and Q_{t+1} are known at the time of signing the contract, the borrower and lender agree on a cutoff $\bar{\omega}_{t+1}^j$ and an implied promised repayment of capital to the lender: $\bar{\omega}_{t+1}^j K_{t+1}^j$. If $\omega_{t+1}^j < \bar{\omega}_{t+1}^j$, the borrower does not have sufficient funds to pay the lender. He declares bankruptcy, the lender incurs the monitoring cost and gets all of the remaining capital, which yields him an income flow of $(1-\mu)\omega_{t+1}^j R_{t+1}^k K_{t+1}^j$. If $\omega_{t+1}^j \geq \bar{\omega}_{t+1}^j$, no monitoring occurs, the borrower repays the promised amount $\bar{\omega}_{t+1}^j K_{t+1}^j$ and holds on to the remaining capital which yields him an income flow of $(\omega_{t+1}^j - \bar{\omega}^j) R_{t+1}^k Q_t K_{t+1}^j$. Note that $\bar{\omega}_{t+1}^j$ implicitly determines an interest rate $R_{t+1}^{def,j}$ earned by the lender that is subject to default risk, defined by: $R_{t+1}^{def,j}(Q_t K_{t+1}^j - N_t^j) = \bar{\omega}_{t+1}^j R_{t+1}^k Q_t K_{t+1}^j$.

In the presence of aggregate uncertainty, however, the optimal contract involves the lender and borrower agreeing upon a schedule of $\{\bar{\omega}_{t+1}^j\}$, with a specific value of the cutoff for each possible realization of the aggregate state. Conditional on having observed aggregate outcomes and thus knowing the implied $\bar{\omega}_{t+1}^j$, the optimality of risky debt, now for each realization of the aggregate state, remains. The CSV problem takes as exogenous the aggregate returns (r_{t+1}, Q_{t+1}) on capital and the opportunity cost of the lender.

Let $\Gamma(\bar{\omega})$ denote the expected gross share of entrepreneur j's held capital, or equivalently the returns on this capital, going to the lender. And let $\mu G(\bar{\omega})$ be the expected monitoring costs:

$$\begin{aligned} \Gamma(\bar{\omega}) &\equiv \int_0^{\bar{\omega}} \omega f(\omega) d\omega + \bar{\omega} \int_{\bar{\omega}}^{\infty} f(\omega) d\omega = \int_0^{\bar{\omega}} \omega f(\omega) d\omega + \bar{\omega}[1 - F(\bar{\omega})] \\ \mu G(\bar{\omega}) &\equiv \mu \int_0^{\bar{\omega}} \omega f(\omega) d\omega \end{aligned}$$

Noting that

$$\begin{aligned} \Gamma'(\bar{\omega}) &= 1 - F(\bar{\omega}) > 0 \\ \Gamma'(\bar{\omega}) - \mu G'(\bar{\omega}) &= [1 - F(\bar{\omega})][1 - \mu\bar{\omega}h(\bar{\omega})] > 0 \text{ if } \bar{\omega} < \bar{\omega}^* \end{aligned}$$

we have that the entrepreneur's expected net share $[1 - \Gamma(\bar{\omega})]$ is decreasing in $\bar{\omega}$ and that of the lender, $[\Gamma(\bar{\omega}) - \mu G(\bar{\omega})]$ increasing.[12]

Let us define and denote entrepreneur j's leverage attained in period t, going into period $t+1$ as: $\kappa_t^j \equiv \frac{Q_t K_{t+1}^j}{N_t^j}$. Then, integrating out the realization of ω_{t+1}^j, conditional on the aggregate

[12] In the above, $h(\omega) \equiv f(\omega)/[1 - F(\omega)]$ is the hazard rate and $\bar{\omega}^*$ is the cutoff value at which the lender's net share is maximized. Assuming that $\frac{\partial[\omega h(\omega)]}{\partial\omega} > 0$ and $\lim_{\omega \to +\infty} \omega h(\omega) > \frac{1}{\mu}$, as will be satisfied by the log-normal distribution employed in the computations, there exists a unique such $\bar{\omega}^*$. At the optimum, it cannot be the case that for any realization of aggregate shocks, $\bar{\omega}^j > \bar{\omega}^*$. Because then, $\bar{\omega}^j$ can be reduced, the borrower made better off and the participation constraint slackened. In the simulations, $\bar{\omega}_t$ will be far below $\bar{\omega}^*$. For example, in the baseline calibration, $\bar{\omega}_t$ will be significantly lower than $\bar{\omega}^*$.

realizations of (r_{t+1}, Q_{t+1}), we can write the $t+1$ equity of the entrepreneur and the lender's return $R^{l,j}_{t+1}$ as:

$$
\begin{aligned}
E^j_{t+1} &\equiv [1-\Gamma(\bar{\omega}^j_{t+1})]R^k_{t+1}Q_tK^j_{t+1} = [1-\Gamma(\bar{\omega}^j_{t+1})]R^k_{t+1}\kappa^j_t N^j_t \\
R^{l,j}_{t+1} &\equiv \frac{[\Gamma(\bar{\omega}^j_{t+1}) - \mu G(\bar{\omega}^j_{t+1})]R^k_{t+1}K^j_{t+1}}{Q_tK^j_{t+1} - N^j_t} = [\Gamma(\bar{\omega}^j_{t+1}) - \mu G(\bar{\omega}^j_{t+1})]R^k_{t+1}\frac{\kappa^j_t}{\kappa^j_t - 1}
\end{aligned}
$$

And denoting dividends paid by entrepreneur j in t as div^j_t, we also have

$$N^j_t = E^j_t - div^j_t$$

Since all entrepreneurs are identical, apart from their equity, the relevant state variable for entrepreneur j will just be E^j_t. Let us denote the value function of an entrepreneur with period t equity E^j_t, before paying dividends to the family by $V_t(E^j_t)$. Given equity, the contracting problem is to choose K^j_{t+1} and the schedule $\{\bar{\omega}^j_{t+1}\}$ subject to the lender's participation constraint, or equivalently one can choose κ^j_t and $\{\bar{\omega}^j_{t+1}\}$. A unit of dividends paid to the family is valued at $\tilde{U}'(C^e_t)$, with C^e_t being each family member's consumption, taken as given by every atomistic entrepreneur j, and $\tilde{U}'$ the derivative of the entrepreneurs' strictly concave momentary utility function $\tilde{U}$.

Because entrepreneur j cannot raise external financing without any net worth, dividends necessarily cannot exceed equity $div^j_t \leq E^j_t$, and to continue operating a capital project, the inequality must be strict. $div^j_t < 0$ is understood as equity injections by the family to the entrepreneur. Entrepreneur j's value function will thus satisfy the Bellman equation:

$$V_t(E^j_t) = \max_{\{\bar{\omega}^j_{t+1}\},\kappa^j_t,div^j_t} \left\{\tilde{U}'\left(C^e_t\right) div^j_t + \beta_e \mathbb{E}_t\left[V_{t+1}(E^j_{t+1})\right]\right\} \tag{12}$$

$$\text{s.t. } \mathbb{E}_t\left[M_{t+1}R^{l,j}_{t+1}\right] = \mathbb{E}_t\left\{M_{t+1}[\Gamma(\bar{\omega}^j_{t+1}) - \mu G(\bar{\omega}^j_{t+1})]R^k_{t+1}\frac{\kappa^j_t}{\kappa^j_t - 1}\right\} \geq \mathbb{E}_t[M_{t+1}R^l_{t+1}] = 1$$

$$E^j_{t+1} = \max\{\omega^j_{t+1} - \bar{\omega}^j_{t+1}, 0\}R^k_{t+1}\kappa^k_t\left(E^j_t - div^j_t\right), \quad div^j_t \leq E^j_t$$

The lender's participation constraint arises as the result of the intermediary being a pass-through entity, combining with the facts that in equilibrium all contracts will offer the same expected return to the lender $R^l_t = R^{l,j}_t, \forall j$ and as elaborated above $R^l_t = R^d_t$ in equilibrium, and finally employing the household's Euler equation. Alternatively, this participation constraint arises as the result of intermediaries' equity value maximization subject to being owned by the household, as presented in Appendix D.

We can now guess that the continuation value function is linear, i.e. $V_{t+1}(E^j_{t+1}) = V_{t+1}E^j_{t+1}$, where, with an abuse of notation, V_{t+1} is now understood to be a *variable* that measures the marginal valuation of an additional unit of equity to the entrepreneurs. Plugging in the law of motion for E^j_{t+1} and applying the law of iterated expectations to integrate out the realization of

ω^j_{t+1}, the Bellman equation becomes:

$$\begin{aligned} V_t(E^j_t) = &\max_{div^j_t \le E^j_t} \left\{ div^j_t \left(\tilde{U}'(C^e_t) - \beta_e \mathbb{E}_t \left[V_{t+1}[1-\Gamma(\bar{\omega}^j_{t+1})] R^k_{t+1} \right] \kappa^j_t \right) \right\} + \\ &+ E^j_t \times \max_{\{\bar{\omega}^j_{t+1}\}, \kappa^j_t} \left\{ \beta_e \mathbb{E}_t \left[V_{t+1}[1-\Gamma(\bar{\omega}^j_{t+1})] R^k_{t+1} \right] \kappa^j_t \right\} \\ &\text{s.t. } \mathbb{E}_t \left\{ M_{t+1}[\Gamma(\bar{\omega}^j_{t+1}) - \mu G(\bar{\omega}^j_{t+1})] R^k_{t+1} \right\} \kappa^j_t \ge \kappa^j_t - 1 \end{aligned}$$

In equilibrium, the constraint $div^j_t \le E^j_t$ could not be binding as, by linearity, it would have to be binding for all entrepreneurs $j \in [0,1]$, implying no net worth were to be left for the entrepreneurs and no capital K_{t+1} could be acquired.[13] The individual div^j_t are thus not pinned down, and in equilibrium it must be the case that:

$$\tilde{U}'(C^e_t) = \beta_e \mathbb{E}_t \left[V_{t+1}[1-\Gamma(\bar{\omega}_{t+1})] R^k_{t+1} \right] \kappa_t$$

Since the participation constraint was already initially written independently of E^j_t, the above clearly verifies the guess that the entrepreneur's value function $V_t(E^j_t)$ is linear in equity and the problem of choosing κ^j_t and $\{\bar{\omega}^j_{t+1}\}$ is independent of entrepreneur j's equity. Thus, given that the optimal choices of κ^j_t, $\{\bar{\omega}^j_{t+1}\}$ are unique, which can be proved rigorously, each entrepreneur chooses the same leverage ratio κ_t and cutoff schedule $\{\bar{\omega}_{t+1}\}$. This implies that the Bellman equation can be written as

$$V_t = \max_{\{\bar{\omega}_{t+1}\}, \kappa_t} \left\{ \beta_e \mathbb{E}_t \left[V_{t+1}[1-\Gamma(\bar{\omega}_{t+1})] R^k_{t+1} \right] \kappa_t \right\} \tag{13}$$

$$\text{s.t. } \mathbb{E}_t \left\{ M_{t+1}[\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] R^k_{t+1} \right\} \kappa_t \ge \kappa_t - 1 \tag{14}$$

further implying that $\tilde{U}'(C^e_t) = V_t$. Given that all entrepreneurs choose the same κ_t and $\{\bar{\omega}_{t+1}\}$, the distribution of wealth across the entrepreneurs does not matter and we need to only track the aggregate level of entrepreneurial wealth. And although the distribution of dividend payments is not pinned down, on the entrepreneurial family level it must be the case that $C^e_t = \int_0^1 div^j_t dj$, with C^e_t satisfying the Euler equation:

$$\tilde{U}'(C^e_t) = \beta_e \mathbb{E}_t \left[\tilde{U}'(C^e_{t+1})[1-\Gamma(\bar{\omega}_{t+1})] R^k_{t+1} \right] \kappa_t \tag{15}$$

And the aggregate entrepreneurial net worth then evolves as

$$N_t = [1-\Gamma(\bar{\omega}_t)] R^k_t \kappa_{t-1} N_{t-1} - C^e_t \tag{16}$$

$$\text{where } \kappa_t \equiv \frac{Q_t K_{t+1}}{N_t} \tag{17}$$

Note that the leverage rate κ_t is simultaneously the inverse of the entrepreneurs' share of financial wealth in the economy. Because each entrepreneur needs a positive amount of net worth to operate its project, I assume that the family provides transfers from others to any entrepreneurs who default and must pay out all capital held to the lender. These transfers are inconsequential

[13] To be precise, one can first establish that the value function is *affine*, and given an affine value function, it must be the case that in equilibrium $\tilde{U}'(C^e_t) = \beta_e \mathbb{E}_t \left[V_{t+1}[1-\Gamma(\bar{\omega}_{t+1})] R^k_{t+1} \right] \kappa_t$, yielding *linearity*.

as the distribution of wealth across entrepreneurs is irrelevant for the aggregates.

Taking the first order conditions to the contracting problem of maximizing the entrepreneur's continuation value in (13) subject to (14) with respect to κ_t and $\{\bar{\omega}_{t+1}\}$, and using these in (13) to yield that the Lagrange multiplier on (14) equals V_t, one can summarize the resulting entrepreneurs' optimality condition as:

$$\frac{\Gamma'(\bar{\omega}_{t+1})}{\Gamma'(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})} = \left(\beta_e \frac{\tilde{U}'(C^e_{t+1})}{\tilde{U}'(C^e_t)}\right)^{-1} M_{t+1} \tag{18}$$

which holds state-by-state, for each realization of aggregate uncertainty in $t+1$. Of course, in the set of equilibrium conditions that determine period t realizations, this condition shows up with time indices lagged by one period compared to (18), in order to pin down the current $\bar{\omega}_t$.

The equilibrium lender return is:

$$R^l_{t+1} = [\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] R^k_{t+1} \frac{\kappa_t}{\kappa_t - 1} \tag{19}$$

R^l_{t+1} most natural variable to capture the degree of aggregate risk sharing. BGG imposed that R^l_{t+1} is predetermined in t and thus constant across realizations of aggregate uncertainty, whereas CFP showed that under optimal contracting, it comoves significantly with R^k_{t+1}.

A thorough analysis of the properties of the privately optimal contract and its implications in the standard BGG framework is presented by Carlstrom et al. (2016), with all the insights extending to the set up presented above. To reiterate, the key optimality condition governing aggregate risk sharing is (18). Given the assumptions in Footnote 12, one can show that the left hand side is strictly increasing in $\bar{\omega}_{t+1}$. Therefore, naturally, whenever the household values wealth relatively more, meaning M_{t+1} is high, all else equal, also $\bar{\omega}_{t+1}$, and thus the lender's net share $[\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})]$ and the lender's return R^l_{t+1} are high, to provide consumption insurance to the households. Conversely, when the value of entrepreneurs' internal net worth captured by $U'(C^e_{t+1}) = V_{t+1}$ is high, the contract calls for a lower $\bar{\omega}_{t+1}$ allowing the borrowers to hold on to more net worth, all else equal.

