Intergenerational Risk Sharing, Stability and Optimality of Alternative Pension Systems*

by

John Hassler

and

Assar Lindbeck

Institute for International Economic Studies‡

This revision: December 12, 1997

JEL Classification number: H5, H6, H55.

Keywords: Pension systems, Pay-as-you-go, intergenerational

Abstract

In an analysis of the risk-sharing properties of different types of pension systems, we show that only a fixed-fee pay-as-you-go (PAYG) pension systems can provide intergenerational risk sharing for living individuals. Under some circumstances, however, other PAYG pension systems can enhance the expected welfare of all generations by reducing intergenerational income variability. We derive conditions for this to occur. We also analyze the stability of actuarially fair PAYG pension systems. It is shown that if an actuarially fair pension with a non-balanced budget system is dynamically stable, its accumulated surpluses will converge to the same fund as in a fully funded system. We also show that the welfare loss due to labor market distortions will, surprisingly, *increase* if the implicit marginal return in a compulsory system is raised above the average return.

---

* We thank Jasmin Häcker for detecting an error in equation A.5 of an earlier version of the paper.

‡ Assar Lindbeck is also a fellow at IUI, Stockholm. Address of authors: Institute for International Economic Studies, Stockholm University, S-106 91 Stockholm, Sweden. Telephone: +46-8-162070, Fax: +46-8-161443, Email: John.Hassler@iies.su.se and Assar@iies.su.se; homepages: http:/www.iies.su.se/

1. Introduction

One of the rationales behind compulsory pension systems (social security) is their contribution to intergenerational risk sharing. Private insurance cannot achieve this unless the contracting parties are alive both when the contract is to be signed *and* after the risky outcome has materialized. In a world with overlapping generations, these conditions may not be met for all types of risk.[1] For example, young individuals saving for their old age may face a risky aggregate return on saving – a risk that they would like to share with future generations. Similarly, returns on human capital may have a risky aggregate component that cannot be pooled within each generation. This raises the potential for intergenerational risk sharing – an issue discussed by, among others, Enders & Lapan (1982) and Gordon & Varian (1988). More recently, Attanasio and Davis (1996) found compelling empirical evidence that the potential for risk sharing between different birth cohorts (and educational groups) is far from fully exploited.

Given the potential for intergenerational risk sharing, how well do different compulsory pension systems contribute to it? Since we focus on transfers *between* different generations, we disregard transfers and insurance within generations. Moreover, important real life problems due to imperfect information, for example adverse selection and moral hazard, are not dealt with in this paper.

In section 2 we describe alternative pension systems and the effects of different shocks on the lifetime budget constraints of individuals. For this purpose we disregard how labor supply is affected by a compulsory pension system by assuming labor supply to be fixed. In section 3 we analyze the risk-sharing properties of the different systems. It is shown that only fixed-fee pay-as-you-go (PAYG) pension systems can provide risk sharing for living individuals – what we call ”true risk sharing” in this paper. Under certain circumstances, all PAYG pension systems can, however, provide mechanisms for reducing intergenerational in-

[1] Also if the generations are linked by altruistic relations as in Barro (1974), there may be a missing insurance market since individuals are generally not constrained by contracts signed by their ancestors.

come variability, which we call *ex-ante* risk sharing behind the famous veil of ignorance. We derive conditions for this to occur.

Some pension systems may be vulnerable to the stochastic environment in which they are supposed to operate. Actuarial PAYG pension systems that promise a certain implicit rate of return may, for instance, generate strongly fluctuating deficits/surpluses. Under what circumstances can different PAYG pension systems redeem promised returns without violating their own dynamic stability? This question is analyzed in section 4. We also show that if an actuarially fair pension system with a non-balanced budget is stable, it will accumulate surpluses that *automatically* converge to a fund of the same size as in a fully funded system.

Under special circumstances, it is, however, possible to construct actuarially fair PAYG pension systems with a balanced budget, which, of course, would be stable by construction. We derive conditions for the existence of such a system in section 4. It turns out that, unless the growth rate of the tax base always coincides with the capital market return, a balanced budget PAYG pension system can only be actuarially fair if it is of a specific size, as defined by the size of the pension fee.

In section 5 we analyze the optimality of pension systems. To do this, we relax the assumption of an inelastic labor supply. The first issue concerns the well-known fact that a compulsory pension system may distort labor supply if the return on pension fees differs from the return on capital markets. In reality, the average implicit rate of return in non-funded pension systems is typically lower than the average return on the capital market (Feldstein, 1996 and Auerbach & Kotlikoff, 1987). It seems reasonable to argue, however, that labor market distortions are connected the *marginal* rather than the average return to fees paid. Despite the lower average return, the system could, however, be constructed so that the marginal return to fees paid equals the capital market return. Can the labor market distortions created by a compulsory pension system then be mitigated by setting the marginal implicit return higher than the average? We show that the answer to this question is, to our surprise, unambiguously no. The marginal and average return on pension fees should instead coincide.

The second issue in section 5 is: how large should a pension system be in order to be optimal from the viewpoint of intergenerational risk sharing? We show that there is a simple optimality condition – the size of the pension system should be such that individuals perceive it as actuarially fair.

Section 6 summarizes and concludes.

2. Risk Characteristics of Balanced Budget Pension Systems

For the purpose of describing the effects of various shocks on the intertemporal and intergenerational distribution of income under alternative types of pension systems, we use a two-period overlapping generations model. The agent's consumption in the first period of life is denoted $c_{1,t}$, where the first subindex denotes whether the person is young or old, while the second subindex denotes the time period under consideration. In t+1, when the same individual is in his second period of life, his consumption is denoted $c_{2,t+1}$. To start with, we assume that the individual supplies inelastically one unit of labor in the first period for which he receives a wage, denoted w_t. Later, when we analyze the optimal size of the pension system, we will take into account labor market distortions by assuming elastic labor supply.

Individuals have access to a capital market with a risky return r_{t+1}. The size of the generation born in t is N_t. We denote the ratio between generations born in t+1 and t, i.e., N_{t+1}/N_t, by $1+n_{t+1}$, so n_{t+1} is the rate of population growth.[2]

At each time period t, three exogenous stochastic variables are realized: w_t the wage of the young generation in t, r_t the rate of return on the investments of the currently old in the preceding period, and n_t the rate of growth of the number of working (young) individuals. We denote the growth rate of the aggregate wage income by g_t so that $1+g_{t+1} \equiv N_{t+1}w_{t+1}/N_t w_t$.

There are several reasons to believe that these stochastic variables are not independent of each other. For example, the growth rate of the population may affect both wages and the realized rate of return on investments by influencing the capital-labor ratio. This mechanism is studied in Smith (1982) where variations in population size are the only exogenous source of uncertainty. In Enders & Lapan (1982), productivity shocks are the source of uncertainty

[2] Here, aggregate longevity risk could be covered by regarding $1+n_t$ as the ratio between the number of working individuals in period t and the number of living retirees. Aggregate longevity risk creates a source of risk that cannot be insured against within each generation.

and affect both wages and the real rate on return on savings in the form of money holdings. The stochastic relations between population growth, wages and capital returns may, of course, be much more intricate. Therefore, we do not restrict these relations, in particular the variances and covariances.

We assume that capital market returns, wages and population growth are exogenous to the model – we allow no interdependency between these variables and the pension system itself. This is, of course, a potentially important limitation, in particular if we consider closed or large economies that cannot take global factor markets as exogenous. Modeling such interdependencies would have called for a general equilibrium model, which would have complicated the analysis. We have a more important reason, however, for not following that route. The relations between implicit pension returns, wages and market returns would have been constrained in a way that had left little freedom for the kind of analysis we have performed in this paper, where we wanted to allow different correlations between these variables.

The PAYG pension systems discussed in this section are assumed to operate under a balanced budget restriction. This implies that either pension benefits or fees have to be adjusted to variations in population and wage growth, while in a fully funded system no government intervention is required to adhere to the budget condition of the pension system. Later on we will analyze the behavior of accumulated debt and the stability of the PAYG pensions system under other assumptions. To highlight how different types of shocks influence the budget restriction of the individual under alternative pension systems, let us start by simply characterizing some different systems.

2.1 A fixed-fee PAYG system

In this case, a *fixed* tax rate τ is applied to the income of the young. The proceeds are transferred to the old in the same period. The per capita pensions transferred to the old in period t are thus $\tau w_t N_t / N_{t-1}$. The benefits are adjusted each period to guarantee budget balance in the pension system.[3] Letting s_t denote savings by the young in t, we get

$$\begin{aligned} c_{1,t} &= (1-\tau)w_t - s_t \\ c_{2,t} &= s_{t-1}(1+r_t) + \tau w_t \; N_t / N_{t-1} \\ &= s_{t-1}(1+r_t) + (1+g_t)\tau w_{t-1}, \end{aligned} \tag{1}$$

assuming a balanced budget in the pension system each period.