2.2.4 Market Clearing and Equilibrium Definition

In equilibrium the households' deposits fund the entrepreneurs' projects:

$$D_{t+1} = Q_t K_{t+1} - N_t$$

Combining this condition, the households' and entrepreneurs' budget constraints, the definition of leverage and the rental and labor market equilibrium conditions with new capital producers' profits, one arrives at the resource constraint:

$$C_t + C^e_t + I_t \vartheta\left(\frac{I_t}{I_{ss}}\right) + \mu G(\bar{\omega}_t) R^k_t Q_{t-1} K_t = A_t K^\alpha_t L^{1-\alpha}_t \tag{20}$$

Definition 1. *A* competitive equilibrium of the representative agent model *is a collection of* *stochastic processes for:*

1. *a price system* $\{r_t, W_t, R^k_t, R^l_t, Q_t\}$,
2. *household's consumption, stochastic discount factor and value function realization* $\{C^e_t$, M_t, $\mathcal{V}_t\}$
3. *entrepreneurial consumption, net worth and leverage quantities and contractual cutoffs* $\{C^e_t$, N_t, κ_t, $\bar{\omega}_t\}$
4. *aggregate labor, investment and capital quantities* $\{L_t, I_t, K_{t+1}\}$

such that equations: (1), (2), (4)–(8), (10), (11), (15)–(20), *with* $R^d_t = R^l_t$, *where applicable, are satisfied, given the stochastic process for* $\{A_t\}$, *and initial conditions* $(K_0, N_{-1}, D_0, \kappa_{-1}, \bar{\omega}_0)$.

2.3 Results on Privately Optimal Risk Sharing

2.3.1 Equivalence with CFP Model

In this section I will argue that the entrepreneurial family construct with logarithmic utility is effectively equivalent, to a first order approximation, to the standard approach used by BGG and CFP. Further details and the precise entrepreneurs' problem in the CFP model are presented in Appendix B.

In the benchmark setup employed by BGG and CFP, the entrepreneurs are assumed to have linear utility from consumption and a time discount factor identical to that of the households. To be precise, let us denote this time discount factor as β_e^{CFP}. Because of linear utility, they need not be members of a family to share idiosyncratic risk and yield optimality of risky debt in the CSV contracting problem. Since the entrepreneurs are financially constrained, it is optimal for them to postpone consumption indefinitely. To keep entrepreneurs saving themselves "out of financial constraints", it is assumed that each faces a constant probability $1-\gamma$ of dying each period. The dying entrepreneurs are replaced by an equal of mass entering ones who get a transfer from the survivors to start operations. It is optimal for entrepreneur to only consume when they die. This means that in each period, fraction $1-\gamma$ is consumed and the remaining fraction of entrepreneurial equity is invested. Using the same notation as above:

$$C^e_t = (1-\gamma)[1-\Gamma(\bar{\omega}_t)]R^k_t \kappa_{t-1} N_{t-1}$$
$$N_t = \gamma[1-\Gamma(\bar{\omega}_t)]R^k_t \kappa_{t-1} N_{t-1}$$

And even though the entrepreneurs have linear utility, their marginal valuation of an extra unit of equity is stochastic because they face time-varying investment opportunities:

$$V_t = (1-\gamma) + \gamma\beta_e^{CFP}\mathbb{E}_t\left\{V_{t+1}[1-\Gamma(\bar{\omega}_{t+1})]R^k_{t+1}\right\}\kappa_t \tag{21}$$

with V_t the marginal valuation of a unit of equity at the beginning of t before the death shock (and consumption) is realized. The participation constraint in the contracting problem is identical across the two models. And the optimality condition for risk sharing through $\bar{\omega}_{t+1}$ in the CFP model is:

$$\frac{\Gamma'(\bar{\omega}_{t+1})}{\Gamma'(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})} = \left(\gamma \beta_e^{CFP} \frac{V_{t+1}}{V_t - (1-\gamma)}\right)^{-1} M_{t+1} \tag{22}$$

As for the model presented in this paper, if the entrepreneurs have logarithmic utility $\tilde{U}(C) = \log C$, then guessing that consumption is a constant fraction of equity, and combining (15) and (16) yields the standard result when an agent has only financial wealth:

$$C_t^e = (1-\beta_e)[1-\Gamma(\bar{\omega}_t)]R_t^k \kappa_{t-1} N_{t-1}$$
$$N_t = \beta_e[1-\Gamma(\bar{\omega}_t)]R_t^k \kappa_{t-1} N_{t-1}$$

So we see right away that if $\beta_e = \gamma$, then the two models imply identical entrepreneurial consumption and net worth accumulation, conditional on all other equilibrium variables.

To fully establish identical dynamics for these models, it remains to be shown that (21) and (22) imply the same risk sharing behavior as do (15) and (18).[14] Appendix B establishes this to a first order approximation as $\beta_e^{CFP} \to 1$.[15] To see why this might be the case, notice the similarities between these pairs of equilibrium conditions. In both cases, the marginal valuation of an extra unit of wealth V_t, which also equals $U'(C_t^e)$ in my setup, must satisfy an Euler equation which determines how the log-deviations of V_t from steady state are related across time. And the similarities between (18) and (22) are evident. When log-linearized, the only difference is the appearance of β_e^{CFP} in both conditions for the CFP model.

2.3.2 The Relevance of Human and Financial Wealth Dynamics

An important determinant of aggregate financial risk sharing in the economy is the behavior of human and financial wealth dynamics. To make the analysis of this idea clear, let us consider the household utility specification $\xi = \psi = 1$, i.e. log-utility from consumption, and similarly log-utility for the entrepreneurs. Also, given that in a first order approximation certainty equivalence applies, let us consider how the economy behaves after a TFP shock has been realized and no future shocks are expected. Under such a household utility specification, we have that the household consumes a constant fraction $(1-\beta)$ of its total wealth:

$$C_t = (1-\beta)\left(R_t^l D_t + \mathcal{H}_t\right) = (1-\beta)\left\{[\Gamma(\bar{\omega}_t) - \mu G(\bar{\omega}_t)]\, R_t^k Q_{t-1} K_t + \mathcal{H}_t\right\} =$$
$$= (1-\beta)\left\{[\Gamma(\bar{\omega}_t) - \mu G(\bar{\omega}_t)]\, \mathcal{F}_t + \mathcal{H}_t\right\}$$
$$\text{where } \mathcal{F}_t \equiv [r_t + (1-\delta)Q_t]K_t$$
$$\mathcal{H}_t \equiv \sum_{j=0}^{\infty} \frac{1}{\mathcal{R}_{t,t+j}^l} W_{t+j} L_{t+j} \text{ with } \mathcal{R}_{t,t+j}^l \equiv \prod_{s=1}^{j} R_{t+s}^l, \text{ and } \mathcal{R}_{t,t}^l = 1$$

[14] As mentioned, the participation constraints, which could be loosely thought of as determining κ_t, are necessarily identical.

[15] The statement must be made in a limiting sense because if $\beta_e^{CFP} = 1$, then V_t is not finite in the CFP model.

with $\mathcal{H}_t$ and $\mathcal{F}_t$ standing for the human and financial wealth in the economy, respectively. Following Section 2.3.1, entrepreneurs consume fraction $(1-\beta_e)$ of their total (financial) wealth:

$$C_t^e = (1-\beta_e)[1-\Gamma(\bar{\omega}_t)]R_t^k \kappa_{t-1} N_{t-1} = (1-\beta_e)[1-\Gamma(\bar{\omega}_t)]\mathcal{F}_t$$

Suppose that the economy is shocked in period t, while previously having been in steady state. Using these optimal consumption policies in the privately optimal risk sharing condition (18) under log-utility, we have:

$$\frac{\Gamma'(\bar{\omega}_t)}{\Gamma'(\bar{\omega}_t) - \mu G(\bar{\omega}_t)} = \frac{[1-\Gamma(\bar{\omega}_t)]\mathcal{F}_t}{[\Gamma(\bar{\omega}_t) - \mu G(\bar{\omega}_t)]\,\mathcal{F}_t + \mathcal{H}_t} \times \frac{1-\beta_e}{1-\beta}\frac{\beta}{\beta_e}\frac{C_{ss}}{C_{ss}^e}$$

Therefore, given that the left hand side is increasing and the right hand side decreasing in $\bar{\omega}_t$, this establishes a negative relationship between $\mathcal{H}_t/\mathcal{F}_t$ and $\bar{\omega}_t$. That is, whenever the human wealth in the economy increases more than the financial wealth, the gains accrue to the household and it is thus optimal to leave a larger share of financial weatlh, implied by a lower $\bar{\omega}_t$, to the entrepreneurs. For example, if $\mathcal{H}_t/\mathcal{F}_t = \mathcal{H}_{ss}/\mathcal{F}_{ss}$, then $\bar{\omega}_t = \bar{\omega}_{ss}$ and the aggregate financial risk is shared perfectly, meaning that R_t^l responds to the shock by the same relative amount as R_t^k.

Of course, $\mathcal{H}_t$ and $\mathcal{F}_t$ are themselves equilibrium objects, dependent on $\bar{\omega}_t$ itself, but this note emphasizes that it is important to keep in mind that shocks which affect human and financial wealth differently, could have markedly different implications for how the aggregate financial risk is to be shared. This motivates the discussion of the importance of TFP shock persistence in Section 3.1 and the model's extension to uninsurable household labor (liquidity) risk in Section 4, which aims to detach the household's consumption, at least partly, from the total human capital in the economy.

3 Quantitative Analysis for the Representative Household Model

3.1 Calibration

In the calibration of model parameters I pursue targets from earlier literature, following BGG and CFP wherever possible for comparability. One time period t is considered to be a quarter. As CFP, I set the capital share in production to be $\alpha = 0.35$, capital price elasticity with respect to investment $\phi_Q = 0.5$ and the depreciation rate $\delta = 0.025$. TFP follows $\log A_t = \rho_A \log A_{t-1} + \varepsilon_t^A$ with $\rho_A = 0.95$, as used by CFP as a benchmark, varied below. ε_t^A is i.i.d normal mean-zero with standard deviation 0.0072, following King and Rebelo (1999).

As is common since Carlstrom and Fuerst (1997), the idiosyncratic entrepreneurial capital shock is log-normal: $\log \omega \sim N\left(-\frac{\sigma^2}{2}, \sigma\right)$. Following the discussion in Section 2.3.1 and the targets set by CFP, the parameters (μ, β_e, σ) pertaining to the entrepreneurial financial frictions are pinned down, jointly with all other parameters, to yield in steady state: (i) a spread of 200 basis points (annualized) between the lender return R_{ss}^{def} subject to default risk and the riskless lender return

R^l_{ss}, both as defined in Section 2.2.3, (ii) a quarterly bankruptcy rate $F(\bar{\omega}_{ss})$ of 0.75%, (iii) a leverage ratio of $\kappa_{ss} = 2$. Exactly as in CFP, this results in $(\mu, \beta_e, \sigma) = (0.63, 0.94, 0.28)$. Based on the discussion in Section 2.3.1, I set $\tilde{U}(C) = \log C$.

Following CFP, I set β to 0.99. As stated above, momentary household utility has the form $u(C, L) = [C\Phi(L)]^{1-\frac{1}{\psi}}$. Like BGG and CFP, I solve the model by log-linearization around the deterministic steady state. Because of this, the properties of the function Φ only affect the equilibrium conditions through $\Phi(L_{ss})$, $\Phi'(L_{ss})$ and $\Phi''(L_{ss})$. More specifically, one needs to determine $\nu_l \equiv -\frac{\Phi'(L_{ss})L_{ss}}{\Phi(L_{ss})} > 0$ which captures the effect of L_t on M_t and $\mathcal{V}_t$, $-\frac{\Phi'(L_{ss})}{\Phi(L_{ss})}$, which pins down L_{ss}. And finally, the determination of labor supply elasticity can be seen by log-linearizing (4) to get:

$$c_t + \underbrace{\left[\frac{\Phi''(L_{ss})L_{ss}}{\Phi'(L_{ss})} + \nu_l\right]}_{\equiv 1/\eta_l} l_t = w_t$$

with lowercase letters denoting the corresponding log-deviations from steady state values. By setting $\eta_l = 3$, one can exactly replicate the labor supply condition as used by CFP.[16] The value of ν_l is determined independently of Φ in steady state. To see this, rewrite the labor market equilibrium condition in steady state as:

$$-\frac{\Phi'(L_{ss})}{\Phi(L_{ss})}L_{ss} = (1-\alpha)\left(\frac{K_{ss}}{L_{ss}}\right)^{\alpha}\frac{L_{ss}}{C_{ss}}$$

Similarly as in a conventional RBC model, $\frac{K_{ss}}{L_{ss}}$ and $\frac{K_{ss}}{L_{ss}}$ are pinned down by the remaining system of equilibrium conditions. This results in $\nu_l \approx 0.958$, close to the recommendation by Uhlig (2010) made based on the fact that $(1-\alpha)\left(\frac{K_{ss}}{L_{ss}}\right)^{\alpha}\frac{L_{ss}}{C_{ss}} = (1-\alpha)\frac{Y_{ss}}{C_{ss}} \approx 1$, with $Y_{ss}/C_{ss} \approx 3/2$. I then choose $-\frac{\Phi'(L_{ss})}{\Phi(L_{ss})}$ to normalize L_{ss} to 1.

As for the parameters ψ and ξ governing the household's intertemporal elasticity of substitution (IES) and risk aversion, respectively, I consider various values below. In the benchmark case of $\psi = \xi = 1$, the model's first order approximation matches the log-utility specification used by CFP. As discussed in detail by Swanson (2015), the ability of households to adjust their labor supply in response to shocks affects their attitude towards risk and thus measures of relative risk aversion, as defined by Arrow (1965) and Pratt (1964) do not exactly equal ξ in this case. Nontheless, ξ equals the coefficient of relative risk aversion when labor were to be held exogenously fixed. And ξ is larger than the *consumption-wealth coefficient of relative risk aversion* with adjusting labor, as defined and shown by Swanson (2015) to be the most adequate measure in explaining equity premia in an RBC model. This means that allowing for a variable labor margin tilts the outcomes *against* less risk sharing between the household and the entrepreneurs as the household's effective risk aversion is less than ξ.

Finally, note that unlike most applications in which the introduction of Epstein-Zin utility with ξ differing from $1/\psi$ has no effect in a first order approximation solution, it does here. Because of certainty equivalence imposed by the solution method and the fact that in standard DSGE model equilibrium conditions in period t, the stochastic discount factor M_{t+j} shows up inside expectation

[16] Further log-linearizations are provided in Appendix C.

terms with $j > 0$, the term $\left(\frac{\mathcal{V}_{t+1}}{\mathbb{E}_t[\mathcal{V}_{t+1}^{1-\xi}]^{\frac{1}{1-\xi}}}\right)^{\frac{1}{\psi}-\xi}$ effectively equals 1 everywhere *ex ante*. However in this case, the agents' stochastic discount factors explicitly appear *ex post*, as M_t in t, in condition (18) to determine how any realized risk is shared. And the implied realization of $\bar{\omega}_t$ then has direct first order effects on the agents' wealth distribution and equilibrium dynamics. Additional details can be seen in Appendix C.

3.2 Second Moments and Impulse Responses

As the analysis of the representative household model focuses on how changes in the household's preferences affect privately optimal aggregate risk sharing in this baseline economy, I concentrate the quantitative analysis on presenting standard deviations and impulse response functions of key balance sheet and real variables for various preference calibrations.

Table 1 below presents results from different combinations of ψ and ξ. The benchmark is $\psi = \xi = 1$, corresponding to the infinitely patient entrepreneur limit of CFP's model. Increasing either ξ or decreasing ψ makes the household less willing to take on aggregate financial risk generated by TFP shocks. Equilibrium increases in household consumption after positive productivity shocks generate larger drops in its stochastic discount factor due to decreased elasticity of intertemporal substitution and increased risk aversion. I consider two main exercises. Firstly, keeping $\psi = 1$, I determine the ξ necessary to yield an optimal contract that implies a non-state-contingent lender's return. Secondly, I do the same while setting $\psi = 1/\xi$, i.e. employ expected utility preferences with momentary utility $\frac{[C\Phi(L)]^{1-\xi}}{1-\xi}$.

The first column of output in Table 1, denoted as $\left(\frac{\partial r_t^l}{\partial \varepsilon_t^A}\right) / \left(\frac{\partial r_t^k}{\partial \varepsilon_t^A}\right)$, refers to the TFP shock responsiveness of the net lender return, $\log(R_t^l)$, relative to that on the borrowers' assets, $\log(R_t^k)$ at impact. For brevity, I use this as the measure of the degree of risk sharing. If this entry is 1.0, then there is perfect aggregate risk sharing between the households and entrepreneurs. This is meant in the sense that in response to an unexpected innovation in TFP at t, $\bar{\omega}_t$ and the households' share in the capital project returns do not respond, making the return on households' financial assets move one for one with that on the entrepreneurs'. If this entry is 0.0, then $\bar{\omega}_t$ responds to eliminate any effects of R_t^k on R_t^l, implying a non-state-contingent lender return – the contract imposed by BGG. And if this entry happens to be negative, it is the entrepreneurs who are providing consumption insurance to the households, increasing payouts in recessions, and vice versa. The following two columns present the standard deviation of log entrepreneurial net worth and leverage relative to that of log output y. The standard deviation of the latter is in the last column. I compute the second moments based on a simulation of 10^6 quarters.

First of all, as demonstrated by CFP, we see that with logarithmic utility and TFP shock persistence of 0.95, there is a considerable degree of aggregate risk sharing, although not exactly close to 1.0, with the measure of risk sharing at approximately 0.82. It is still enough to generate small leverage fluctuations of about 5% of that of output, and net worth volatility at the same magnitude of numeraire output – evidence of a significant dampening of the financial accelerator mechanism.

Table 1: Relative impulse response of lender return, relative standard deviations of log entrepreneurial net worth and leverage, absolute standard deviation of log output (in percentages), in representative household model; simulation of 10^6 quarters.

ρ_A	ξ, ψ	$\left(\frac{\partial r_t^l}{\partial \varepsilon_t^A}\right) \Big/ \left(\frac{\partial r_t^k}{\partial \varepsilon_t^A}\right)$	$\frac{\text{std}(n)}{\text{std}(y)}$	$\frac{\text{std}(\hat{\kappa})}{\text{std}(y)}$	std(y), %
0.95	1.0, 1.0	0.825	0.947	0.050	2.626
	13.2, 1.0	0.000	1.498	0.495	2.875
	5.92, $1/\xi$	0.000	1.597	0.424	3.403
0.99	1.0, 1.0	0.752	0.982	0.029	6.170
	4.34, 1.0	0.000	1.100	0.206	6.311
	2.57, $1/\xi$	0.000	1.099	0.156	6.493

Increasing ξ to 13.2 yields a high enough aversion to risk for the household not to be willing to take on aggregate financial risk and the contract imposed by BGG becomes the optimal private contract. An unexpected increase in household consumption in response to a positive TFP shock increases the household's utility $\mathcal{V}_t$, making the stochastic discount factor M_t drop significantly. The relative volatility of entrepreneurial net worth is about 1.5 times higher and that of leverage almost 10 times higher than under logarithmic household utility. There is also slight amplification of output fluctuations, with its standard deviation increasing about 10%.

Similarly, one can arrive at less risk sharing with lower household's IES. Setting ξ to 5.92 and $\psi = 1/5.92$, again leads to non-state-contingent lender returns. The increase in the relative volatility of net worth is slightly larger, and smaller for leverage. The implied increase in output volatility is significantly larger, almost 30%. The causes of these differences become clearer from the impulse responses below. Although the aim of the analysis here is not to match the simulated moments to the data, it is worthwhile noting that if one HP-filters the simulated series for comparability, the relative net worth and leverage volatilities generated under non-state-contingent lender returns are remarkably close to those of US non-financial corporates and noncorporates during 1976Q1–2015Q3. These results are shown in Table 3 in Appendix A.

To illustrate the model's dynamics in more detail, Figure 2 below presents impulse responses to a 1% positive TFP shock over 28 quarters for the three utility calibrations under $\rho_A = 0.95$. The lower left panel shows the relative response of the quarterly lender return r_t^l. As seen above, the benchmark case exhibits a non-trivial degree of risk sharing with more than 80% of the innovation in capital returns, seen in the middle right panel, paid out to the households. And the engineered non-state-contingency of the lender returns are seen for the two other calibrations. Since in the latter two cases the entrepreneurs hold on to relatively more wealth, their net worth increases and leverage decreases significantly more at impact. Higher net worth facilitates investment, which increases the price of capital, in turn increasing returns to capital and net worth – the financial accelerator mechanism. And because of logarithmic utility for the entrepreneurs, the response of C^e follows that of net worth. With less risk sharing, the household's wealth increases less, leading to lower consumption and a weaker positive wealth effect, increasing labor supply and output.

The key difference between the two parametrizations with non-state-contingent lender returns are the transitional dynamics of investment, net worth and output. In both cases, at the time of the shock, there is significant amplification. Yet for the case with high risk aversion and unitary

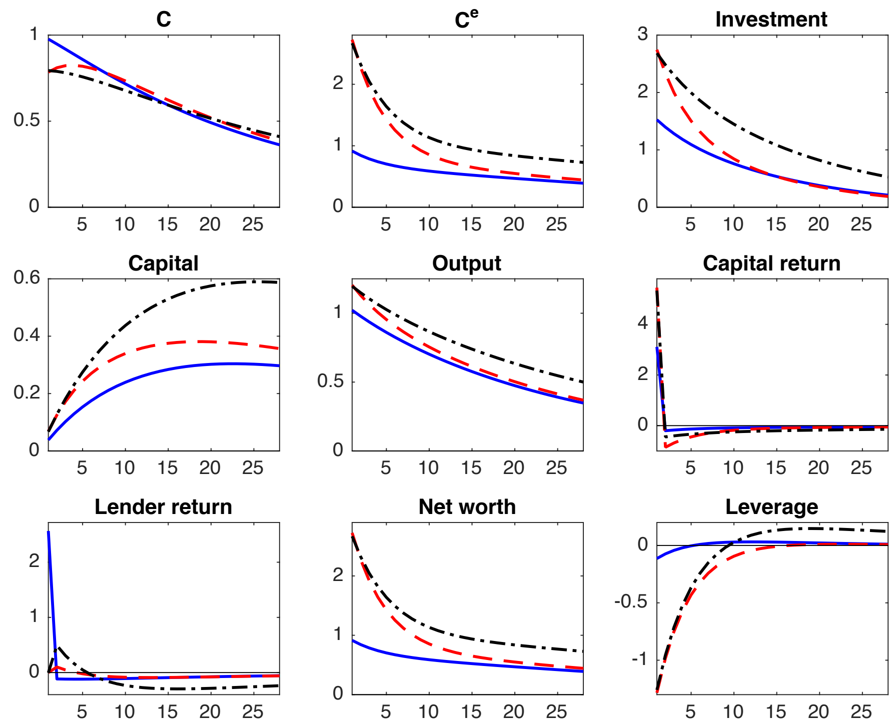

Figure 2: Impulse responses to 1% positive TFP shock in representative household model, with $\rho_A = 0.95$, log-deviations from steady state, returns annualized; Horizontal axis – quarters; Blue solid – log-utility, red dashed – $\xi = 13.2$, black dash-dotted – $\xi = 1/\psi = 5.92$.

IES, investment and net worth fall significantly faster, leading to smaller capital accumulation and a faster reversion in output. The reason is that with a low IES, the household prefers a flatter consumption profile, inducing it to save more throughout the first few periods after the shock. The extra savings flow through entrepreneurs into investment, boosting entrepreneurial net worth. The fact that higher household savings should increase their wealth share, and thus entrepreneurial leverage, is counteracted by the fact that the high investment increases capital prices and entrepreneurial net worth through capital returns. Yet we do see that over time, leverage recovers faster under low IES, reflecting the household's larger accumulated wealth share. The sustained high investment yields larger capital accumulation and higher output throughout the transition path.

Note that the appearance of labor growth – which is negative along the transition path – in the household's stochastic discount factor when $\psi < 1$ makes it willing to take on a steeper consumption profile and dampens this added stimulus to the financial accelerator mechanism from low IES. If one were to exogenously set $\nu_l = 0$ for the sake of the argument, the desire for a flat profile would be stronger, the initial household savings even larger and the amplification in the low IES parametrization more significant. A $\nu_l > 0$ also explains how negative consumption growth can appear alongside lender returns above steady state values in the first few quarters after the shock.

When the persistence of the producvitity shocks is increased, less of the aggregate financial risk associated with these shocks is taken on by the households, evident for $\xi = \psi = 1$. This reflects the importance of human and financial wealth dynamics for aggregate financial risk sharing in the economy, as discussed in Section 2.3.2. An increase in productivity shock persistence increases the household's human capital responsiveness as the gains or losses from the discounted labor income throughout all future periods are accrued. This effect on human wealth is counteracted by the fact that with a less persistent shock, the transition path of household consumption is steeper, implying larger swings in the lender return used for discounting human wealth income.

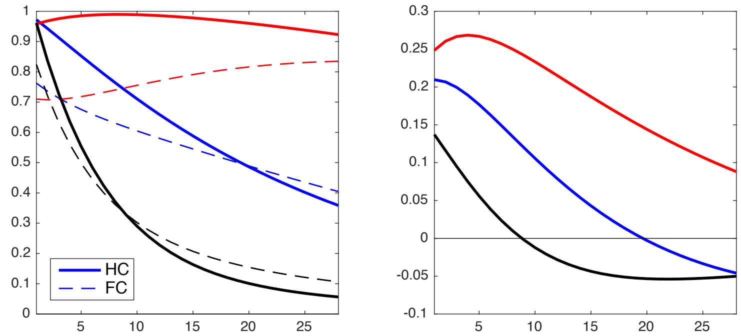

Figure 3: Left panel: Impulse responses of human (HC) and financial wealth (FC) to 1% positive TFP shock, percentage deviations from steady state, baseline calibration. Right panel: difference in HC and FC percentage deviations; Horizontal axis – quarters; Red – $\rho_A = 0.99$, blue – $\rho_A = 0.95$, black – $\rho_A = 0.85$.

At the same time, a more persistent positive TFP shock does not necessarily lead to a larger innovation in financial wealth. When shocks have less persistence, consumption smoothing motivates agents to save more of the initial windfall, boosting investment. As the price of capital is directly tied to investment, this channel decreases financial wealth responsiveness when TFP shocks become more persistent. The left panel of Figure 3 depicts the impulse responses of human and financial wealth, as defined in Section 2.3.2, for three different degrees of TFP shock persistence in the $\xi = \psi = 1$ calibration. All three specifications yield almost identical initial responses of human wealth, implying that for less persistent shocks a larger drop in the discount rate compensates for the lower duration of higher labor income. On the other hand, increased shock persistence does lead to lower initial responsiveness of financial wealth, resulting in larger differences in human and financial capital responses, seen in the right panel of the figure.

Because of the increased relative volatility of human wealth under $\rho_A = 0.99$, smaller changes in ξ and ψ are required to yield non-state-contingent lender returns, as households become more averse to any given changes in consumption. One either needs $\xi = 4.34$ and $\psi = 1$, or $\xi = 1/\psi = 2.57$. Yet there is also an extra effect arising from Epstein-Zin preferences whenever $\xi \neq 1/\psi$. Persistently higher consumption brings about large innovations in household lifetime utility $\mathcal{V}_t$, making the household additionally averse to taking on the financial risk related to persistent TFP shocks. This discussion of course implies that for *less* persistent productivity shocks, ξ and ψ would have to deviate significantly from unity in order for non-state-contingent lender returns to arise. For example, for $\rho_A = 0.90$, the financial accelerator mechanism implies that financial wealth responds relatively more than human wealth on impact, and $\xi \approx 27$ while $\psi = 1$ is necessary for non-state-contingent lender return optimality.

4 The Heterogeneous Households Extension

To study the relevance of uninsurable idiosyncratic lender risk, I append the BGG financial frictions framework as outlined in Section 2 to an underlying baseline environment which features endogenous unemployment, uninsurable household liquidity risk and limited cross-sectional heterogeneity. In its essence, this underlying environment is a real business cycle model with a frictional labor market and a cash-in-advance constraint on consumption. The resulting model with heterogeneous households and financial frictions is a slight variation of the representative agent model presented above whenever households can fully insure against idiosyncratic risk. This can happen, for example, thanks to government unemployment insurance, "sufficient" sources of credit or complete markets in idiosyncratic employment outcomes. In the absence of entrepreneurial financial frictions, it becomes a real business cycle model with limited household heterogeneity.

4.1 The Environment

As in the representative household model, time is discrete and infinite. Yet now, each time period t is split into two subperiods, labeled "I" and "II". There are two types of goods. The *numeraire good* is used for consumption and investment to form productive capital. It is storable intra-period and, in the form of capital, across periods with depreciation rate $\delta \in (0, 1]$. I assume that capital cannot be consumed, but in what is to follow, this restriction will never be binding. There is also *good* X that is only used for consumption and is perishable. Also, labor vacancy posting costs and monitoring costs arising from the CSV friction are incurred in terms of the numeraire. In addition to the agents present in the representative household model, there are also competitive labor intermediaries, discussed below.