The budget restriction for individuals born at time t is

$$c_{2,t+1} = \big((1-\tau)w_t - c_{1,t}\big)(1+r_{t+1}) + (1+g_{t+1})\tau w_t\,. \tag{2}$$

From (2) we see that the implicit return on individual pension fees paid in period t is g_{t+1}, i.e., it is determined by income growth between the period when the fees were paid and the period when benefits are received. This is a well-known result.[4]

The budget restriction for the government is satisfied since

$$N_t(1+g_{t+1})\tau w_t = N_{t+1}\tau w_{t+1}, \tag{3}$$

where the LHS denotes expenditures and the RHS revenues of the pension system in period t+1.

Now consider instead the case where a benefit level for the old follows a fixed rule. We consider two alternative rules, either that pensions are based on what the individual earned while young or that they are fixed in absolute values.

2.2 An earnings-based PAYG system

In an earnings-based system, benefits paid to the old at time t+1 are a fixed fraction β of their earnings in the preceding period. We still assume that the pension system has a balanced budget. This means that the tax rate τ_t has to be adjusted each period so that payments from the currently young exactly balance the predetermined benefits paid out in the same period. The balanced budget restriction in such a PAYG system, with benefits based on earnings when young, is

---

[3] An alternative version of a fixed-fee system is where benefits to an individual are calculated as fees paid multiplied by a fixed interest rate factor. Such systems, which are often called “contribution based” are not generally consistent with budget balance. We return to non-balanced budget pension systems in section 4.

[4] The implicit rate of return is defined as the ratio of benefits received by an individual of a specific generation to previously paid fees, minus one.

$$\begin{aligned} \beta N_t w_t &= N_{t+1}\tau_{t+1} w_{t+1} \\ \Rightarrow \tau_{t+1} &= \frac{\beta N_t w_t}{N_{t+1} w_{t+1}} = \frac{\beta}{1+g_{t+1}}. \end{aligned} \tag{4}$$

This implies that consumption is

$$\begin{aligned} c_{1,t} &= (1 - \frac{\beta}{1+g_t})w_t - s_t \\ c_{2,t} &= s_{t-1}(1+r_t) + \beta w_{t-1}. \end{aligned} \tag{5}$$

The fees and benefits of individuals born in period t are $\beta w_t/(1+g_t)$ and βw_t respectively, implying that the implicit rate of return on pension fees paid in t is g_t. Note the difference between this and the fixed-fee system, which has an implicit return of g_{t+1}. The budget restriction for individuals born in time t is

$$c_{2,t+1} = \left((1 - \frac{\beta}{1+g_t})w_t - c_{1,t} \right)(1+r_{t+1}) + \beta w_t. \tag{6}$$

2.3 A fixed-benefit PAYG system

When benefits are fixed in absolute terms, the government budget restriction is

$$\begin{aligned} N_{t-1}B &= N_t \tau_t w_t \\ \Rightarrow \tau_t &= \frac{B}{(1+n_t)w_t}, \end{aligned} \tag{7}$$

where τ is endogenous as in the earnings-based case rather than exogenous as in the fixed-fee case. This gives

$$\begin{aligned} c_{1,t} &= (1 - \frac{B}{(1+n_t)w_t})w_t - s_t \\ c_{2,t} &= s_{t-1}(1+r_t) + B. \end{aligned} \tag{8}$$

Now, fees and benefits of individuals born in period t are $B/(1+n_t)$ and B, so the implicit rate of return on pension fees paid in t is n_t. The budget restriction for individuals born at time t is

$$c_{2,t+1} = \left((1 - \frac{B}{(1+n_t)w_t})w_t - c_{1,t} \right)(1+r_{t+1}) + B. \tag{9}$$

2.4 A fully funded system

We assume, for analytical simplicity, that the return to the individual in a fully funded compulsory system is the same as on voluntary private savings.[5] A fully funded system implies that all pension fees are invested in the market. The fee τ is exogenous as in the fixed-fee PAYG system. Consumption of the young and old is now

$$\begin{aligned} c_{1,t} &= w_t(1-\tau) - s_t \\ c_{2,t} &= (w_{t-1}\tau + s_{t-1})(1+r_t). \end{aligned} \tag{10}$$

Consequently, the budget restriction for individuals born at time t is

$$c_{2,t+1} = \big((1-\tau)w_t - c_{1,t} + \tau w_t\big)(1+r_{t+1}). \tag{11}$$

The government pension budget is not balanced each year if the economy is growing. The income in the pension system at time t+1 consists of the net return on investment of the fees paid by the young in t plus the fees paid by the young in t+1. Income minus spending of the pension system is thus

$$\begin{aligned} &N_t r_{t+1}\tau w_t + N_{t+1}\tau w_{t+1} - N_t(1+r_{t+1})\tau w_t = \\ &N_t\tau w_t(r_{t+1} + 1 + g_{t+1}) - N_t(1+r_{t+1})\tau w_t = N_t\tau w_t g_{t+1}. \end{aligned} \tag{12}$$

The size of the fund in period t is $N_t\tau w_t$ and its yearly change is $N_t\tau w_t g_{t+1}$. In a growing economy the fund in a fully funded system will also grow.

3. Risk Sharing in Different Pension Systems

3.1 Two risk concepts

Before analyzing the risk-sharing properties of different pension systems, it should be noted that we can identify two conceptually different forms of risk sharing. One type is the sharing of risk that does not materialize until later in life. Any mechanism that reduces such uncertainty may provide a valuable benefit to risk-averse individuals. We call this "true risk sharing" or "true insurance". Consider an individual in our model that is born in period *t.*

---

[5] As evidenced by several historical examples, this may not necessarily be a good description of actual pension systems. The return on the pension fund may be lower on average and also have different stochastic properties than the return on the market portfolio. A reason for this is that the pension fund's investments may be influenced by political considerations.

This individual knows his wage, so the wage risk has already materialized. His lifetime utility is, however, still uncertain since his future consumption opportunities depend on stochastic variables that are not realized until period $t+1$, i.e., the return on savings. In our model, the only source of uncertainty for *living individuals*, excluding any risk generated by the pension system, is uncertainty about the return on savings – return risk.

A second type of risk is due to uncertainty about variables that are realized immediately at birth. When such risk is reduced, we refer to this as "*ex-ante* risk sharing", or "*ex-ante* insurance". More specifically, when an individual is born, his expected lifetime utility can be calculated. Let this value be denoted V_t and assume that $V_t = V(y_t)$ where y_t is a state variable that affects expected lifetime utility and is known in t but not earlier. In our model y_t is the individual's wealth, i.e., the present discounted value of his lifetime income. Now consider the expected value of V_t calculated at some date s prior to t, denoted $E_s V(y_t)$. If an intergenerational transfer scheme increases $E_s V(y_t)$ by reallocating income from generations with high expected income to generations with lower expected income, we will define it as a system that provides "ex-ante risk-sharing".

We should note that *ex-ante* risk sharing requires compensatory transfers to (from) individuals in their *first* period of life who are born with low (high) levels of wealth. True risk sharing instead implies compensatory transfers to (from) individuals in their *second* period of life who receive low (high) returns on their investments.

Although not true risk sharing, as defined above, *ex-ante* risk sharing has been the focus of several papers on intergenerational risk sharing. One example is Smith (1982), where each generation makes consumption and saving decisions under certainty and thus faces no risk after being born. In our model, one source of *ex-ante* risk is wage risk since the wage is realized as a generation is born. Under a PAYG pension system, the size of an individual's own generation may also influence his life-time budget constraint, creating another source of *ex-ante* risk.

Ex-ante risk sharing should perhaps be regarded as a redistribution scheme among generations rather than as true risk sharing. Such a redistribution scheme can be rationalized if we want to maximize a social welfare function with equal weights for all individuals. An alternative might be the celebrated philosophical idea of maximizing expected utility behind a "veil of ignorance" (Vickrey, 1946). We could also interpret our two-period OLG model as a simplified version of a three-period model, where individuals work only in the second period. In the first period, individuals in that model would perceive future risks and may participate in risk-reducing contracts. With this particular interpretation, *ex-ante* risk sharing could, in fact, be thought of as true risk sharing. Even though the distinction between true risk and *ex-ante* risk is general and fundamental, the exact line of demarcation between the two concepts is model specific.

3.2 Sources of income in alternative pension systems

In our framework, the lifetime budget restriction of the individual is affected by variations in wages, interest rates and population growth. Consider first a situation without any pension system at all. In this case, the consumption and utility of each generation are determined by the wage received and the return on savings, i.e., by w_t and r_{t+1} for a generation born in t. Any risk sharing between generations via a compulsory system must be derived from letting the consumption of a generation born in t depend also on other variables than w_t and r_{t+1}.