The households, indexed by $i \in [0, 1]$, are infinitely-lived and *ex ante* identical. They have a discount factor $\beta \in (0, 1)$ between "II" in t and "I" in $t + 1$. I now assume that the households have expected utility preferences. They consume both the numeraire and good X. Numeraire consumption c yields momentary utility $U(c)$ only in subperiod "I". The consumption of X yields momentary utility $\Upsilon(X)$. Each household can produce good X subject to per unit linear disutility D in subperiod "II". Since X is non-storable, its consumption must also take place in "II". The relative price of good X is p_t. Production of the numeraire is discussed below. I assume that U and Υ are strictly increasing, strictly concave and continuously differentiable. And suppose that there exists $X^* \in (0, \infty)$ such that $\Upsilon'(X^*) = D$. In subperiod "I", each household can supply H units of labor in a market subject to a Mortensen and Pissarides (1994)-type search and matching friction. There is no disutility from searching nor providing this labor. Thus, all households search and, when matched with a vacancy posted by a labor intermediary, inelastically supply all their labor. They take the wage per efficiency unit in period t, w_t, as given. All jobs are assumed to last for the duration of each subperiod "I". In each subperiod "II" in t, households deposit their savings, d_{it+1}, in a financial intermediary. These deposits yield gross returns R^{ld}_{t+1} and R^{nd}_{t+1} in "I" and "II" in $t + 1$, respectively. The reason for two distinct return streams comes from the key assumption that productive capital cannot be used as a means of payment in "I", discussed in more detail below. Again, these returns are not predetermined in t and are realized at $t + 1$, depending on aggregate shocks. Finally, in each "I" in t, households can finance their purchases of the numeraire with returns from deposits $R_t^{ld} d_{it}$, labor income $w_t H$ whenever employed, and with sales of intra-period bonds $-b_{it}$ at a price q_t^b. These bonds are redeemed in the upcoming "II", and are subject to a Kiyotaki and Moore (1997)-type borrowing constraint allowing to borrow up to a fraction $\chi \in [0, 1]$ of returns to be received in "II", i.e. $\chi R_t^{nd} d_{it}$, plus some potentially non-zero $\bar{B}$.

Analogously as in the representative household model, the representative financial intermediary accepts deposits from households and extends loans, between "II" in t and "I" in $t + 1$, to the continuum of entrepreneurs. Now however, because of the illiquidity of productive capital, the intermediary's portfolio yields separate payments in "I" and "II", which are passed on to the households R_{t+1}^{ld} and R_{t+1}^{nd}. Appendix D provides a more rigorous foundation for these statements and the financial intermediary's optimal behavior based on households' ownership of the intermediary. Households cannot write contracts contingent on their idiosyncratic employment realizations and the intermediary can make payments only contingent on the amount of deposits a household has deposited.

The preferences and technology of the entrepreneurs, who are indexed by $j \in [0, 1]$ and belong to a family, are a direct adaptation of those presented in Section 2.1 to the two subperiod case. They only consume the numeraire good, receiving utility $\tilde{U}(C^e)$, and do so in subperiod "II" – this assumption is without loss of generality as explained below. The entrepreneurs have a time discount factor $\beta_e \in (0, 1)$ between "II" in t and "I" in $t + 1$. They are the only agents who can hold capital between subperiods "II" in t and "I" in $t + 1$. They purchase physical capital in each subperiod "II" using net worth and external financing from the financial intermediary.

The idiosyncratic shock ω_{t+1}^j scales entrepreneur j's capital holdings K_{t+1}^j at the beginning of "I" in $t + 1$, distributed and observed as explained in Section 2.1. Again, only one-period contracts between the entrepreneurs and the financial intermediary are feasible. Similarly as in the representative household case, I assume that once agents enter "I" in $t+1$ and entrepreneurial capital shocks are realized, the entrepreneurs pay back the intermediaries in the form of capital, as dictated by the previously signed contract, and the latter then receive the proceeds from the ownership of their share of the capital. Returns to capital are derived from capital gains in the price of capital when selling it in "II" and renting the capital out to a representative numeraire good producer in subperiod "I". The total returns to capital in period $t + 1$ are R_{t+1}^k as defined in (1). The monitoring costs are still a proportion μ of the realized gross payoff to a given entrepreneur's capital, yet whenever the lender monitors entrepreneur j, a cost in the size of a fraction μ of the rental returns to entrepreneur j's capital is incurred in "I" and a fraction μ of the market value of this capital in subperiod "II".

The key underlying assumption regarding the illiquidity of productive capital is that it cannot be used as a method of payment in subperiod "I". That is, in this subperiod, both the entrepreneurs and the financial intermediary only acquire rental returns from their capital ownership. Capital is traded in "II". In "I", entrepreneurs store their rental proceeds or acquire bonds in the intra-period bond market. In "II" they are assumed to liquidate all their capital, pay dividends to their family, buy consumption in "II" and repurchase capital for investment. As in the representative household model, entrepreneurs have limited liability in that each individual entrepreneur's project cannot make payments in excess of $\omega_{t+1}^j K_{t+1}^j$ – equity injections or dividend payments are made in subperiod "II".

The representative numeraire producer runs a Cobb-Douglas production function in aggregate labor L_t and capital K_t producing $A_t K_t^{\alpha} L_t^{1-\alpha}$ in subperiod "I". It rents capital from entrepreneurs and financial intermediaries, for rental rate r_t, and labor from labor intermediaries for wage rate W_t per efficiency unit, both in frictionless centralized markets operating in "I". Shocks to the stationary AR(1) TFP process A_t are still the only source of aggregate uncertainty, publicly observed at the beginning of period t. The new capital producers owned by households operate in "II" using exactly the same technology to transform the numeraire into capital as assumed in Section 2.1. I assume the distribution of ownership is uniform across $i \in [0, 1]$.

Labor services are sold to the numeraire producer by competitive labor intermediaries who hire labor from households in a market with search frictions. This labor market operates in subperiod "I". Labor intermediaries post vacancies at unit cost κ_v, pay w_t per efficiency unit to each hired household, and sell these labor services to the representative producer for W_t. There is a matching function $\mathcal{M}(s_t, v_t)$ that generates successful matches from the mass of households searching, $s_t = 1$, and posted vacancies v_t. Labor intermediary profits are zero in equilibrium as I assume that there is free entry in labor intermediation. w_t is taken as given by all market participants and its determination in equilibrium is discussed in Section 4.2.2.

4.1.1 Discussion of Assumptions

Before we continue, a word is in order on some of the assumptions made above. First of all, I now assume that the household has expected utility preferences just because of the specific exercise I am conducting. The key focus is to study the effects of uninsurable idiosyncratic lender risk while keeping the momentary utility from numeraire consumption identical for households and entrepreneurs. This way the implications for risk sharing do not get obscured by differing aversion to risk or intertemporal substitution due to preferences and derive from underlying idiosyncratic risk.

The assumption that all filled jobs last for only one period is not required to yield tractability in the households' wealth distribution. But it guarantees a well-defined optimal contract in the financing of entrepreneurs. Linearity in the disutility of producing good X will yield that all households are identical at the end of each "II" in equilibrium, so there is a unique Euler equation with respect to deposits across $i \in [0, 1]$.

In BGG and CFP, it is assumed that entrepreneurs borrow in t to acquire capital that yields returns $\omega_{t+1}^{j} R_{t+1}^{k}$, per unit of numeraire spent, to the entrepreneurs in $t+1$. And these returns then determine whether the entrepreneurs can repay their lender the agreed upon amounts. Because of constant returns to scale in production, it does not matter whether the entrepreneurs are assumed to hire labor and run the production themselves, or rent out the capital in a competitive market to a representative producer. Furthermore, because of the linear returns to capital, linear monitoring costs in gross capital returns and perfect observability of R_{t+1}^{k} it is equivalent in the BGG framework to assume that the borrower and lender agree upon repayment in terms of *capital* after all shocks are realized, yet *before* the rental returns and capital gains on each unit of capital are received. That is, suppose that the entrepreneur and the lender agree upon some (idiosyncratic and aggregate) state contingent repayment scheme $P_j(\omega^j_{t+1}, \mathcal{S}_{t+1})$ in the standard BGG interpretation *after* the entrepreneur has reaped the per unit revenues $R^k_{t+1}Q_t$ on his $\omega^j_{t+1}K^j_{t+1}$ units of capital. They can then equivalently agree upon the repayment of $\frac{P_j(\omega^j_{t+1},\mathcal{S}_{t+1})}{Q_t R^k_{t+1}}$ units of capital *before* rental returns and capital gains on it are received without changing any payoffs to the counterparties. I use $\mathcal{S}_{t+1}$ as shorthand for the aggregate state. The lender can then himself rent out the capital and liquidate it, to end up with the effective repayment of $\frac{P_j(\omega^j_{t+1},\mathcal{S}_{t+1})}{Q_t R^k_{t+1}}Q_t R^k_{t+1}$. In the standard BGG framework this would just be relabeling. However, once one splits period t into subperiods with capital gains being received in subperiod "II", the CSV problem between the lender and the entrepreneur would become significantly more complicated, already due to the dynamic nature of the game over several (sub-)periods. Therefore, assuming that the entrepreneurs must make all repayments in the form of capital at the beginning of "I" keeps the lender-entrepreneur interaction a static, one-shot problem, and minimizes any departures from the treatment by BGG and CFP that would introduce unnecessary complications obscuring the focus of the paper. The assumption of splitting monitoring costs across subperiods, even though the decisions to monitor are taken in "I", is done just for algebraic simplicity and does not carry any significant relevance because the calibrated monitoring costs are small.

The assumption of capital illiquidity in "I" could be relaxed by allowing households to pay stochastic transaction costs to access their "illiquid wealth" in the form of capital to be used as a means of payment, and sell intra-period bonds backed by a fraction of their remaining capital. However, if one takes the cross-sectional average consumption drop experienced due to unemployment as a calibration target, as I do in Section 5.1 below, at least some transaction cost realizations would be calibrated high enough such that there exist unemployed agents who choose not to pay them. This would effectively cause a mean-preserving spread in the consumption distribution of the unemployed and only amplify any quantitative results presented below.

It is without loss of generality that there are no zero-net-supply financial assets bought and sold between households because the resulting degenerate portfolio distribution in equilibrium would imply zero holdings of such assets for each $i \in [0, 1]$. Similarly, allowing households to trade in the ownership of new capital producers in "II" would not change the assumed, uniform distribution of ownership.

4.2 Equilibrium

I now present the agents' problems and derive their equilibrium optimality conditions to the extent that they differ from those in the representative household model of Section 2.

4.2.1 Households

I present household i's problem in recursive form, focusing on decisions made in "I" and "II" separately.

At the beginning of subperiod "I" in t, the relevant idiosyncratic state variables for household i are the deposits in the financial intermediary chosen in the previous period, denoted d_{it}, and the household's employment status $\theta_{it} \in \{0, 1\}$, valued 1 if employed. Let $\mathcal{V}_t(d, \theta)$ be the value function for a household with idiosyncratic state (d, θ) at the beginning of subperiod "I" in t. For brevity of notation, I again assume that the aggregate state is encompassed by allowing for an aggregate state contingent value function $\mathcal{V}_t(\cdot)$. Let s_{it} denote intra-period storage of the numeraire and b_{it} intra-period bonds bought at price q^b_t. The household's Bellman equation at the beginning of "I" is then:

$$\begin{gathered}\mathcal{V}_t(d_{it}, \theta_{it}) = \max\nolimits_{c_{it}, s_{it}, b_{it}} \{U(c_{it}) + \mathcal{W}_t(d_{it}, s_{it}, b_{it})\} \\ \text{s.t. } c_{it} + s_{it} + q^b_t b_{it} = R^{ld}_t d_{it} + w_t H \theta_{it} \\ c_{it} \geq 0; \quad q^b_t b_{it} \geq -\chi R^{nd}_t d_{it} - \bar{B}; \quad s_{it} \geq 0\end{gathered}$$

where $\mathcal{W}_t(d, s, b)$ is the value function for a household with idiosyncratic state (d, s, b) at the beginning of subperiod "II" in t. θ_{it} is a Bernoulli random variable that equals 1 with probability $(1 - u_t)$, where u_t is the unemployment rate, determined in equilibrium.

Assuming $\lim\limits_{c \to 0} U'(c) = +\infty$, the non-negativity constraint on consumption will not bind in equilibrium. We can plug in the budget constraint for c_{it} and let the Lagrange multiplier on the credit constraint be μ^b_{it} and that on non-negativity of storage μ^s_{it}. Then, the first order necessary and envelope conditions yield:

$$\text{FOC:} \quad (s_{it}): \quad U'(c_{it}) = \mathcal{W}^s_t(d_{it}, s_{it}, b_{it}) + \mu^s_{it} \tag{23}$$

$$(b_{it}): \quad q^b_t U'(c_{it}) = \mathcal{W}^b_t(d_{it}, s_{it}, b_{it}) + q^b_t \mu^b_{it} \tag{24}$$

$$\text{EC:} \quad (d_{it}): \quad \mathcal{V}^d_t(d_{it}, \theta_{it}) = R^{ld}_t U'(c_{it}) + \chi R^{nd}_t \mu^b_{it} + \mathcal{W}^d_t(d_{it}, s_{it}, b_{it}) \tag{25}$$

where $\mathcal{V}^x$ and $\mathcal{W}^x$ denote the partial derivatives of $\mathcal{V}$ and $\mathcal{W}$ with respect to any argument x.

Household i's Bellman equation at the beginning of "II" is:

$$\begin{gathered}\mathcal{W}_t(d_{it}, s_{it}, b_{it}) = \max\nolimits_{(d_{it+1}, X^d_{it}, X^s_{it}) \geq 0} \left\{\Upsilon(X^d_{it}) - D X^s_{it} + \beta \mathbb{E}^i_t[\mathcal{V}_{t+1}(d_{it+1}, \theta_{it+1})]\right\} \\ \text{s.t. } d_{it+1} + p_t(X^d_{it} - X^s_{it}) = R^{nd}_t d_{it} + s_{it} + b_{it} + \Pi^I_t\end{gathered}$$

where X^d_{it} denotes the consumption and X^s_{it} the production of good X by household i. Π^I_t are profits of new capital producers. $\mathbb{E}^i_t[\cdot]$ refers to the conditional expectation operator with respect to both aggregate and idiosyncratic uncertainty in "I" in $t+1$, conditional on information known in period t. The non-negativity constraint on d_{it+1} will not bind in equilibrium. An interior solution for X^d_t is guaranteed under standard assumptions on Υ. As for X^s_{it}, we can assume interiority, characterize the equilibrium and then check that $X^s_{it} \geq 0$ is indeed satisfied.

By plugging in for X^s_{it} from the budget constraint, the Bellman equation becomes:

$$\begin{aligned}\mathcal{W}_t(d_{it}, s_{it}, b_{it}) \quad = \quad & \frac{D}{p_t}\left(R^{nd}_t d_{it} + s_{it} + b_{it} + \Pi^I_t\right) + \max_{X^d_{it} \geq 0}\left\{\Upsilon(X^d_{it}) - D X^d_{it}\right\} + \\ & + \max_{d_{it+1} \geq 0}\left\{-\frac{D}{p_t} d_{it+1} + \beta \mathbb{E}^i_t[\mathcal{V}_{t+1}(d_{it+1}, \theta_{it+1})]\right\}\end{aligned}$$

As is common in this type of models, we see that $\mathcal{W}_t$ is linear in (d_{it}, s_{it}, b_{it}), and the choice of

X_{it}^d and d_{it+1} is independent of the individual portfolio.

The first order necessary and envelope conditions are then:

$$\text{FOC:} \quad (X_{it}^d): \quad \Upsilon'(X_{it}^d) = D \Rightarrow X_{it}^d = X^* \tag{26}$$

$$(d_{it+1}): \quad \frac{D}{p_t} = \beta \mathbb{E}_t^i [\mathcal{V}_{t+1}^d(d_{it+1}, \theta_{it+1})] \tag{27}$$

$$\text{EC:} \quad (d_{it}): \quad \mathcal{W}_t^d(d_{it}, s_{it}, b_{it}) = R_t^{nd} \frac{D}{p_t} \tag{28}$$

$$(s_{it}, b_{it}): \quad \mathcal{W}_t^s(d_{it}, s_{it}, b_{it}) = \mathcal{W}_t^b(d_{it}, s_{it}, b_{it}) = \frac{D}{p_t} \tag{29}$$

Combining the above first order necessary and envelope conditions, we get two equations, that, with the respective complementary slackness conditions, characterize numeraire consumption and the Lagrange multipliers, and an Euler equation governing deposits:

$$U'(c_{it}) = \frac{D}{p_t} + \mu_{it}^s \tag{30}$$

$$U'(c_{it}) = \frac{1}{q_t^b} \frac{D}{p_t} + \mu_{it}^b \tag{31}$$

$$\frac{D}{p_t} = \beta \mathbb{E}_t^i \left[R_{t+1}^{ld} U'(c_{it+1}) + \chi R_{t+1}^{nd} \mu_{it+1}^b + R_{t+1}^{nd} \frac{D}{p_{t+1}} \right] \tag{32}$$

Firstly, (30) and (31) imply, as one would expect, that if $q_t^b > 1$, then $\mu_{it}^b > 0$ and all households (and entrepreneurs) are at the intra-period borrowing constraint. As the implied intra-period net interest rate is negative, it is beneficial to borrow in "I" and store the numeraire. Whenever the agents face a non-zero borrowing constraint, this cannot happen in equilibrium, as no-one would be lending whenever storage is available and the bond market cannot clear.

On the other hand, if $q_t^b < 1$, then $\mu_{it}^s > 0$. Buying bonds yields a higher return than storage, so neither the households nor entrepreneurs store any of the numeraire. As will be seen below, storage by these two types of agents is the only source of input to producing investment goods in "II". Therefore, in any steady state of the model, it cannot be that $q^b < 1$, as the capital stock would be decreasing due to depreciation.

Since I will again be using perturbation methods around the deterministic (zero aggregate shocks) steady state to solve the model, from this point on I will conjecture that $q_t^b = 1$ for all t and then verify that at this price the bond market clears in each t.[17] When $q_t^b = 1$, it is the case that $\mu_{it}^b = \mu_{it}^s$ and any agent for whom the constraints are not binding is indifferent between storing or acquiring bonds. Similarly, an agent's borrowing constraint is binding if and only if the non-negativity constraint on storage is binding. So in general, only an individual's "net storage" $\tilde{s}_{it} = b_{it} + s_{it}$ is pinned down in equilibrium.

Following a similar derivation as presented by Lagos and Wright (2005), one can verify that if in equilibrium, households in t face a non-zero probability of $\mu_{it+1}^b > 0$, then $\mathbb{E}_t[\mathcal{V}_{t+1}(d_{it+1}, \theta_{it+1})]$

[17]The existence of a steady state of the underlying real business cycle framework can be established by conventional methods. In general, the uniqueness of the steady state cannot be guaranteed because of the liquidity services provided by capital. However, in all of the quantitative applications presented in this paper, I have verified the uniqueness of the steady state numerically.

is strictly concave in d_{it+1}, and all choose the same level of deposits. Also, whenever $q_t^b = 1$, then given the calibration of χ and $\bar{B}$ I use, it must be that there is a non-zero mass of households for whom $U'(c_{it}) = \frac{D}{p_t}$, as otherwise all households would be at their borrowing constraints and the entrepreneurs' supply of funds in "I" would not be enough to clear the intra-period bond market – a condition that can be verified quantitatively below. Thus, because in equilibrium the only difference between the employed and unemployed in "I" is that the former have higher income, we have that at least for the employed households, $\mu_{it}^s = \mu_{it}^b = 0$ and their identical consumption level, denoted c_{wt}, satisfies:

$$U'(c_{wt}) = \frac{D}{p_t} \tag{33}$$

Then, after solving and simulating the model, one can check that in each subperiod "I", it is the case that the net storage of the employed and the entrepreneurs is strictly higher than the borrowing capacity of the unemployed. This then verifies that the intra-period bond market clears at the price of $q_t^b = 1$, and any "excess net storage" above the supply of bonds in the intra-period bond market is kept in the form of physical storage.

By using the fact that in equilibrium $U'(c_{wt}) = \frac{D}{p_t}$ and $\mu_{it}^b = U'(c_{it}) - \frac{D}{p_t}$, we can rewrite household i's Euler equation as:

$$1 = \mathbb{E}_t^i \left[(R_{t+1}^{ld} + \chi R_{t+1}^{nd})\beta \frac{U'(c_{it+1})}{U'(c_{wt})} + (1-\chi) R_{t+1}^{nd} \beta \frac{U'(c_{wt+1})}{U'(c_{wt})} \right]$$

And finally, because all households choose the same deposits d_{t+1} in equilibrium and therefore also the same level of consumption whenever unemployed, c_{ut}, there is a unique Euler equation which, after applying the law of iterated expectations and integrating out the idiosyncratic unemployment risk, can be written as:

$$1 = \mathbb{E}_t \left[(R_{t+1}^{ld} + \chi R_{t+1}^{nd}) M_{t+1} + (1-\chi) R_{t+1}^{nd} Z_{t+1} \right] \tag{34}$$

$$\text{where } M_{t+1} \equiv \beta \frac{(1-u_{t+1})U'(c_{wt+1}) + u_{t+1} U'(c_{ut+1})}{U'(c_{wt})} \tag{35}$$

$$Z_{t+1} \equiv \beta \frac{U'(c_{wt+1})}{U'(c_{wt})} \tag{36}$$

$$\text{and } c_{ut} = \begin{cases} c_{wt} = U'^{-1}\left(\frac{D}{p_t}\right) & \text{if } U'((R_t^{ld} + \chi R_t^{ld})d_t + \bar{B}) \le \frac{D}{p_t} \\ (R_t^{ld} + \chi R_t^{ld})d_t + \bar{B} & \text{otherwise} \end{cases} \tag{37}$$

The households thus use two separate stochastic discount factors M_{t+1} and Z_{t+1} to price returns in subperiod "I" and "II", respectively. And it is through M_{t+1} that (countercyclical) idiosyncratic risk affects asset pricing and households' willingness to bear aggregate risk in this framework.

Note that if one lets χ or $\bar{B}$ become large, the unemployed become unconstrained and consume the same amount of the numeraire as the employed. There will be idiosyncratic risk in the amount an individual has to produce of the X good, but this only leads to constant marginal disutility and does not affect any outcomes regarding the numeraire. I will refer to the case in which the unemployed are unconstrained and able to reach $c_{ut} = c_{wt}, \forall t$, as the *representative household* case of the model with frictional labor markets. This is because all outcomes regarding the numeraire will be identical to an economy where there is perfect idiosyncratic risk sharing and no heterogeneity among the households.

And finally, since all households consume X^* of good X, their budget constraints in "II" imply:

$$X_{it}^s = X^* + \frac{1}{p_t}\left(d_{t+1} - R_t^{nd} d_t - \tilde{s}_{it} - \Pi_t^I\right) \tag{38}$$

with the net storage quantities satisfying:

$$\tilde{s}_{it} = R_t^{ld} d_t + w_t H \theta_{it} - c_{it} \tag{39}$$

4.2.2 Final Goods and New Capital Producers and Labor Intermediaries

The representative numeraire producer's and capital producers' problems are identical to those in the representative household of Section 2 and thus equilbrium conditions (7)–(11) apply.

Since all matches in the labor market only last for one subperiod, the value of a successful match to the labor intermediary is $(W_t - w_t)H$. Denoting the probability of filling a vacancy as π_t, the free entry condition for labor intermediaries then implies $\pi_t(W_t - w_t)H = \kappa_v$. Given the matching function $\mathcal{M}(s_t, v_t)$ and $s_t = 1$, in equilibrium $\pi_t = \dfrac{\mathcal{M}(1, v_t)}{v_t}$. Thus, posted vacancies are pinned down by:

$$\frac{\mathcal{M}(1, v_t)}{v_t}(W_t - w_t)H = \kappa_v \tag{40}$$

In equilibrium, the unemployment rate and the labor hired by the final goods producer must be consistent with the number of successful matches:

$$u_t = 1 - \mathcal{M}(1, v_t) \tag{41}$$

$$L_t = \mathcal{M}(1, v_t)H \tag{42}$$

Because of frictions in the labor market, there is a bargaining set of wages over which a labor intermediary and a worker find it mutually profitable to be matched. Because there is no disutility from labor supply by assumption and due to the assumption of one period lasting matches, this bargaining set is simply $w_t \in [0, W_t]$. As pointed out by Hall (2005), from a bargaining theoretical perspective, any wage within this set could possibly be the outcome of a bargain if one utilizes a demand-game auction mechanism. This serves to generate wage rigidities which help towards matching empirical unemployment dynamics, such as those explored by Shimer (2005). I follow Challe et al. (2017) and pick a specific equilibrium process for w_t, to be verified to lie the bargaining set:

$$w_t = w_{t-1}^{\gamma_w}\left[w_{ss}\left(\frac{1-u_t}{1-u_{ss}}\right)^{\psi_n}\right]^{1-\gamma_w} \tag{43}$$

γ_w is the degree of indexation to past wages, and ψ_n is the sensitivity of wages to the business cycle. w_{ss} and u_{ss} are the steady state values of w_t and u_t and will be calibrated below.

It is worthwhile to foreshadow the quantitative results in Section 5.2 and point out that because of the simplistic structure of the labor market, the equilibrium dynamics of numeraire output are relatively simple and almost independent of the aggregate risk sharing between entrepreneurs and households in this model specification. To see why, let us plug (7), (41), (42) and (43) into (40), and suppose that $\gamma_w = 0$:

$$\frac{\mathcal{M}(1, v_t)}{v_t} \left\{ (1-\alpha) A_t K_t^{\alpha} [\mathcal{M}(1, v_t) H]^{-\alpha} - \bar{w} \left(\frac{\mathcal{M}(1, v_t)}{1 - \bar{u}} \right)^{\psi_n} \right\} = \kappa_v$$

This equation determines equilibrium vacancies v_t as some function $\Psi_v(K_t, A_t)$ of the current level of capital and TFP. Most importantly, Ψ_v only depends on model primitives, i.e. assumptions on production and matching technologies and the law of motion of w_t, and is completely independent of any other variables or parameters. So if one considers the effect of a TFP shock in t, then conditional on K_t we have that v_t, L_t and numeraire output always respond the same way, no matter how wealth shares and consumption or investment levels respond. It is only through different investment responses in t affecting K_{t+j}, that two different aggregate risk sharing schemes can have differing output responses in $t + j$, for $j \geq 1$. One could take this to be a virtue, rather than a flaw, of the current setup as it allows one to focus on the specific research question of comparing optimal aggregate risk sharing outcomes across calibrations without general equilibrium forces, such as wealth effects in labor supply, obscuring the analysis. I will discuss extensions of the model that allow aggregate risk sharing and the financial accelerator mechanism to affect numeraire output in this version of the model in Section 6.

4.2.3 Entrepreneurs and the Loan Contract

For entrepreneurs, the analysis from Section 2.2.3 applies almost directly. Because in equilibrium the intra-period net interest rate is zero, in each subperiod "I", entrepreneurs just collect their rental returns and carry these over to "II", either through storage or the intra-period bond market. In "II" they receive capital gains from liquidating their capital share, pay dividends to their family and consume C_t^e. Thus, by again denoting the value function of an entrepreneur with equity E_t^j in subperiod "II" in t, before paying dividends to the family, by $V_t(E_t^j)$, the Bellman equation (12), with an alternative participation constraint shown below applies. One can follow the exact same steps to show that optimal contract is still risky debt, and all entrepreneurs choose the same $\{\bar{\omega}_{t+1}\}$ and κ_t.

The only difference now is the participation constraint of the lender when contracting with j. Given the risky debt structure, for $\omega_{t+1}^j < \bar{\omega}_{t+1}^j$, the entrepreneur declares bankruptcy, the lender incurs the monitoring cost and gets all of the remaining capital, which yields him income flows of $(1-\mu)\omega_{t+1}^j r_{t+1} K_{t+1}^j$ and $(1-\mu)\omega_{t+1}^j (1-\delta) Q_{t+1} K_{t+1}^j$ in "I" and "II", respectively. If $\omega_{t+1}^j \geq \bar{\omega}_{t+1}^j$, no monitoring occurs, the borrower repays the promised amount $\bar{\omega}_{t+1}^j K_{t+1}^j$ and holds on to the remaining capital which yields him a total income flow of $(\omega_{t+1}^j - \bar{\omega}^j)[r_{t+1} + (1-\delta) Q_{t+1}] K_{t+1}^j = (\omega_{t+1}^j - \bar{\omega}^j) R_{t+1}^k Q_t K_{t+1}^j$ across "I" and "II". Again, we can define the implicit interest rate $R_{t+1}^{def,j}$ earned by the lender over "I" and "II" that is subject to default risk, by: $R_{t+1}^{def,j}(Q_t K_{t+1}^j - N_t^j) = \bar{\omega}_{t+1}^j R_{t+1}^k Q_t K_{t+1}^j$.