The generation that is old when a PAYG pension system is introduced, of course, gets a windfall gain. However, we focus on the risk sharing properties of a pension system in steady state. Certainly, the intergenerational transfers that occur during a transition period after a PAYG pension system is introduced or dismantled may be very large, see, for example Feldstein (1996). The analysis of such transfers is, however, outside the scope of this paper.

Table 1 Sources of income under alternative pension systems

Pension System	Income received when young	Pension received when old	Interest on private savings	Implicit return on pension fees
Fixed-fee	$(1-\tau)w_t$	$(1+n_{t+1})\tau w_{t+1}$	r_{t+1}	g_{t+1}
Earnings-based	$w_t - \beta w_t/(1+g_t)$	βw_t	r_{t+1}	g_t
Fixed-benefit	$w_t - Bw_t/(1+n_t)$	B	r_{t+1}	n_t
Fully funded actuarial	$(1-\tau)w_t$	$(1+r_{t+1})\tau w_t$	r_{t+1}	r_{t+1}

The variables that affect the utility of individuals may be classified in three categories: income received in the first period, non-capital income received in the second period and interest on savings received in the second period. Since we assume that no wage income is received in the second period, pension benefits are the only non-capital income in that period. The four pension systems are classified in *Table* 1.

3.3 True risk sharing

It is necessary that pension benefits are stochastic for a pension system to provide true risk sharing. To see this, recall that we assume that individuals have access to a perfect capital market. Under this assumption, it is clear that a pension benefit with a value that is known in the first period has the same effect on utility and consumption as a first-period transfer with a present value equal to the pension benefit. The pension benefit is then known in advance and the present value should be calculated using the risk-free interest rate, which we denote r^f, rather than the risky capital market return r.[6] Equivalently, forced saving at the market interest rate is irrelevant for consumption and utility since individuals can exactly offset any unwanted saving in the pension system. Thus, when analyzing consumption and utility, pension benefits can always be represented by the corresponding present values in the earnings-based, fixed benefit and fully funded pension systems. This is shown in *Table* 2. There we also see that it is only under the fixed-fee system that the future pension amount is unknown to young individuals. The reason is, of course, that wages and cohort size of the

[6] If no risk-free interest rate exists, we could still calculate a hypothetical risk-free interest rate and use it to discount the guaranteed pension benefits.

subsequent generation in such a system automatically influence the benefits that the currently young get when they become old. Thus, the conclusion is that only the fixed-fee pension system can potentially provide true insurance. This conclusion would not change if we added other sources of risk, for example, work in the second period with corresponding wage uncertainty.

We can think of the fixed-fee system as providing a diversification mechanism. Private saving and public pensions both imply uncertain returns. Under certain circumstances, PAYG pensions may reduce the overall risk by providing a hedge against the risky return on private saving. In our model, where each generation at birth gets to know its wage, a fixed-fee pension system may be beneficial even if the expected implicit return is lower than the expected market return. In sections 4 and 5 we will derive conditions for this to occur.

According to Table 2, the earnings-based, fixed benefit and fully funded pension systems differ only in terms of how much discounted income they generate to the young. We note that the earnings-based pension system incurs a gain (loss) to the individual if g_t is larger (smaller) than r_{t+1}^f. The same is true in the fixed benefit system if n_t is larger (smaller) than r_{t+1}^f. The table also indicates that the possibility of reducing intergenerational income instability in the earnings-based will depend on the correlation between w_t and g_t, and between w_t and n_t in the fixed benefit system.

In the earnings-based and fixed-benefit PAYG pension systems, both benefits and contributions are known to the young generation. Among these pension systems, the one that maximizes the present value of lifetime income for the currently young thus also maximizes expected utility at birth of that generation. The present value of lifetime income in the earnings-based and fixed-benefit pension systems depends on the particular realization of wage and population growth. This relation, which is given in Table 2, will determine the *ex-ante* risk sharing properties of the pension systems. Before analyzing this, we should note that a pension system that only provides *ex-ante* insurance but no true risk sharing, is bound to be bad *ex-post* for some young generations which then would prefer to opt out of the system. What is preferred by one young generation may thus not be preferred by the next.

In the fully funded system, individual wealth is not directly affected by variations in cohort sizes and wages of other generations.

Table 2 Present discounted value of income known in first and second period of life

Pension System	First period	Second period
Fixed-fee	$(1-\tau)w_t$	$(1+n_{t+1})\tau w_{t+1}$
Earnings-based	$w_t\left(1+\beta\left(\frac{g_t - r^f_{t+1}}{(1+g_t)(1+r^f_{t+1})}\right)\right)$	0
Fixed Benefit	$w_t + B\left(\frac{n_t - r^f_{t+1}}{(1+n_t)(1+r^f_{t+1})}\right)$	0
Fully funded actuarial	w_t	0

Note that r^f_{t+1} denotes the safe interest rate between period *t* and *t*+1

3.4 *Ex-ante* risk sharing

In section 0 we showed that only the fixed-fee pension system can provide true risk sharing. We will now consider how alternative pension systems affect *ex-ante* risk. This is of interest if we want to evaluate the performance of these pension systems as perceived behind a "veil of ignorance".

Fixed-fee pension system

Consider first the fixed-fee PAYG pension system. For each dollar paid in fees to this system, $1+g_{t+1}$ dollars of benefits are generated in the next period. As noted in the previous subsection, these benefits are stochastic when viewed from period *t,* since the growth rate of aggregate wage income between *t* and *t*+1 is unknown in period *t.* We can, however, use a capital asset pricing model to compute the market price in period *t* of a claim to $1+g_{t+1}$ dollars in *t*+1. This price, denoted P_t, expresses the period *t* value of $1+g_{t+1}$ dollars in *t*+1 and will depend on the safe interest rate and the covariance between g_{t+1} and r_{t+1}. Note that if $P_t < 1$, the young in period *t* will have to be forced to participate in the pension system since for each dollar they pay in fees, they receive benefits with a value less than a dollar.[7]

Letting y_t denote the wealth of an individual born in period *t*, we have

[7] We will derive conditions for $P_t \geq 1$ in section 4.3.

$$y_t = (1-\tau)w_t + P_t \tau w_t \\ = w_t\left(1+\tau(P_t-1)\right). \quad (13)$$

As seen in (13), we can think of P_t-1 as a measure of the excess value of pension benefits. P_t affects the wealth of the generation born in period t if τ>0 and the contribution to wealth by a marginal increase in τ is given by $w_t(P_t\text{-}1)$. Whether the pension system can help reduce *ex-ante* risk will thus depend on how variations in P_t are related to variations in w_t. If the wealth contribution of the pension system, given by $w_t(P_t\text{-}1)$, tends to be high when w_t is low, a fixed-fee pension system may help mitigate intergenerational income variation. Let us now analyze this possibility.

The expected lifetime utility at birth of an individual born in period t is determined by his wealth, taking the distribution of r_{t+1} as given, i.e., $V_t = V(y_t)$. We consider the case when individuals are risk averse. Thus, we assume that V is concave.

Constant valuation of pension benefits. First assume that P_t, i.e., the value of future pension benefits, is constant over time at some value P.[8] Then, the fixed-fee pension system does not provide ex-ante risk sharing. Clearly, if P is constant, a generation that is born with low (high) wealth is not compensated by being promised pensions with a high (low) value. Thus, no *ex-ante* insurance is provided.

Stochastic valuation of pension benefits. Now consider the case where P_t i stochastic. We then have

$$\frac{\partial}{\partial\tau|\tau=0}EV(y) = \frac{\partial}{\partial\tau|\tau=0}EV(y_t) \\ = EV'(w_t)\left((P_t-1)w_t\right) \\ = \left((\overline{P}-1)\overline{w} + \text{cov}(w_t, P_t)\right)EV'(w_t) + \text{cov}\left(V'(w_t), w_t(P_t-1)\right), \quad (14)$$

where $\overline{P}$ and $\overline{w}$ denote the expected values of P_t and w_t.

The first term in the last line of (14) can be interpreted as the value of expected addition to wealth provided by a marginal pension system. As we see, this is determined both by the expected value of P_t and its covariance with w_t. If $\left((\overline{P}-1)\overline{w} + \text{cov}(w_t, P_t)\right)$ >0, the pension system provides an expected addition to wealth to all generations. By multiplying

[8] Using a standard CAPM, it is straightforward to show that if the safe interest rate is constant and g_{t+1} and r_{t+1} are drawn from distributions that are independent and constant over time, P_t would be constant.

this by the expected marginal utility of wealth, *EV'*, we get the expected value of this marginal addition. This is a stochastic analogue to the effect discussed by Samuelson (1958) and Aaron (1966) who showed that a PAYG pension system can be Pareto improving if aggregate income growth is higher than the interest rate. In a non-stochastic version of our model, this implies that *P* would be greater than unity.