Denoting the lender's returns from lending to j paid in "I" and "II" in $t + 1$, respectively, as

$R^{ll,j}_{t+1}$ and $R^{nl,j}_{t+1}$ we have:

$$R^{ll,j}_{t+1} \equiv \frac{[\Gamma(\bar{\omega}^j_{t+1}) - \mu G(\bar{\omega}^j_{t+1})] r_{t+1} K^j_{t+1}}{Q_t K^j_{t+1} - N^j_t} = [\Gamma(\bar{\omega}^j_{t+1}) - \mu G(\bar{\omega}^j_{t+1})] \frac{r_{t+1}}{Q_t} \frac{\kappa^j_t}{\kappa^j_t - 1}$$
$$R^{nl,j}_{t+1} \equiv \frac{[\Gamma(\bar{\omega}^j_{t+1}) - \mu G(\bar{\omega}^j_{t+1})](1-\delta) Q_{t+1} K^j_{t+1}}{Q_t K^j_{t+1} - N^j_t} = [\Gamma(\bar{\omega}^j_{t+1}) - \mu G(\bar{\omega}^j_{t+1})] \frac{(1-\delta) Q_{t+1}}{Q_t} \frac{\kappa^j_t}{\kappa^j_t - 1}$$

The relevant lender participation constraint is therefore:

$$\begin{aligned}
&\mathbb{E}_t \left[(R^{ll,j}_{t+1} + \chi R^{nl,j}_{t+1}) M_{t+1} + (1-\chi) R^{nl,j}_{t+1} Z_{t+1} \right] = \\
&= \mathbb{E}_t \left\{ [\Gamma(\bar{\omega}^j_{t+1}) - \mu G(\bar{\omega}^j_{t+1})] \left[\frac{(r_{t+1} + \chi(1-\delta) Q_{t+1}) M_{t+1} + (1-\chi)(1-\delta) Q_{t+1} Z_{t+1}}{Q_t} \right] \frac{\kappa^j_t}{\kappa^j_t - 1} \right\} \\
&\geq \mathbb{E}_t [(R^{ll}_{t+1} + \chi R^{nl}_{t+1}) M_{t+1} + (1-\chi) R^{nl}_{t+1} Z_{t+1}] = 1
\end{aligned}$$

It again follows as the result of the intermediary being a pass-through entity, combining with the facts that in equilibrium all contracts will offer the same expected return to the lender $R^{ll}_t = R^{ll,j}_t, R^{nl}_t = R^{nl,j}_t, \forall j$ and as elaborated above $R^{ll}_t = R^{ld}_t, R^{nl}_t = R^{nd}_t$ in equilibrium, and finally employing the household's Euler equation. Alternatively, this participation constraint can be derived as the result of the intermediary's equity value maximization subject to being owned by the households, as presented in Appendix D.

Furthermore, by defining an "adjusted" household stochastic discount factor $\tilde{M}_{t+1}$, the participation constraint can written exactly as in the representative household case:

$$\mathbb{E}_t \left\{ [\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] R^k_{t+1} \tilde{M}_{t+1} \kappa_t \right\} \geq \kappa_t - 1 \tag{44}$$
$$\text{where } \tilde{M}_{t+1} \equiv \frac{(r_{t+1} + \chi(1-\delta) Q_{t+1}) M_{t+1} + (1-\chi)(1-\delta) Q_{t+1} Z_{t+1}}{r_{t+1} + (1-\delta) Q_{t+1}} \tag{45}$$

Note that in the representative household case, $\tilde{M}_{t+1} = M_{t+1} = Z_{t+1}$. (44) has exactly the same form as the participation constraint in the benchmark representative household case in Section 2.2.3, only with an effective stochastic discount factor $\tilde{M}_{t+1}$ that is an average of M_{t+1} and Z_{t+1}, weighted by the returns in "I" and "II", respectively. Therefore, the characterization of the optimal contract and the entrepreneurs' problem from Section 2.2.3 applies directly, implying that the equilibrium conditions (15)–(17) are unchanged and (18) is replaced with

$$\frac{\Gamma'(\bar{\omega}_{t+1})}{\Gamma'(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})} = \left(\beta_e \frac{\tilde{U}'(C^e_{t+1})}{\tilde{U}'(C^e_t)} \right)^{-1} \tilde{M}_{t+1} \tag{46}$$

The equilibrium lender returns are:

$$R^{ll}_{t+1} = [\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] \frac{r_{t+1}}{Q_t} \frac{\kappa_t}{\kappa_t - 1} \tag{47}$$
$$R^{nl}_{t+1} = [\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] \frac{(1-\delta) Q_{t+1}}{Q_t} \frac{\kappa_t}{\kappa_t - 1} \tag{48}$$

Although the timing of the returns matters for the lending households, it is still useful to focus on the dynamics of the lenders' total $t+1$ return R^l_{t+1}, exactly as in the representative household case:

$$R^l_{t+1} \equiv R^{ll}_{t+1} + R^{nl}_{t+1} = [\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] R^k_{t+1} \frac{\kappa_t}{\kappa_t - 1}$$

R^l_{t+1} is again a central measure of the degree of aggregate risk sharing, and characterizes the dynamics of entrepreneurial liabilities and net worth.

4.2.4 Market Clearing and Equilibrium Definition

Again, in equilibrium, households' deposits fund the entrepreneurs' projects:

$$\int_0^1 d_{it+1} di = d_{t+1} = Q_t K_{t+1} - N_t \tag{49}$$

Combining this condition, the households' and entrepreneurs' budget constraints with the labor intermediaries' free entry condition, new capital producers' profits, good X market clearing and the intra-period bond market clearing, one arrives at the numeraire market clearing conditions in "I" and "II", respectively:

$$\int_0^1 c_{it} di + \int_0^1 s_{it} di + s^e_t + \kappa_v v_t + \mu G(\bar{\omega}_t) r_t K_t = A_t K^\alpha_t L^{1-\alpha}_t$$
$$C^e_t + I_t \vartheta \left(\frac{I_t}{I_{ss}}\right) + \mu G(\bar{\omega}_t)(1-\delta) Q_t K_t = \int_0^1 s_{it} di + s^e_t$$

where s^e_t is aggregate entrepreneurial storage. Since agents with positive net storage are indifferent between buying intra-period bonds and storing when $q^b_t = 1$, only aggregate storage $\int_0^1 s_{it} di + s^e_t$ is pinned down in equilibrium. One can combine the two resource constraints and use the fact that all employed and unemployed households consume the same amounts, to get the effective period t numeraire resource constraint:

$$(1-u_t) c_{wt} + u_t c_{ut} + C^e_t + \kappa_v v_t + I_t \vartheta \left(\frac{I_t}{I_{ss}}\right) + \mu G(\bar{\omega}_t) R^k_t Q_{t-1} K_t = A_t K^\alpha_t L^{1-\alpha}_t \tag{50}$$

which is sufficient to define and solve for the equilibrium. I will employ an equilibrium definition in which $q^b_t = 1$ and which already accounts for the fact that all employed and unemployed households are identical.

Definition 2. *A* competitive equilibrium of the heterogeneous household model *is a collection of stochastic processes for:*

1. *a price system* $\{r_t, W_t, w_t, p_t, R^k_t, R^{ll}_t, R^{nl}_t, Q_t\}$,
2. *household consumption, production and deposit quantities* $\{c_{wt},\ c_{ut},\ d_{t+1},\ X^d_{wt},\ X^d_{ut},\ X^s_{wt},\ X^s_{ut}\}$ *and net storage quantities* $\{\tilde{s}_{wt}, \tilde{s}_{ut}\}$
3. *stochastic discount factors* $\{M_t, Z_t, \tilde{M}_{t+1}\}$
4. *entrepreneurial consumption, net worth and leverage quantities and contractual cutoffs* $\{C^e_t,\ N_t,\ \kappa_t,\ \bar{\omega}_{t+1}\}$

5. *unemployment and vacancy levels* $\{u_t, v_t\}$

6. *aggregate labor, investment and capital quantities* $\{L_t, I_t, K_{t+1}\}$ *and capital producer profits* $\{\Pi_t^I\}$

such that equations: (1), (7)–(11), (15)–(17) (26), (33)–(43), (45) – (50), *with* $R_t^{ld} = R_t^{ll}$, $R_t^{nd} = R_t^{nl}$ *and* $i \in \{e, u\}$, *where applicable, are satisfied, given the stochastic process for* $\{A_t\}$, *and initial conditions* $(K_0, N_{-1}, d_0, \kappa_{-1}, \bar{\omega}_0, w_{-1})$.

5 Quantitative Analysis for the Heterogeneous Household Model

5.1 Calibration

Compared to the representative agent model's calibration in Section 3.1, I follow the same set of calibration targets as closely as possible. And I extend the set of targets to accommodate the different labor market structure. Also, the households' and entrepreneurs' utility function parameters considered vary.

Again, one time period t is considered to be a quarter. The following values follow directly from Section 3.1: $\alpha = 0.35$, $\phi_Q = 0.5$, $\delta = 0.025$. The benchmark TFP process is unchanged, with $\rho_A = 0.95$ in all that is to follow. (μ, β_e, σ) are chosen to match the same targets for a default premium, bankruptcy rates and leverage as in Section 3.1. The specific values vary in the different calibrations considered.

Households' and entrepreneurs' utility from consumption of the numeraire is CRRA: $U(c) = \frac{c^{1-\xi_h}}{1-\xi_h}$, $\tilde{U}(c) = \frac{c^{1-\xi_e}}{1-\xi_e}$, with various values of (ξ_h, ξ_e) considered below. I normalize H to 1. Because of idiosyncratic risk present in steady state without aggregate risk, the households' stochastic discount factor does not equal β in steady state. I set β so that the stochastic discount factor for returns in "I", M_t, is 0.99 in steady state, building on the analogy that in the representative household case, $M_{ss} = \beta = 0.99$.

The matching function is Cobb-Douglas: $\mathcal{M}(s_t, v_t) = \bar{m} s_t^{\psi_m} v_t^{1-\psi_m}$. I normalize $\bar{m}$ to 0.1, just so that with $s_t = 1$, the implied number of matches is sure to be smaller than that of vacancies, but the value of $\bar{m}$ has no effect on the analysis below. I set ψ_m to a non-controversial 0.5, used for example by Pissarides (2009), based on empirical literature covered in Petrongolo and Pissarides (2001). w_{ss} and κ_v are calibrated to yield a steady state unemployment rate of 6% and a share of vacancy posting costs in numeraire output of 1%, the latter following Challe et al. (2017).

The parameters γ_m and ψ_n, governing the wage paid to households, do not affect the steady state and must be calibrated based on labor market dynamics. Ideally, one would like to target second moments that have been the focus of a long discussion in the literature, starting with Shimer (2005). However, because of the stylized nature of the current specification of the model, this is not exactly straightforward. For example, in the standard Mortensen-Pissarides setting, one measures labor market tightness based on v_t/u_t since the unemployed are the ones looking for work. In this model, every worker in the economy is searching for a job ($s_t = 1$) and u_t will just be those who fail to find a successful match. Thus, labor market tightness in period t is $v_t/s_t = v_t$. Also, with multi-period job contracts, u_t is a stock variable that changes due to inflows and outflows, inheriting at least some persistence. In the current setting, it can naturally be a lot more volatile as u_t is determined only by period t outcomes. Nontheless, I set $\psi_n = 1.57$ to yield $\dfrac{\mathrm{std}(\log u_t)}{\mathrm{std}(\log p_t)} \approx 10$, where p_t is labor productivity measured as average output per worker, with the empirical target found by Shimer (2005). Since there is currently no strong reason why extra wage rigidity is needed, I set $\gamma_w = 0$. This yields a first order autocorrelation of $\log(u_t)$ of approximately 0.96, reasonably close to 0.94 found by Shimer (2005).

With CRRA utility, the quantitative relevance of idiosyncratic risk is captured by the probability of becoming unemployed, covered above, and the relative loss in consumption when one falls from employment to temporary unemployment. The key parameters that govern this consumption loss are χ and $\bar{B}$. For now, I set $\bar{B} = 0$ and calibrate χ to a value such that the steady state loss of consumption due to temporary unemployment is 21%, i.e. $c_{w,ss}/c_{e,ss} = 0.79$. This is a calibration target taken from estimation results by Chodorow-Reich and Karabarbounis (forthcoming), found to be the average decline in expenditures of nondurable goods and services during unemployment in the Consumer Expenditure Survey data, also used for example by Challe et al. (2017). In the benchmark heterogeneous household calibration with $\xi_h = \xi_e = 1.0$, this yields $\chi \approx 0.17$.

Because my analysis will focus on the allocations and interest rates regarding the numeraire, I normalize $D = 1.0$ and choose $\Upsilon(X) \equiv \phi_X \log(X)$. I set $\phi_X = 0.12$ which is large enough to always yield an interior solution for X^s_{it}, while keeping the value of aggregate X production relatively small, about 7% of aggregate output in the economy.

For a quantitative exercise, I also consider increasing the severity of idiosyncratic risk by allowing for the households' borrowing limit χ to move over the business cycle. As a simple test, I will simply tie the value of χ_t to that of TFP by imposing $\chi_t = \chi A_t^{\phi_\chi}$, with $\phi_\chi > 0$, where χ is the steady state calibrated value as explained above. A positive ϕ_χ is a reduced form stand-in for potentially more relaxed consumer credit standards in booms.[18] I then ask the question: how much procyclicality in the borrowing constraint, captured by ϕ_χ must there be so that for a given combination of (ξ_h, ξ_e), the optimal entrepreneurial financing contract implies a non-state-contingent lender's return.

5.2 Second Moments and Impulse Responses

Similarly as for the representative household model, the focus of the analysis is on whether and how severely the introduction of lender idiosyncratic risk affects the properties of privately optimal

[18]Given the very stylized structure of household finance in this model, households only borrow to finance consumption when they become unemployed. Because unemployment is countercyclical, total household credit also turns out to increase in recessions. In US data, unsecured household credit is procyclical, as for example pointed out by Nakajima and Ríos-Rull (2014). A procyclical borrowing limit allows to dampen this discrepancy between my model and the data. Although consumer credit is usually unsecured, in a more elaborate setting with an optimizing lender, borrower default risk and default costs, a household's access to credit could still depend on its financial position – which the Kiyotaki and Moore (1997)-type borrowing constraint allows to capture in a simplistic way. For a survey of potential theories explaining banks' countercyclical credit standards, see for example Berger and Udell (2004).

aggregate risk sharing. In the following I thus again report the standard deviations and impulse response functions of central balance sheet and real variables.

Table 2 below presents results from different calibrations, corresponding to various combinations of households' and entrepreneurs' coefficients of relative risk aversion. As mentioned above, the representative agent (RA) model is one in which χ is high enough that the unemployed households are not constrained in their consumption of the numeraire in any period. The heterogeneous agent (HA) model refers to the baseline calibration in which all unemployed households consume the same amount of the numeraire and are liquidity constrained in doing so in every period. And finally, the heterogeneous agent case with a time-varying borrowing limit χ_t is denoted by the value of ϕ_χ that is required to yield non-state-contingent lender returns in response to TFP shocks.