The second term can be interpreted as the value of the *ex-ante* insurance provided by the system. If the insurance term is positive, the pension system provides *ex-ante* risk sharing. The value of the insurance term is, of course, zero if individuals are risk neutral, since marginal utility of wealth, *V'*, is then constant. In this case, (14) is positive only if $\left((\overline{P}-1)\overline{w}+\operatorname{cov}(w_t,P_t)\right)>0$.

To analyze the potential for *ex-ante* risk sharing further, let us evaluate the insurance term by making a standard approximation of the last term of (14)[9]

$$\operatorname{cov}\left(V'(w_t), w_t(P_t-1)\right) \approx V''(\overline{w})\operatorname{cov}(w_t, w_t(P_t-1)). \tag{15}$$

Assuming risk aversion, i.e., that $V''<0$, we see that the sign of the insurance term depends on the sign of the covariance term, i.e., $\operatorname{cov}(w_t, w_t(P_t\text{-}1))$.

It is useful to consider two cases – one where the valuation of future pension benefits is independent of the wage realization and another when it is correlated with the wage realization. Starting with the first case, we can easily establish the following proposition.

Proposition 1 *Assume that P_t and w_t are independent. Increasing τ from zero in a fixed-fee pension system can then reduce ex-ante risk if and only if it also reduces the expected wealth of all generations in steady state, i.e., iff $\overline{P}$ is smaller than unity.*

Proof: See Appendix.

The intuition for this result is that when P_t and w_t are independent and $\overline{P}>1$, the variance of y_t increases in τ, as can be seen by calculating the variance of y_t from (13). Then, a fixed fee pension system increases the expected wealth as well as its variance. A necessary condition for risk sharing is thus that $\overline{P}<1$.[10] Thus, there is no “free lunch” in this case;

---

[9] The approximation is exact if utility is quadratic.

[10] In the appendix we show that it is also sufficient.

lower intergenerational volatility (*ex-ante* risk sharing) has to be acquired at the expense of lower expected income. Whether this is beneficial depends on the degree of risk aversion.

Now consider the second case, when P and w are correlated. Let us focus on the case where the relationship is linear, so that $E\left(P_t | w_t\right) = \overline{P} + \phi(w_t - \overline{w})$. We can interpret ϕ as the regression coefficient in a regression of P_t on w_t. The coefficient ϕ could be negative if, for example, growth is negatively autocorrelated. In this case, a lower than expected w_t means that g_{t+1} is expected to be high. This, in turn, increases the value of pension benefits to individuals born in period t. This provides a potential for *ex-ante* risk sharing – a generation that is born with a low (high) level of w_t receive claims to future pension benefits that have a high (low) value. To analyze this further, we use the assumption of a linear relationship between P and w in the expression of the approximated insurance term, i.e., the RHS of (15). We show in the appendix that we can then write this as

$$V''(\overline{w})\left((\overline{P}-1)\operatorname{var}(w_t) + \phi\left(\operatorname{skew}(w_t) + \overline{w}\operatorname{var}(w_t)\right)\right), \tag{16}$$

where skew(w_t) denotes the skewness of the distribution of w_t. As we see, the effect of different ϕ on *ex-ante* insurance depends on the skewness of the distribution of w. From an empirical point of view, it seems reasonable that the skewness is positive, i.e., that the distribution has a long tail to the right. A natural base case is a log-normal distribution with a positive skewness. In this case, a negative correlation tends to reduce *ex-ante* risk, i.e., the value of (15) falls in ϕ.

Now let us investigate if it is possible that a fixed-fee pension system can reduce *ex-ante* risk without reducing expected wealth. We can see in (16) that if ϕ is negative, it is possible that the insurance value is positive also if the expected excess value, $(\overline{P}-1)$, is positive, since $\left(\operatorname{skew}(w_t) + 2\overline{w}\operatorname{var}(w_t)\right)$ is positive. In this case, both terms of (14) may be positive. Then, the introduction of a fixed-fee pension system is beneficial regardless of the degree of risk aversion. The condition under which this happens is stated in the following proposition.

Proposition 2. *Increasing τ from zero in a fixed-fee pension system may increase expected wealth and reduce ex-ante risk for all generations in steady state. This requires that*

$$\begin{aligned}&\bar{w}(\bar{P}-1)+\phi\,\mathrm{var}(w)>0,\ \text{and}\\&(\bar{P}-1)\,\mathrm{var}(w)+\phi\big(\mathrm{skew}(w)+\bar{w}\,\mathrm{var}(w)\big)<0.\end{aligned} \tag{17}$$

Corollary: If the skewness is non-negative, (17) is satisfied iff

$$\frac{-\bar{w}(\bar{P}-1)}{\mathrm{var}(w)}<\phi<\frac{-(\bar{P}-1)\,\mathrm{var}(w)}{\mathrm{skew}(w)+\bar{w}\,\mathrm{var}(w)}. \tag{18}$$

Proof: See Appendix.

It is clear that there exist parameters such that the range defined in (18) is non-empty. We see that both inequalities in (17) cannot be satisfied unless $\bar{P}$ -1 and ϕ have different signs. The intuitive reason is that if they were both positive (negative), the introduction of a fixed-fee pension system would tend to increase (decrease) both risk and the expected income. In this case we cannot be sure of if the system is beneficial unless the degree of risk aversion is specified.

Earnings-based pension system

We saw in the previous subsection that the possibility of *ex-ante* insurance under the fixed-fee pension system depends on how the evaluation of future pension benefits varies with wage income. Under the earnings-based and fixed-benefit system, it is instead variations in the level of pension *fees* that provide a possible source of *ex-ante* insurance. Consider first the earnings-based pension system. We know from *Table 1* that the implicit return on pension fees is safe and equal to g_t for a generation born in period *t.* The safe return on savings between period *t* and *t*+1 is denoted r^f_{t+1} Since the pension benefits are known in advance under the earnings-based pension system, they should be discounted by the excess implicit return, given by $g_t - r^f_{t+1}$. Let us define $r^e_{t+1} \equiv (g_t - r^f_{t+1})/\big((1+g_t)(1+r^f_{t+1})\big)$ and note that this is proportional to the excess implicit return. A positive (negative) value of r^e_{t+1} means that the pension system is more (less) than actuarially fair. From *Table* 2 and using this definition, we see that under the earnings-based pension system, the wealth of an individual at birth can be written

$$y_t = w_t(1+\beta r^e_{t+1}) \tag{19}$$

We also see that (19) is identical to (13) if we substitute β for τ and r^e_{t+1} for $(P_t\text{-}1)$. The contribution to wealth generated by a marginal increase in β is now $w_t\, r^e_{t+1}$. The analysis of the fixed-fee pension system thus carries over to the earnings-based pension system. In particular, proposition 2 apply also the earnings-based pension system if r^e_{t+1} is substituted for $(P_t\text{-}1)$.

Letting $\bar{r}^e$ denote expected value of r^e_{t+1}, we then find the effect on expected utility of introducing a small earnings-based pension system by evaluating the analogue of (14), i.e.,

$$\frac{\partial}{\partial\beta|\beta=0}EV(y_t)=\left(\overline{w}\bar{r}^e+\operatorname{cov}(w_t,r^e_{t+1}\right)EV'(w_t)+\operatorname{cov}\left(V'(w_t),w_t r^e_{t+1}\right). \tag{20}$$

As in (14), we can interpret the first term in the last line of (20) as the value of the expected extra monetary benefit provided by the system. If expected income growth is higher than the safe interest rate, i.e., $\bar{r}^e>0$, this term is positive provided the covariance between wages and excess return is not too negative. The last term in the third line of (20) is the insurance value of the system. The insurance value is thus positive if low realizations of the wage, and thus high values of marginal utility, are associated with high values of the extra wealth contribution from the pension system, as given by $w_t\, r^e_{t+1}$.

Applying proposition 1, we find that if r^e_{t+1} and w_t are independent, the introduction of earnings-based pension system provides *ex-ante* insurance only if the expected excess implicit return is negative. Furthermore, proposition 2 in this case provides the combinations of parameters in the distribution of r^e_{t+1} and w_t such that the introduction of a small earnings-based pension system is always beneficial, regardless of the degree of risk aversion.

Fixed-benefit system

According to Table 2, the excess implicit return in the fixed-benefit system is determined by the difference between population growth and the safe interest rate. We thus replace g_t by n_t in the expression for the excess return in this pension system. We now redefine $r^e_{t+1} \equiv (n_t\text{-}\, r^f_{t+1})/\big((1+n_t)(1+ r^f_{t+1})\big)$ with exactly the same interpretation as before. In contrast to the earning-based pension system, wage income is not multiplied by r^e_{t+1} in the expression for wealth. Instead we have

$$y_t = w_t + Br^e_{t+1}. \tag{21}$$

The marginal effect of introducing a fixed-benefit pension system can now be calculated as

$$\begin{aligned}\frac{\partial}{\partial B|B=0} EV(y_t) &= EV'(w_t) \\ &= \bar{r}^e EV'(w_t) + \operatorname{cov}\left(V'(w_t), r^e_{t+1}\right). \\ &\approx \bar{r}^e EV'(w_t) + V''(\bar{w}) \operatorname{cov}(w_t, r^e_{t+1})\end{aligned} \tag{22}$$

where we have used (15) to obtain the approximation in the last line. Certainly, both terms in (22) can be positive.

Proposition 3 *A small fixed-benefit pension system provides ex-ante risk reduction iff* $\operatorname{cov}(w,r^e) < 0$.

Corollary: *If* $\operatorname{cov}(w,r^e) < 0$ *and* $\bar{r}^e > 0$ *the introduction of a small fixed-benefit increase expected wealth and reduces ex-ante risk of all generations in steady state.*

Proof: Follows immediately from (22).

Comparing propositions 2 and 3 we see that the requirements in the former are more restrictive. This is due to differences in how the excess return enters into the equation for individual wealth. Under the fixed-fee and the earnings-based system, the wage is multiplied by $(P_t$-1) and r^e_{t+1}, respectively, as seen in (13) and (19). This tends to increase *ex-ante* risk. In the fixed benefit system r^e_{t+1} is instead additive, as seen in (21), in which case this effect does not arise.

If the safe interest rate r^f is constant over time, the covariance term in (22) becomes $-V''(\bar{w})\operatorname{cov}(w, 1/(1+n))$. If low realizations of the wage are associated with high realizations of population growth, $\operatorname{cov}(w, 1/(1+n)) > 0$. In this case, the pension system provides insurance against an uncertain wage income. The reason is that a low wage tends to be associated with a large cohort size and thus with low pension fees. It is not unlikely that $\operatorname{cov}(w,1/(1+n)) > 0$ in the real world. A possible mechanism behind this is that wage fluctuations are driven by demographic factors. In this case we may expect a large generation to have to work with a low capital-labor ratio and thus receive a low wage. This is the case in, for example, Smith (1982).

The Fully funded system

As seen in Table 2, the fully funded system is identical to no pension system at all in our framework. This is due to the fact that we assume that individuals have access to a capital market with a return that always coincides with the implicit return in the pension system. As noted in footnote 5, the implicit return on pension fees may in reality be lower than the market return. In any case, the compulsory fully-funded pension system cannot do anything that the agents cannot do themselves on the capital market, unless it has access to investment opportunities that individuals cannot use. In the latter case, the fully funded pension system can be analyzed using the methods discussed in section 4.3.

4. Stability and Actuarial Fairness

So far, we have assumed that either taxes or benefits are adjusted to balance the budget in each period. We now modify this assumption in order to examine various forms of "actuarial fairness" in the PAYG system, which imposes other restrictions on the relation between fees and contributions. Moreover, allowing deviations from budget balance raises the question of the stability of a pension system. Under what conditions will the accumulated debt of a PAYG pension be non-explosive? These considerations imply that it is useful to discuss the issues of actuarial fairness and stability simultaneously. We will analyze whether a PAYG pension system can be both actuarially fair and stable. We start the analysis by fixing the implicit rate of return in the pension system to an actuarially fair level and derive conditions for stability of such a pension system. Then we turn the analysis around by assuring stability by focusing on a balanced budget pension system and derive conditions for the actuarial fairness of such a system.

The usual definition of actuarial fairness is that the expected present values of pension benefits and contributions are equal.[11] In the non-stochastic case, there is a single market interest rate – the safe interest rate. It is then straightforward to derive the restrictions on pension system implied by actuarial fairness; the implicit rate of return in the pension system

[11] The term "actuarial fairness" is sometimes used to refer to systems where pensions are proportional but not necessarily equal to previously paid fees, so that no redistribution take place within generations.

should simply coincide with the market interest rate. If instead pensions are stochastic, the market rate for safe investments is not the appropriate discount rate for future pensions. The correct discount rate is then the expected market rate for an asset (possibly hypothetical) with risk characteristics similar to those of the pension system. Actuarial fairness then implies that the expected implicit rate of return in the pension system equals the expected market rate of that asset. If a pension system is actuarial and if the pension benefits could actually be sold, the market value today of the benefits generated by one dollar in contributions would equal one dollar.

In subsection 4.1, we focus on stability in the non-stochastic case. In subsection 4.2 we instead allow stochastic elements in the rate of income growth, capital market returns and the implicit return in the pensions system. We assume that the pension system can lend and borrow at a rate equal to the market rate for safe one-period assets. It turns out that the stability properties of a pension system, under the circumstances studied in this paper, are independent of the rate of return on pension fees. We can therefore analyze stability for an arbitrary implicit rate of return in the pension system.

In subsection 4.3, we first impose stability by focusing on a balanced budget fixed-fee PAYG pension system. Such a system is stable by construction, since no debt is accumulated, but can it be actuarially fair? This would mean that the present discounted value of the pension fees coincide with the amount of pension fees paid. In an economy with assets having different risk characteristics and thus different market rates, it is necessary to decide which return to use when calculating the expected net present value of the pension benefits. To do this we need to specify the risk characteristics of the pension benefits. An answer to the question of whether a balanced budget actuarially fair PAYG system exists thus necessarily involves calculating the appropriate discount rate for the uncertain pension benefits in such a system. In general, this return will differ from the return on the capital market. Moreover, we find that it will depend on the size of the pension system. We analyze the circumstances under which a fixed-fee PAYG pension system of some endogenous positive size could be actuarially fair.

4.1 The deterministic case

The pension benefit in period t for each old individual in a PAYG pension system can be expressed as $\left(1+r_t^p\right)\tau w_{t-1}$ where r_t^p is the implicit rate of return in the system. The deficit in the pension system can then be written

$$N_{t-1}(1+r_t^p)\tau w_{t-1} - N_t \tau w_t \,. \tag{23}$$

Now express this as a share of the wage bill of the currently young by dividing by $N_t w_t$.

$$\frac{N_{t-1}(1+r_t^p)\tau w_{t-1}}{N_t w_t} - \tau = \tau\left(\frac{1+r_t^p}{1+g_t} - 1\right) = \tau\left(\frac{r_t^p - g_t}{1+g_t}\right). \tag{24}$$

The RHS of (24), denotes the deficit share which, of course, is non-zero if the rate of return in the pension system differs from the growth rate of the economy. There is a deficit if r_t^p is larger than g_t and a surplus (negative deficit) otherwise.

Let us now consider what happens if the return in the system differs from the growth rate. First let D_t, denote the debt share, i.e. the accumulated deficit share. The time path of the debt share is obviously

$$D_t = D_{t-1}\frac{(1+r_t)}{(1+g_t)} + \frac{\tau\left(r_t^p - g_t\right)}{(1+g_t)}\,, \tag{25}$$

where r_t is the market interest rate. In the non-stochastic case with constant income growth and interest rates:

$$D_t = D_{t-1}\frac{1+r}{1+g} + \tau\frac{r^p - g}{1+g}\,. \tag{26}$$

Equation (26) has one steady-state D that satisfies

$$D = -\tau\frac{r^p - g}{r-g}\,. \tag{27}$$

If the debt share D_t satisfies (27), it remains constant at that level. So if the PAYG system is actuarially fair in the sense that $r^p = r$, a debt share of -τ is a steady state. A debt share or $-\tau$ implies that the pension system has accumulated surpluses equal to $\tau N_t w_t$. These are in equal in size to the fund in the fully funded system, which by construction is each period's pension fees, i.e., $\tau N_t w_t$. This leads to the following proposition.

Proposition 4. *In an economy with constant income growth and a constant capital market return, an actuarially fair PAYG system is consistent with a constant debt share only if it is fully funded.*

Proof: Follows from the analysis above.

An actuarial PAYG system with assets equal in size to the fund in the fully funded pension system is, in fact, identical to a fully funded pension system even if the pension payments are not formally tied to the return on the fund in the system. We can thus say that any pension system that is actuarially fair and has a constant fund (or debt), expressed as a share of GDP, has to be fully funded.

Now consider the stability of the system. The evolution of the debt share given by (26) is stable iff the absolute value of $(1+r)/(1+g)$ is smaller than unity. This is the case if the economy is dynamically inefficient, i.e., iff $r<g$. In that case, and that case only, will an actuarially fair pension system with $r^p = r$ be sustainable. By contrast, in an dynamically efficient economy, where $r>g$, the debt becomes positive in the period immediately after the system has started, and the debt share will explode.[12] The conclusion is:

Proposition 5. *In an economy with constant growth and constant capital market return, an actuarially fair PAYG system is only viable if the economy is dynamically inefficient, i.e., if the capital market return is lower than the growth rate. In that case the PAYG system converges automatically to a fully funded system.*

Proof: Follows from the analysis above.