Table 2: Relative impulse response of lender return, relative standard deviations of log entrepreneurial net worth and leverage, absolute standard deviation of log numeraire output; simulation of 10^6 quarters.

ξ_h, ξ_e	Model	$\left(\frac{\partial r_t^l}{\partial \varepsilon_t^A}\right) \Big/ \left(\frac{\partial r_t^k}{\partial \varepsilon_t^A}\right)$	$\frac{\text{std}(n)}{\text{std}(y)}$	$\frac{\text{std}(\hat{\kappa})}{\text{std}(y)}$	std(y), %
1.0, 1.0	RA	0.813	0.952	0.049	3.752
	HA	0.763	0.897	0.066	3.660
	HA, $\phi_\chi = 15.9$	0.000	0.705	0.306	3.319
3.0, 3.0	RA	0.774	1.403	0.085	4.252
	HA	0.474	1.279	0.285	3.816
	HA, $\phi_\chi = 2.96$	0.000	1.296	0.520	3.617
2.73, 1.0	RA	0.323	1.294	0.237	4.062
	HA	0.000	1.197	0.371	3.714

RA – representative household; HA – heterogeneous households.

Firstly, focusing on the representative household specification with $\xi_h = \xi_e$, either at 1.0 or 3.0, we again see a considerable amount of risk sharing. As for the model with a representative household with disutility from labor supply, the relative responsiveness of the lenders' return is close to 0.8 if the entrepreneurs' CRRA parameter is equal to that of the households. Increasing both agents' ξ implies lower intertemporal elasticity of substitution, more numeraire consumption smoothing and thus more volatile investment and wealth levels. This explains why the relative volatility of net worth compared to numeraire output increases with higher ξ. The absolute increase in numeraire output volatility can be attributed to the large investment responses leading to larger swings in the capital stock, and thus a more amplified output process.

It is worthwhile pointing out that, regarding numeraire allocations and prices, the only difference between the representative household special case of the specification with a frictional labor market, as presented in Section 4, and the representative agent model presented in Section 2 is the nature of labor supply. In Section 2, the representative household's disutility from labor yields a constant elasticity of labor supply. In the case of a frictional labor market, an upward sloping labor supply curve effectively derives from the assumed law of motion for w_t and the free entry condition of labor intermediaries. Because of this, there is no wealth effect on labor supply in the latter model. Nontheless, as can be seen comparing the first lines in Tables 1 and 2, with $\xi_h = \xi_e = 1.0$, the two specifications yield very similar results, apart from the wealth effects affecting labor supply and thus numeraire output volatility. Because I calibrate relative unemployment volatility to match that reported by Shimer (2005), employment is more volatile in the latter case and directly independent of consumption levels and risk sharing, as discussed in Section 4.2.2.

As for introducing uninsurable idiosyncratic liquidity risk driven by unemployment fluctuations for the lending households, we see that at $\xi_h = \xi_e = 1.0$, the impact is not too large compared to the representative household case. Although the lender return responds relatively less at impact, the drop in the measure of risk sharing is only about 5 percentage points. The changes are small because unemployment fluctuations are calibrated to match unconditional second moments and are thus not too large in response to productivity shocks. As seen below, in response to a positive 1% TFP shock starting from steady state, employment drops by about 0.5 percentage points. Because the steady state probability of becoming unemployed is at a relatively small calibrated 6%, the first order effect of such an increase in the households' probability of suffering a one-period drop of 21% in consumption is small, especially when $\xi_h = 1$.

The slightly lower degree of risk sharing can be seen to lead to a barely higher relative volatility of leverage, and *decreases* in the relative volatility of net worth and absolute volatility of output. The latter two effects can be attributed to the fact that counter-cyclical precautionary motives tend to stabilize a real business cycle model, as for example discussed in more depth by Challe et al. (2017). When unemployment drops in response to a positive productivity shock, then because of the shock's persistence, employed agents face a lower risk of becoming unemployed in the following period, decreasing their motives to save, and increasing current consumption. As more resources in the economy are devoted to the employed households' consumption, relatively less is left to fund the entrepreneurs' investment, dampening the financial accelerator mechanism and capital accumulation over the cycle. Put differently, net worth fluctuations become relatively less volatile because there are smaller swings in *total financial wealth* in the economy, while the volatility of leverage (the reciprocal of the entrepreneurial wealth share) increases because entrepreneurs take on relatively more of the, now smaller, aggregate financial risk. Yet even though at shock impact, entrepreneurs hold on to more net worth, over time their net worth reverts faster as households require higher returns from lending and lower investment levels keep capital prices and returns dampened. This extra motive to consume in booms is absent for a representative household.

A pro-cyclical borrowing limit χ_t leads to less financial risk sharing because the households who end up unemployed can reach higher consumption of the numeraire by borrowing more, and thus valuing payouts from entrepreneurs less. Under log-utility, non-state-contingent lender returns are reached with a borrowing limit elasticity of $\phi_\chi = 15.9$, implying that a 1% increase in TFP leads χ_t to increase from about 0.17 in steady state to 0.197 at the impact of the productivity shock. However, this pro-cyclical force reducing consumption losses from unemployment spells leads to even larger counter-cyclical fluctuations in precautionary savings motives, decreasing the relative volatility of net worth and the absolute volatility of output. These dampening effects from counter-cyclical precautionary motives become more evident in the impulse response functions below.

The next collection of results, with $\xi_h = \xi_e = 3.0$ demonstrates that when all agents in the economy become more averse to fluctuations in consumption, the effects of introducing idiosyncratic lender risk becomes consequential for aggregate financial risk sharing. Compared to the representative household case under the identical preferences, the measure of risk sharing drops by 30 percentage points. This is because the same, calibrated unemployment fluctuations matter a lot more for households with $\xi_h = 3.0$. When unemployment drops in booms, fewer households face the costly drop of approximately 21% in consumption and thus the valuation of payouts (the discount factor $\tilde{M}_t$) drops relatively more, compared to when these consumption drops are less costly under $\xi_h = 1.0$. More net worth is left in the hands of entrepreneurs at the impact of a positive productivity shock, leading to an approximately 3.3-fold increase in the relative volatility of leverage. However, the dampening precautionary savings motives remain, leading to decreases in the relative volatility of net worth and absolute volatility of output, compared to the representative household case. As expected, a significantly lower borrowing limit elasticity is required to reach non-state-contingent lender returns. A ϕ_χ of 2.96 means that a 1% increase in TFP is accompanied by an increase in χ_t from about 0.255, which is the steady state χ_t for this calibration, to about 0.262.

Although increasing $\xi_h = \xi_e$ further, all else equal, decreases the degree of financial risk sharing, it does so at a diminishing rate and a fully non-state-contingent lender return does not arise for values of ξ below 8.0, with $\phi_\chi = 0.0$. This happens because the increased desire of (employed) households and entrepreneurs to smooth consumption leads to amplified investment and capital price swings. These inflate the relative importance of financial wealth fluctuations compared to those in human wealth, which still matter for the consumption choices of the employed households. As discussed in Section 2.3.2, this generates a motive for more aggregate financial risk sharing.

The last combination of ξ_h and ξ_e in Table 2 asks what ξ_h must be, while keeping entrepreneurs at log-utility, to yield optimality of the non-state-contingent lender return – an exercise analogous to those executed in Section 3.2. A relatively low value of ξ_h will do in this case, yielding an amplification of about 1.33 times relative net worth volatility, 5.6 times for relative leverage volatility and a relative 1.5% increase in absolute output volatility, compared to the $\xi_h = \xi_e = 1.0$ case under household heterogeneity. Although for the representative household with $\xi_h = 2.73$ there is also significant dampening of risk sharing compared to the log-log utility case similarly as in the model of Section 2, one needs a higher ξ_h to reach non-state-contingent lender returns.

Figure 4 presents impulse responses to a 1% positive TFP shock over 28 quarters in the frictional labor market model for the representative household, heterogeneous household, and the heterogeneous household with pro-cyclical borrowing constraint $\phi_\chi = 2.96$, specifications. The lower left panel again shows the relative response of the quarterly lender return r_t^l. As mentioned above, the representative agent case exhibits a non-trivial degree of risk sharing with about 80% of the innovation in capital returns, seen in the middle right panel, paid out to the households. And the engineered non-state-contingency of the lender return is seen for the pro-cyclical borrowing constraint specification. The effects of less aggregate risk sharing on the entrepreneurs' balance sheets in the cases with heterogeneous households can be seen by the amplified short-run responses of enrepreneurial net worth and leverage.

In the representative household case, the unemployed can borrow at will to benefit from the relatively abundant and cheap numeraire supply and reach the expanding consumption level of that of the employed. For the baseline heterogeneous household case, the borrowing constraint is fixed and the fluctuations in lender returns received in "I" not too significant, so the expansion in the consumption of the unemployed remains modest. In the case with a pro-cyclical χ_t, one sees how the unemployed are able to borrow more, increasing the consumption per unemployed

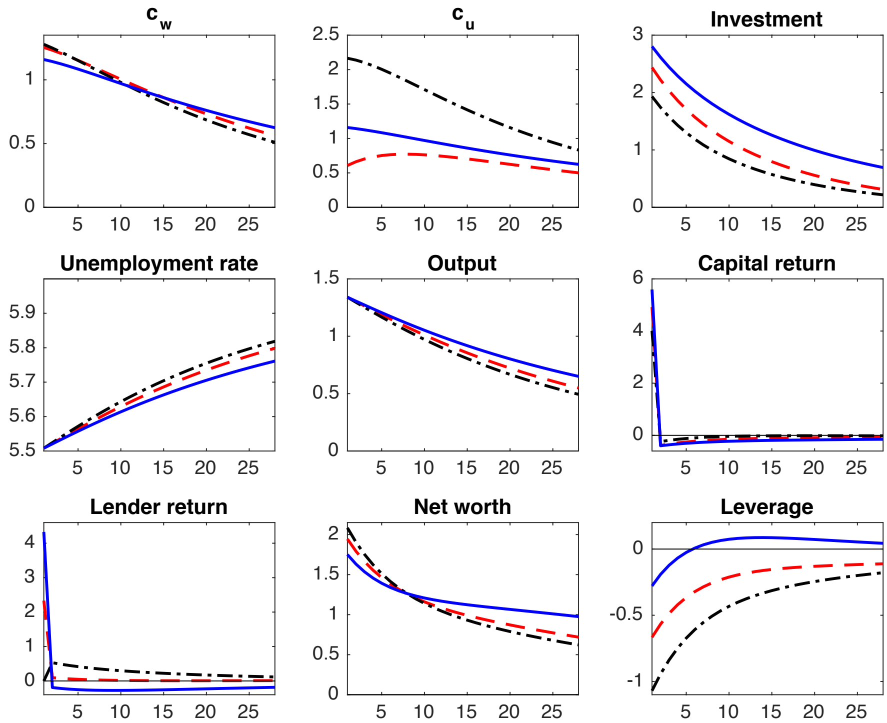

Figure 4: Impulse responses to 1% positive TFP shock in heterogeneous household model, with $\xi_h = \xi_e = 3.0$, $\rho_A = 0.95$; all except unemployment rate in log-deviations from steady state, returns annualized; Horizontal axis – quarters; Blue solid – RA, red dashed – HA, black dash-dotted – HA with $\phi_\chi = 2.96$.

household. The fact that at shock impact, the unemployment rate and output respond in exactly the same manner across all calibrations is a reflection of the straightforward labor market clearing and absence of any wealth effects in labor supply, as discussed in Section 4.2.2. Differences in unemployment and output can only arise due to differing capital accumulation.

The dampening effects from counter-cyclical precautionary savings motives in the heterogeneous household specifications are evident. At shock impact, the employed households' consumption increases relatively more because of a decreased risk of becoming unemployed, decreasing their savings and resources left to finance entrepreneurs' investment. Furthermore, for the case of a pro-cyclical χ_t, the drop in precautionary savings is larger, and also the unemployed can increase their consumption more, leading to even further dampening of investment responses. As discussed above, the drop in precautionary savings motives also shapes the economy's transition back to steady state. The higher relative path of the equilibrium lender return in the periods after the shock's impact is evidence of the decreased willingness of households to save as much in the heterogeneous agent cases. They thus require the entrepreneurs to pay out a relatively larger share of capital returns, drawing down the net worth of the latter faster, speeding up the transition of investment, unemployment and output.

To sum up, the above therefore illustrates how a variation of the benchmark representative household framework that does not feature wealth effects in labor supply can further justify privately optimal contracts in which entrepreneurs borrowing from households take on a large share of the aggregate financial risk. Fully non-state-contingent lender returns are reached under plausible degrees of households' relative risk aversion, while keeping entreprenurs at log-utility, or introducing relatively small pro-cyclical household borrowing-limit fluctuations.

6 Conclusion

This paper revisits the analysis of privately optimal aggregate financial risk sharing between borrowers and lenders in the workhorse Bernanke et al. (1999) (BGG) framework with financial frictions. By modelling the borrowing entrepreneurs explicitly as agents with strictly concave momentary utility, while being able to share idiosyncratic risk among each other, the setup is more naturally cast as a two-agent risk sharing problem. I then show that the standard formulation of independent risk neutral entrepreneurs, as set up by BGG and analyzed by Carlstrom et al. (2016) (CFP) is, to a first order, equivalent to a reformulation in which entrepreneurs have logarithmic utility. This directly implies that if one were to study the BGG model under lender preferences that are not logarithmic, the high degree of risk sharing result found by CFP might not necessarily follow.

I endow the representative lending household, who in addition owns human wealth, with Epstein and Zin (1989)-type preferences. Quantitative results from a calibration exercise demonstrate that in response to total factor productivity shocks, non-state-contingent lender returns, as initially imposed by BGG, are privately optimal if entrepreneurs have logarithmic utility and the household a coefficient of risk aversion parameter of 13.2 with unitary elasticity of intertemporal subsitution. Alternatively, the same result follows when the household has CRRA-type expected utility preferences with a coefficient of 5.92.

The analysis also illustrates how the sharing of financial risk is affected by the fluctuations in human wealth relative to total financial wealth in the BGG economy, which in general tend to comove closely in a real business cycle framework. To separate households' consumption choices from the aggregate human wealth, I build on the insights of Lagos and Wright (2005), and introduce uninsurable idiosyncratic lender risk brought about by liquidity constraints and temporary unemployment spells in a frictional labor market. This introduces limited household heterogeneity that yields analytical tractability and allows to study the effects of added counter-cyclical idiosyncratic risk on aggregate risk sharing. Under expected utility preferences, the benchmark calibration of the heterogeneous lender model yields non-state-contingent lender returns in response to TFP shocks for households' relative risk aversion levels of less than 3.0.