4.2 Stochastic growth and interest rate

When growth and interest rates are stochastic it is much more difficult to derive general conditions for stability. We therefore concentrate on the simplest case where the ratio $(1+r_t^f)/(1+g_t)$ is exogenous, independent of its previous realizations and identically

[12] The fees paid by the young when the system is started are assumed to be paid to the currently old. There is then no deficit in the first period. In the actuarially fair pension system and under the assumption that the economy is efficient, we have $r^p = r>$g. It the follows from (26) that the debt is positive and exploding.

distributed over time.[13] We assume that the pension system can lend and borrow at the safe interest rate r_t^f.[14]

The expected deficit share each period is given by a direct analogue of (24), namely

$$\tau E\left(\frac{1+r_t^p}{1+g_t}-1\right)=\tau E\left(\frac{r_t^p-g_t}{1+g_t}\right). \tag{28}$$

Now consider the expected value of D_{t+1} as a function of D_t. This function is simply the stochastic analogue of (25) and it has only one fixed point D^* at which $E\left[D_t \middle| D_{t-1}\right]=D_{t-1}$. This point is given by

$$D^*=-\tau\frac{E\left[(r^p-g)/(1+g)\right]}{E\left[(r^f-g)/(1+g)\right]}. \tag{29}$$

If the PAYG system provides safe benefits, actuarial fairness requires that $r_t^p = r_t^f$. Then the RHS of (29) is just -τ. Thus, as in the non-stochastic case, the only steady-state debt share is equal to the fund in the fully funded system, i.e., $\tau N_t w_t$.

Now let us consider the stability of the PAYG system in this setting. We assume that $E[(1+r_t^f)/(1+g_t)]$ and $E[(r_t^p-g_t)/(1+g_t)]$ are constant over time and denoted $\bar{\mu}$ and $\bar{d}$. By iterating on (25) and using the i.i.d. assumption, it is straightforward to show that

$$E\left[D_{t+s}\middle|D_t\right]=D_t\bar{\mu}^s+\tau\bar{d}\sum_{i=0}^{s-1}\bar{\mu}^i . \tag{30}$$

Equation (30) defines a converging sequence if $\bar{\mu}$ is smaller than unity. In the case when r^p is set equal to r^f, implying that $\bar{d}=\bar{\mu}-1$, the limit of (30) when s goes to infinity simplifies to -τ. This leads to

Proposition 6. *In an economy where the ratio* $(1+r_t^f)/(1+g_t)$ *follows an i.i.d. stochastic process, an actuarially fair PAYG system with safe benefit can only be viable if the expected value of* $(1+r_t^f)/(1+g_t)$ *is below unity.*

Corollary: If the actuarial PAYG system is viable, its expected accumulated fund in the stochastic case also converges to that of the fully funded system.

---

[13] The assumption of independence over time can be relaxed quite easily, however.

[14] In principle, we could assume that the pension system invests at a risky capital market. This would, however, imply that the lending and borrowing rates for the pension system are different, which would complicate the analysis.

Proof: Follows from the analysis above.

Proposition 6 is, of course, a stochastic analogue to proposition 5. The condition that a viable pension system must have a non-exploding expected debt share is necessary but not sufficient. The latter also requires that the debt share is weakly stationary, i.e., has a non-exploding variance. It turns out that there is a simple sufficient condition for this type of stationarity if we assume that the process $(1+r_t)/(1+g_t)$, in addition to being i.i.d., can also take on only a finite number of values.

Proposition 7. *Under the assumption that* $(1+r_t^f)/(1+g_t)$ *is i.i.d. and can take only a finite number of values, the debt share of an actuarially fair PAYG system that provides an implicit return equal to the market return has a non-exploding variance if*[15]

$$E\left[\left(\frac{1+r}{1+g}\right)^2\right] < 1. \tag{31}$$

Proof: See Appendix.

It should be noted that this condition for stability is not identical to the condition that the PAYG pension system with a return equal to the market return runs an average surplus, i.e., that $E[((1+r_t^f)/(1+g_t))] < 1$. Dynamic inefficiency in the sense of having an expected surplus in the PAYG system does not necessarily imply stability. On the other hand, stability implies inefficiency since $E[((1+r_t^f)/(1+g_t))^2] = E[((1+r_t^f)/(1+g_t))]^2 + \text{Var}[(1+r_t^f)/(1+g_t)]$ so $E[((1+r_t^f)/(1+g_t))^2]<1 \Rightarrow E[((1+r_t^f)/(1+g_t))]<1$. The intuitive explanation behind the more stringent conditions for viability in the stochastic case than in the non-stochastic is the following: The stochastic analogue to the stability condition in the non-stochastic case is that the debt does not explode *on average,* i.e., that the expected debt converges. It is clear, however, that more is required in the stochastic case, namely that the debt does not explode in any states of the world that has positive probability.

[15] Results in Warne (1996) suggest that the condition (31) is also necessary for stationarity. It is also straightforward to relax the assumption of i.i.d. to the assumption that the distribution of $(1+r_t)/(1+g_t)$ depends on a finite number of previous states of the world.

4.3 Balanced budget pension systems

Let us now look at the case when stability is insured a priori. We will restrict the analysis to pension systems with a balanced budget in each period.[16] Can such a system be actuarially fair? In the earning-based and fixed benefit system, the answer is obvious since the implicit return in these pension systems is non-stochastic; actuarial fairness simply requires that g_t and n_t, respectively, equal the safe rate of return r^f_{t+1}.

Next, look at the same issue for the fixed-fee PAYG pension system with budget balance in each period. This pension system is stable. To be actuarially fair, it has to provide an expected implicit return that is equal to the market return on an asset which shares the risk characteristics of the pension system.

A standard CAPM can be used to calculate the price in period t of a hypothetical asset which yields $1+g_{t+1}$ dollars in period $t+1$, i.e., the same as the implicit return in the fixed-fee PAYG system. The return on this asset is denoted r^h_{t+1} and equals $(1+g_{t+1})/P_t - 1$, where P_t is the price of the asset. The pension system is actuarially fair if the expected return $E_t r^h_{t+1}$ equals $E_t g_{t+1}$. Equivalently, the system is actuarially fair if $P_t=1$, i.e., if the market price of the future benefits generated by one dollar in fees is one dollar.

As we shall see, the valuation of the pension asset depends on the size of the supply of the asset, or equivalently, the size of the pension system. We set $\tau=0$ and introduce a small amount of an asset that in period $t+1$ pays $1+g_{t+1}$.

From the CAPM we know that for any market asset i we have

$$E_t\left[r^i_{t+1}\right] = r^f_{t+1} - \frac{\operatorname{cov}\left[U'\left(c_{2,t+1}\right), r^i_{t+1}\right]}{E_t\left[U'\left(c_{2,t+1}\right)\right]}, \tag{32}$$

where r_{t+1} is the safe interest rate. Assuming constant relative risk aversion, (32) can be approximated by

$$E_t\left[r^i_{t+1}\right] = r^f_{t+1} + \gamma \operatorname{cov}\left[c_{2,t+1}/c_{1,t}\ , r^i_{t+1}\right], \tag{33}$$

[16] Note, however, that a period here is a generation. Relaxing budget balance and only imposing a no-Ponzi condition would complicate the analysis substantially.

where γ is a constant that depends on risk aversion and the expected growth of consumption.[17] Now, if τ is zero, $c_{2,t+1} = (w_t - c_{1,t})(1 + r_{t+1}^m)$ where r_{t+1}^m is the stochastic return on the market portfolio. This implies that consumption in t+1 is linear in r_{t+1}^m which, in turn gives

$$E_t\left[r_{t+1}^i\right] = r_{t+1}^f + \gamma\omega_t \operatorname{cov}\left[r_{t+1}, r_{t+1}^i\right], \tag{34}$$

where $\omega_t=(w_t-c_{1,t})/c_{1,t}$, i.e., the ratio of savings to current consumption of the young. Equation (34) gives the required expected returns on the market asset and the pension asset, respectively:

$$\begin{aligned} E_t\left[r_{t+1}^m\right] &= r_{t+1}^f + \gamma\omega_t \operatorname{var}\left[r_{t+1}^m\right] \\ E_t\left[r_{t+1}^p\right] &= r_{t+1}^f + \gamma\omega_t \operatorname{cov}\left[r_{t+1}^m, r_{t+1}^p\right]. \end{aligned} \tag{35}$$

The last line of (35) gives the rate of return on the pension asset that is required to make the agents voluntarily hold a small amount of it. The implicit rate of return in a fixed-fee PAYG system equals $g_{t+1.}$ So if $E_t[g_{t+1}] = r_{t+1}^f + \gamma\omega_t \operatorname{cov}\left[r_{t+1}^m, g_{t+1}\right]$ in a small PAYG pension system, we may say that the system is actuarially fair. Thus, the expected return on pension fees required for actuarial fairness may be lower than the expected return on the capital market. It will even be lower than the risk-free rate, if the covariance between income growth and the market return is negative.[18] This leads to the following proposition where σ denotes standard deviations.

Proposition 8. *If*

$$E_t r_{t+1}^m - E_t g_{t+1} < \sigma_{r_{t+1}^m} \sigma_{g_{t+1}} \gamma\omega_t \left[\frac{\sigma_{r_{t+1}^m}}{\sigma_{g_{t+1}}} - corr\left[r_{t+1}^m, g_{t+1}\right]\right], \tag{36}$$

[17] More specifically, we have $\gamma = \bar{\gamma}U'(c_t) / E_t U'(c_{t+1}) = \bar{\gamma}(1 + r_{t+1}^f) / (1 + \theta)$. where $\bar{\gamma}$ is the constant of relative risk aversion, r is the safe interest rate and θ is the rate of time preference.

[18] These considerations are of substantial importance for evaluating whether pension systems are actuarially fair or not. As is well known, there are large variations in the average return on different financial investments. While the average real return on bonds is on the order of 1% per year, the real return on the stock market is around 6%. Varying the discount rate in this range will have dramatic effects on the PDV of future pension benefits. As an illustration, discounting one dollar 40 years at 1% yields a PDV of 67 cents, while a discount rate of 6% yields less than 10 cents.

there exist fixed-fee balanced budget pension systems with strictly positive tax rates τ*, providing a return at least equal to what is required for actuarial fairness.*[19] *If also*

$$\frac{d}{d\tau}\left(\operatorname{cov}(c_{2,t+1}/c_{1,t}, r_{t+1}^{m}) - \operatorname{cov}(c_{2,t+1}/c_{1,t}, g_{t+1})\right) < 0, \tag{37}$$

then there will exist a fixed-fee balanced budget pension system with a strictly positive tax rate τ *that provides a return exactly equal to what is required for actuarial fairness.*

Proof: The first part of the proposition follows directly from subtracting the second equation in (35) from the first, multiplying and dividing by $\sigma_{r_{t+1}^m}\sigma_{g_{t+1}}$, rearranging terms and noting that the RHS must be continuous in τ. The second part follows directly.

The RHS of the first condition in Proposition 8 can be interpreted as the insurance value of introducing a small fixed-fee PAYG pension system. This value is always positive when the standard deviation of the return in the capital market exceeds the standard deviation of growth. The LHS represents the cost in terms of lower expected return on a marginal pension payment than the return on an investment in the capital market. If the condition in Proposition 8 is satisfied, the insurance value of a small pension system is larger than its cost.

If the second condition in Proposition 8 is satisfied, an increase in the size of the pension system, measured by τ, decreases the marginal insurance value. This is likely to be the case in the real world. When the pension system is increased, agents are likely to reduce their private savings. This reduces the risk associated with the risky capital market return and the value of reducing it further.[20] At some value of τ, equality is achieved between the insurance value of the pension system and its cost. A balanced budget fixed-fee pension system with exactly this value of τ is actuarially fair. In other words, the expected present value of benefits coincides with the fees if benefits are discounted by the market interest rate on assets having the same stochastic properties as the pension system.

---

[19] A similar expression in Lagerlöf (1994) is shown to be a necessary and sufficient condition for the existence of a Pareto improving PayGo pension system.

[20] As shown in the appendix, a sufficient condition for the second condition in proposition 8 is that increases in pension fees are offset one-to-one by decreases in private savings.

It should be noted that the conditions we have derived for the existence of an actuarially fair fixed-fee pension system with balance budget applies to a particular generation. It is, in fact, likely that a tax rate that achieves exact actuarial fairness for individuals in one generation implies actuarially *unfair* implicit returns for other generations. What is regarded as actuarially fair may vary if the distribution functions of income growth rates and market returns vary over time. Similar complications arise if attitudes towards risk vary over time, for example if they depend on individual wages. Moreover, if tax rates vary stochastically over time, budget balance will in general not be achieved in each period.

5. Welfare and Optimal Pensions

A compulsory pension system that is non-actuarial will create a wedge between the wage and the value of leisure and hence distort the labor supply decision. The system could be designed, however, so that a *marginal* contribution yields a return different from the average. This creates a possibility to make the system actuarially fair *on the margin*. The discussion in this section is confined to a balanced budget pension system with an average implicit return that is exogenous. As we have seen, an example of this is the fixed-fee PAYG pension system with an implicit average return equal to g_{t+1}. The purposes are to find the optimal *marginal* return on pension fees and the optimal size of the pension system, i.e., the optimal value of τ.

5.1 The non-stochastic case

In a previous paper (Hassler & Lindbeck, 1997) we analyzed balanced budget pension systems in a non-stochastic model with liquidity constraints. We assumed that a policymaker could let the marginal return, faced by the individuals in the pension system, diverge from the average by introducing a positive or negative lump-sum base pension. We showed that to maximize the utility of the individual, the policymaker should never use this opportunity but instead set the marginal return equal to the average implicit return in the system.[21] Thus, the

---

[21] We also investigated the case where the policymaker had a lower subjective discount rate than the individuals. In this case, the policymaker should set the marginal return higher by a factor equal to the ratio of the two discount factors.

entire pension should be paid in proportion to wage income, without any lump-sum transfers or taxes.

Here we want to show that this conclusion holds also when there are no binding liquidity constraints. We start with the non-stochastic case. Suppose individuals solve

$$\begin{gathered} U \equiv \max_{c_1, c_2, l_1} u(c_1, -l_1) + (1+\theta)^{-1} u(c_2, 0) \\ wl_1(1-\tau) + \alpha\tau wl_1 + \frac{T}{1+r} - c_1 - \frac{c_2}{1+r} \geq 0. \end{gathered} \tag{38}$$

Consumption in the two periods is denoted c_i, labor supply in the first period l_1 and the wage w.[22] A compulsory social security fee of τ is levied on wage income in the first period, i.e., on wl_1. The fee finances the benefit (pension) in the second period. The benefit is paid in two parts. One part depends on previously paid fees and equals $(1+r)\alpha\tau wl_1$; r is the market rate of return, and α is the link between a marginal contribution and the present discounted value of the marginal benefit. Clearly, the implicit *marginal* rate of return on pension fees is r if $\alpha=1$, in which case the pension system is actuarially fair *on the margin*.

The other part of the benefit, T, is a lump-sum, positive or negative, transfer which is adjusted to respect the government's budget constraint. θ is the rate of time preference. The restriction in (38) is the intertemporal budget constraint of the individual. The associated shadow value is denoted λ.

As the government adjusts the transfer T to satisfy budget balance,

$$\tau wl_1\left(1+r^p\right) = (1+r)\alpha\tau wl_1 + T, \tag{39}$$

or

$$T = \tau wl_1(1+r)\left(\frac{1+r^p}{1+r} - \alpha\right) \tag{40}$$

The RHS of (39) represents the benefits paid to the currently old. r^p is the average implicit rate of return in the pension system. In the case of a fixed-fee PAYG system, we know that $r^p = g$, the growth rate of the tax base. From (40) we see that the lump-sum transfer, T, is positive (negative) if α is smaller (larger) than $(1+r^p)/(1+r)$.

---

[22] To simplify notation we disregard work in the second period by setting it to zero. None of the results depends on this.

The first-order conditions for the individual are:

$$
\begin{aligned}
u_{c_1} &= \lambda \\
u_{-l} &= w(1-\tau+\alpha\tau)\lambda, \\
u_{c_2} &= \frac{1+\theta}{1+r}\lambda.
\end{aligned}
\tag{41}
$$

Now consider a government that wants to maximize individual welfare over α for a given size of the pension system, expressed by τ. The derivative of individual utility with respect to α is

$$
\frac{\partial U}{\partial \alpha} = u_{c_1}\frac{\partial c_1}{\partial \alpha} - u_{l_1}\frac{\partial l_1}{\partial \alpha} + \frac{u_{c_2}}{1+\theta}\frac{\partial c_2}{\partial \alpha}. \tag{42}
$$

Substituting the government budget restriction into the budget constraint and differentiating yields

$$
w\left(1-\tau+\tau\frac{1+r^p}{1+r}\right)dl - dc_1 - \frac{dc_2}{1+r} = 0. \tag{43}
$$

We can now use (41) and (43) to eliminate u_{-l} , u_{c2} and dc_2 in (42). This gives

$$
\begin{aligned}
\frac{\partial U}{\partial \alpha} &= u_{c_1}\frac{\partial c_1}{\partial \alpha} - w(1-\tau+\alpha\tau)u_{c_1}\frac{\partial l_1}{\partial \alpha} + u_{c_1}\left(w\left(1-\tau+\tau\frac{1+r^p}{1+r}\right)\frac{\partial l_1}{\partial \alpha} - \frac{\partial c_1}{\partial \alpha}\right) \\
&= \left(\left(1-\tau+\tau\frac{1+r^p}{1+r}\right) - (1-\tau+\alpha\tau)\right)u_{c_1} w\frac{\partial l_1}{\partial \alpha}.
\end{aligned}
\tag{44}
$$

The derivative $\partial U/\partial \alpha$ is zero at $\alpha = (1+r^p)/(1+r)$. The derivative $\partial l_1/\partial \alpha$ is positive since increasing α has a positive substitution effect on labor supply but no income effect[23] (since the size of the pension system, as determined by τ, is given). This implies that (44) is positive (negative) for α smaller (larger) than $(1+r^p)/(1+r)$. Thus, $\alpha = (1+r^p)/(1+r)$ is a necessary and sufficient condition for optimal α. From the government's budget restriction (40) we know that this corresponds to the case of no lump-sum taxes or transfers *T*.