For the purpose of presenting the framework and illustrating the relevance of lender idiosyncratic risk in aggregate risk sharing, I have refrained from adding additional shocks or details to the model. Because of its stylized nature, this benchmark heterogeneous household setting is missing the standard amplification effects of financial frictions, which in the representative household approach arise due to wealth effects in labor supply. An extension of this model with wage bargaining between households and their employers reintroduces these effects because worker bargaining positions will depend on how aggregate risk is shared in the financial contract. Also, it is straightforward to introduce nominal rigidities and additional shocks to study whether the discussed reduction in aggregate risk sharing implies a quantitatively significant amplification mechanism in more elaborate settings. Preliminary results for some alternative conventional business cycle shocks seem to indicate that a significant degree of aggregate risk sharing remains even if households are made comparatively more averse to consumption flucutations than the entrepreneurs.

Appendix

A HP-filtered Second Moments from the Representative Household Model and US Data

Table 3: Relative standard deviations of entrepreneurial net worth and leverage, absolute standard deviation of output (in percentages), in representative agent model; HP-filtered model data from simulation of 10^6 quarters; US data on non-financial corporate and noncorporate net worth and leverage from FOFA, output as GVA of non-farm business sector from NIPA-BEA; HP parameter 1,600.

ρ_A	ξ, ψ	$\frac{\text{std}(n)}{\text{std}(y)}$	$\frac{\text{std}(\hat{\kappa})}{\text{std}(y)}$	$\text{std}(y), \%$
	Data (76Q1–15Q3, corporate)	1.944	0.823	1.884
	Data (76Q1–15Q3, noncorporate)	2.424	1.087	
0.95	1.0, 1.0	0.845	0.109	0.958
	13.2, 1.0	2.098	0.996	1.107
	5.92, $1/\xi$	2.069	1.005	1.115
0.99	1.0, 1.0	0.838	0.152	0.940
	4.34, 1.0	1.941	0.941	1.062
	2.57, $1/\xi$	1.549	0.767	0.989

B Entrepreneurs' Problem in the CFP Model and Equivalence to the log-Utility Entrepreneurial Family

B.1 CFP Model

B.1.1 Entrepreneurs' equilibrium conditions

For more details on the entrepreneurs' problem in the CFP model see Carlstrom et al. (2016). The bottom line is that in the CFP model, the optimality conditions for an entrepreneur's problem can be combined into the following Bellman equation, laws of motion and first order condition in the equilibrium variables $\{V_t, C_t^e, N_t, \bar{\omega}_t, \kappa_t\}$:

$$V_t = (1-\gamma) + \gamma\beta_e^{CFP}\mathbb{E}_t\left\{V_{t+1}[1-\Gamma(\bar{\omega}_{t+1})]R_{t+1}^k\right\}\kappa_t \tag{51}$$

$$C_t^e = (1-\gamma)[1-\Gamma(\bar{\omega}_t)]R_t^k\kappa_{t-1}N_{t-1} \tag{52}$$

$$N_t = [1-\Gamma(\bar{\omega}_t)]R_t^k\kappa_{t-1}N_{t-1} - C_t^e \tag{53}$$

$$\frac{\Gamma'(\bar{\omega}_t)}{\Gamma'(\bar{\omega}_t)-\mu G(\bar{\omega}_t)} = \left(\gamma\beta_e^{CFP}\frac{V_t}{V_{t-1}-(1-\gamma)}\right)^{-1} M_t \tag{54}$$

Plus the participation constraint (14), which effectively determines κ_t. Since these and all other equilibrium conditions are necessarily identical across the two models, I will not focus those. For brevity, let us denote the left hand side of (54) with the increasing function $\Psi(\bar{\omega}_t)$.

B.1.2 Steady state

In the non-stochastic steady state, combining (52) and (53) gives:

$$1 = \gamma[1 - \Gamma(\bar{\omega})]R^k\kappa \tag{55}$$

And using this in (51) yields:

$$V = (1-\gamma) + \gamma\beta_e V[1 - \Gamma(\bar{\omega})]R^k\kappa \Rightarrow V = \frac{1-\gamma}{1-\beta_e} \tag{56}$$

And (54), combined with (51) yields:

$$\Psi(\bar{\omega}) = M\left(\frac{\gamma\beta_e V}{V-(1-\gamma)}\right)^{-1} = M\left(\frac{\gamma\beta_e V}{\beta_e V}\right)^{-1} = M\gamma^{-1} \tag{57}$$

Thus, (56) separately determines V, and (55) and (57) alongside the remaining equilibrium conditions determine the rest of the steady state values.

B.1.3 First order dynamics

As mentioned in Section 2.3.1, (52) and (53) are exactly identical in the two models, given $\gamma = \beta_e$, so their equivalence follows trivially. Also, to save on notation, I will denote $X_t \equiv [1-\Gamma(\bar{\omega}_t)]R_t^k\kappa_{t-1}$ in log-linearizing (51), as this product shows up in the same manner in both of the models.

Log-linearizing (51) gives, using the fact that in steady state $\gamma X = 1$:

$$v_t = \beta_e^{CFP}\mathbb{E}_t\{v_{t+1} + x_{t+1}\} \tag{58}$$

And log-linearizing (54), using the fact that in steady state $V - (1-\gamma) = \beta_e V$, yields:

$$\frac{\Psi'(\bar{\omega})\bar{\omega}}{\Psi(\bar{\omega})}\hat{\omega}_t = m_t - \left(v_t - \frac{1}{\beta_e}v_{t-1}\right) \tag{59}$$

B.2 Entrepreneurial family model

B.2.1 Entrepreneurs' equilibrium conditions

Following the analysis in Sections 2.2.3 and 2.3.1, if the entrepreneurs have logarithmic utility $\tilde{U}(C) = \log C$, one can write the equilibrium conditions determining $\{V_t, C_t^e, N_t, \bar{\omega}_{t+1}, \kappa_t\}$ as:

$$V_t = \frac{1}{C_t^e} \tag{60}$$

$$C_t^e = (1-\beta_e)[1-\Gamma(\bar{\omega}_t)]R_t^k\kappa_{t-1}N_{t-1} \tag{61}$$

$$N_t = [1-\Gamma(\bar{\omega}_t)]R_t^k\kappa_{t-1}N_{t-1} - C_t^e \tag{62}$$

$$\Psi(\bar{\omega}_t) = M_t\left(\beta_e\frac{V_t}{V_{t-1}}\right)^{-1} \tag{63}$$

Plus the participation constraint (14), which again effectively pins down κ_t. As discussed in Section 2.3.1, the result that under log-utility, consumption is a constant fraction of equity can be reached by employing (62) and the entrepreneurs' Euler equation (15), with the latter now being replaced by (61). As mentioned, (61) and (62) are identical across the two models.

B.2.2 Steady state

In steady state combining (61) and (62) implies:

$$1 = \beta_e[1 - \Gamma(\bar{\omega})]R^k \kappa \tag{64}$$

And (63) implies

$$\Psi(\bar{\omega}) = M\beta_e^{-1} \tag{65}$$

which are identical to (55) and (57) whenever $\gamma = \beta_e$, so the two models have *exactly* the same steady states, apart from the value of V which in this case is pinned down by

$$V = \frac{1}{C^e} \tag{66}$$

B.2.3 First order dynamics

Log-linearizing (63) directly yields:

$$\frac{\Psi'(\bar{\omega})\bar{\omega}}{\Psi(\bar{\omega})}\hat{\omega}_t = m_t - (v_t - v_{t-1}) \tag{67}$$

which is equivalent to (59) whenever $\beta_e^{CFP} \to 1$.

And finally, because the Euler equation for the entrepreneur must still be satisfied by V_t, even though now redundant, it is necessarily the case that V_t satisfies

$$\begin{aligned} V_t &= \beta_e \mathbb{E}_t \left\{ V_{t+1}[1 - \Gamma(\bar{\omega}_{t+1})]R^k_{t+1} \right\} \kappa_t \\ \Rightarrow v_t &= \mathbb{E}_t \left\{ v_{t+1} + x_{t+1} \right\} \end{aligned} \tag{68}$$

which is equivalent to (59) whenever $\beta_e^{CFP} \to 1$.

So we have established the equivalence of the five equilibrium conditions in these two log-linearized models whenever $\beta_e^{CFP} \to 1$ and $\gamma = \beta_e$.

C Log-linearization of Epstein-Zin Utility for Representative Household

To shed light on how the Epstein-Zin preferences (2) assumed in the representative household model affect equilibrium conditions in a first order approximation, I provide log-linearized versions of $\mathcal{V}_t$ and the stochastic discount factor M_t as defined by (2) and (6), respectively. The optimality condition for labor supply (4) is already covered in Section 3.1.

For convenience, let us denote $\tilde{\mathcal{V}}_t \equiv \mathbb{E}_t[\mathcal{V}_{t+1}^{1-\xi}]^{\frac{1}{1-\xi}}$. Using lowercase letters to denote the log-deviations of corresponding variables from their steady state values, the log-linearization of (2) can then be rewritten as:

$$v_t = (1-\beta)[c_t - \nu_l l_t] + \beta \tilde{v}_t$$
$$\text{where } \tilde{v}_t = \mathbb{E}_t[v_{t+1}]$$

with $\nu_l = -\frac{\Phi'(L_{ss})L_{ss}}{\Phi(L_{ss})}$ defined as in Section 3.1. As for M_t, we have

$$m_t = \left(\frac{1}{\psi} - \xi\right)(v_t - \mathbb{E}_{t-1}[v_t]) - \frac{1}{\psi}(c_t - c_{t-1}) - \left(1 - \frac{1}{\psi}\right)\nu_l(l_t - l_{t-1})$$

Therefore, as mentioned in Section 3.1, whenever the stochastic discount factor appears in period t (log-)linearized equilibrium conditions in reference to future values, we get

$$\mathbb{E}_t[m_{t+1}] = -\frac{1}{\psi}\mathbb{E}_t(c_{t+1} - c_t) - \left(1 - \frac{1}{\psi}\right)\nu_l \mathbb{E}_t(l_{t+1} - l_t)$$

and ξ plays no role. Yet because $v_t \neq \mathbb{E}_{t-1}[v_t]$ whenever there are unexpected aggregate shocks, ξ matters for the realization of m_t and equilibrium dynamics through the risk sharing implied by (18).

By the above it is also clear why the disutility from labor $\Phi(L)$ only affects the log-linearized equilibrium conditions through ν_l and η_l, as covered in Section 3.1. Finally, when $\psi = \xi = 1$, M_t does not depend on $\mathcal{V}_t$ nor L_t. Specifying a log-linearized optimality condition for labor supply as covered in Section 3.1 then implies equivalence to a preference specification where the household has momentary log-utility from consumption separable from labor disutility, exactly as used by BGG and CFP.

D Ownership-Based Financial Intermediary Optimization

Suppose there is a representative financial intermediary, owned by the households. And suppose that the returns on households' deposits in the intermediary in equilibrium were *predetermined* and taken as given by the agents. Denote these returns as R_t^{ld} and R_t^{nd}, to be paid in "I" and "II" in t, respectively, and determined in $t-1$. The households' Euler equation for deposits is still:

$$\mathbb{E}_t[(R_{t+1}^{ld} + \chi R_{t+1}^{nd})M_{t+1} + (1-\chi)R_{t+1}^{nd}Z_{t+1}] = 1$$

Also, suppose the intermediary pays dividends div_t^l and div_t^n, in "I" and "II" in t, respectively. By definition, the dividends are:

$$\begin{aligned} div_t^l &= \left(R_t^{ll} - R_t^{ld}\right) d_t \\ div_t^n &= \left(R_t^{nl} - R_t^{nd}\right) d_t \end{aligned}$$

where d_t are aggregate deposits, deposited in $t-1$ and R_t^{ll} and R_t^{nl} are the aggregate returns on the entrepreneurial lending portfolio, received by the financial intermediary in "I" and "II", respectively. The dividends are allowed to be negative, being interpreted as equity injections in the financial intermediary.

Because the lender is owned by the households, it chooses deposits d_{t+1} to maximize the value of its dividend stream:

$$\mathbb{E}_t \sum_{j=1}^{\infty} \left[(div_{t+j}^l + \chi div_{t+j}^n)M_{t+j} + (1-\chi)div_{t+j}^n Z_{t+j}\right] =$$
$$= \mathbb{E}_t \sum_{j=1}^{\infty} \left\{\left[R_{t+j}^{ll} - R_{t+j}^{ld} + \chi\left(R_{t+j}^{nl} - R_{t+j}^{nd}\right)\right] d_{t+j}M_{t+j} + (1-\chi)\left(R_{t+j}^{nl} - R_{t+j}^{nd}\right) d_{t+j}Z_{t+j}\right\}$$

Because of free entry, it must be the case that expected intermediary profits are zero:

$$\begin{aligned}
&\mathbb{E}_t \left\{\left[R_{t+1}^{ll} - R_{t+1}^{ld} + \chi\left(R_{t+j}^{nl} - R_{t+j}^{nd}\right)\right] M_{t+1} + (1-\chi)\left(R_{t+1}^{nl} - R_{t+1}^{nd}\right) Z_{t+1}\right\} = 0 \\
\Rightarrow\ &\mathbb{E}_t \left[(R_{t+1}^{ll} + \chi R_{t+1}^{nl})M_{t+1} + (1-\chi)R_{t+1}^{nl} Z_{t+1}\right] = \\
&\qquad = \mathbb{E}_t \left[(R_{t+1}^{ld} + \chi R_{t+1}^{nd})M_{t+1} + (1-\chi)R_{t+1}^{nd} Z_{t+1}\right] = 1
\end{aligned}$$

where the last equality follows from the households' Euler equation.

The households' total returns from deposits and dividends, in "I" and "II", respectively, are:

$$R_t^{ld}d_t + div_t^l = R_t^{ll}d_t$$
$$R_t^{nd}d_t + div_t^n = R_t^{nl}d_t$$

So, the households are effectively still getting exactly the return from the entrepreneurial lending portfolio, but now the returns are just split into the sum of a predetermined return on deposits and risky dividends from intermediary ownership.

When the intermediary is contracting with entrepreneur j, it takes the opportunity cost of its funds, the equilibrium market returns R_{t+1}^{ll} and R_{t+1}^{nl} as given, and requires that the return provided by entrepreneur j is sufficiently high:

$$\mathbb{E}_t \left[(R_{t+1}^{ll,j} + \chi R_{t+1}^{nl,j})M_{t+1} + (1-\chi)R_{t+1}^{nl,j} Z_{t+1}\right] \geq$$
$$\geq \mathbb{E}_t \left[(R_{t+1}^{ll} + \chi R_{t+1}^{nl})M_{t+1} + (1-\chi)R_{t+1}^{nl} Z_{t+1}\right] = 1$$

where $R_{t+1}^{j,ll}$ and $R_{t+1}^{j,nl}$ as functions of $\bar{\omega}_{t+1}^j$ and κ_t^j are defined in Section 4.2.3, and the last equality follows because of free entry and the households' Euler equation.

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