The previous result may seem counter-intuitive in analogy with the well-known result that a given level of government expenses is best financed by a lump-sum tax. This is the wrong analogy, however. The transfer *T* is in effect not lump-sum if α differs from

[23] To prove this, assume the opposite, that labor supply decreases in α. Then consumption has to decline so λ in (41) increases. But then the RHS of the second equation in (41) has to increase in α and thus also the marginal disutility of work, which contradicts the initial assumption of decreasing labor supply.

$(1+r^p)/(1+r)$. It is true that an increase in α is perceived as an increase in the marginal implicit return in the pension system. However, when the individual changes his labor supply, T is adjusted so that the *actual* implicit return is always $(1+r^p)$. Regardless of the value of α, the pension system budget constraint implies that a marginal increase in labor supply always generate pension benefits with a present value of $w\tau(1+r^p)/(1+r^p)$. In a welfare optimum, the marginal value of foregone leisure must be equal to the value generated by an additional unit of working time. The latter is equal to the value of the wage, net of pension fees, plus the value of the generated pension benefits. This means that the value α should be chosen so that

$$u_{-l} = w(1-\tau)u_{c_1} + w\tau\frac{1+r^p}{1+r}u_{c_1}, \tag{45}$$

where the LHS is the marginal utility of leisure, the first term of the RHS is the marginal utility of net wages and the second term is the marginal utility of generated pension benefits. Then, we know from the individual first-order condition in (41) that the marginal utility of forgone leisure is set equal to the privately perceived return of working, i.e.,

$$u_{c_1} = w(1-\tau)u_{c_1} + w\tau\alpha u_{c_1}. \tag{46}$$

Clearly, (45) is then satisfied iff $\alpha = (1+r^p)/(1+r)$. We should think of this as the *constrained* first best, where the constraint is the existence of the compulsory pension system with a given τ. If α differs from $(1+r^p)/(1+r)$, an externality is created and this externality is not internalized by the individual who behaves atomistically. To achieve the unconstrained optimum, we also have to choose τ optimally. To analyze this is the purpose of section 5.3.

It is important to note that we *do not* say that a compulsory pension system with an implicit return lower than the market rate is harmless to the generations affected by it. Rather, as we have seen, any disadvantage of such a system to individuals *cannot* be mitigated by setting the marginal degree of actuarial fairness higher than the average.

5.2 The stochastic case

Now consider the case where both the return on the capital market (r) and the implicit return in the pension system (r^p) are stochastic. The problem of the individual is then

$$U = \max_{c_1, c_2, l_1} u(c_1, -l_1) + (1+\theta)^{-1} Eu(c_2, 0)$$
$$c_2 = (wl_1(1-\tau) - c_1)(1+r) + (\tilde{\alpha} wl_1\tau + T)(1+r^p). \tag{47}$$

The parameter $\tilde{\alpha}$ corresponds to α in the non-stochastic case; it denotes the share of the pension fees that is distributed in proportion to the individual's labor income. The remainder, $(1-\tilde{\alpha})wl1\tau$, is distributed as lump-sum pension transfers (or taxes if $\tilde{\alpha}>1$) which the individual takes as independent of his work effort. As we see, the rate of return on pension fees facing the individual is $\tilde{\alpha}(1+r^p)$. Both r and r^p are stochastic and unknown to the individual when he decides on l_1 and c_1.

We allow $\tilde{\alpha}$ to be smaller or larger than unity. The expected marginal return on a dollar paid in pension fees is equal to the expected return on the capital market if $\tilde{\alpha}$ is equal to Er/Er^p. This is larger than unity if the expected return on the capital market is larger than the expected average return in the pension system.

The first-order conditions for the individual can be written

$$\begin{aligned} u_{c_1} &= \frac{1}{1+\theta} E\left[u_{c_2}(1+r)\right] \\ u_{l_1} &= \frac{1}{1+\theta} w(1-\tau)E\left[u_{c_2}(1+r)\right] + \frac{1}{1+\theta}\tilde{\alpha} w\tau E\left[u_{c_2}(1+r^p)\right] \\ \Rightarrow u_{l_1} &= wu_{c_1} + \frac{1}{1+\theta} w\tau E\left[u_{c_2}(\tilde{\alpha} r^p - r)\right]. \end{aligned} \tag{48}$$

The budget restriction in the pension system is now

$$T = (1-\tilde{\alpha})wl_1\tau. \tag{49}$$

Differentiating U with respect to $\tilde{\alpha}$ yields

$$\frac{\partial U}{\partial \tilde{\alpha}} = u_{c_1}\frac{\partial c_1}{\partial \tilde{\alpha}} - u_{l_1}\frac{\partial l_1}{\partial \tilde{\alpha}} + \frac{1}{1+\theta} Eu_{c_2}\frac{\partial c_2}{\partial \tilde{\alpha}}. \tag{50}$$

Inserting (49) in the individual's budget constraint gives

$$\begin{aligned} c_2 &= (wl_1(1-\tau) - c_1)(1+r) + wl_1\tau(1+r^p) \\ \Rightarrow dc_2 &= (1+r)(w(1-\tau)dl_1 - dc_1) + (1+r^p)w\tau dl_1. \end{aligned} \tag{51}$$

Substituting from (51) into (50), yields

$$
\begin{aligned}
&\frac{\partial U}{\partial \tilde{\alpha}} \\
&= u_{c_1}\frac{\partial c_1}{\partial \tilde{\alpha}} - u_{l_1}\frac{\partial l_1}{\partial \tilde{\alpha}} + \frac{1}{1+\theta}\left(w(1-\tau)\frac{\partial l_1}{\partial \tilde{\alpha}} - \frac{\partial c_1}{\partial \tilde{\alpha}} \right) Eu_{c_2}(1+r) + \frac{1}{1+\theta} w\tau \frac{\partial l_1}{\partial \tilde{\alpha}} Eu_{c_2}(1+r^p) \\
&= u_{c_1}\frac{\partial c_1}{\partial \tilde{\alpha}} - \left(\frac{1}{1+\theta} w(1-\tau)E\left[u_{c_2}(1+r)\right] + \frac{1}{1+\theta}\tilde{\alpha} w\tau E\left[u_{c_2}(1+r^p)\right] \right)\frac{\partial l_1}{\partial \tilde{\alpha}} \\
&\quad + \frac{1}{1+\theta}\left(w(1-\tau)\frac{\partial l_1}{\partial \tilde{\alpha}} - \frac{\partial c_1}{\partial \tilde{\alpha}} \right) Eu_{c_2}(1+r) + \frac{1}{1+\theta} wt \frac{\partial l_1}{\partial \tilde{\alpha}} Eu_{c_2}(1+r^p), \\
&= (1-\tilde{\alpha}) w\tau \frac{1}{1+\theta}\frac{\partial l_1}{\partial \tilde{\alpha}} E\left[u_{c_2}(1+r^p)\right]
\end{aligned}
\tag{52}
$$

where the second FOC in (48) is used in the second equality. To obtain the third equality we have used the first FOC in (48) and collected terms. As in the non-stochastic case, the derivative of l_1 with respect to $\tilde{\alpha}$ is positive since the substitution effect of a higher wage is positive. The expectations term in (52) is, of course, also positive. Thus, the derivative in (52) is positive (negative) for $\tilde{\alpha}$ smaller (greater) than unity. We then have

Proposition 9. *In a balanced budget fixed-fee pension system, financed with a proportional tax rate on wage income, the optimal degree of marginal actuarial fairness is such that the entire pension is paid in proportion to wage income. There should be no lump-sum transfers or taxes in the system.*

Note that the result in proposition 9 relates to *marginal* as opposed to *average* actuarial fairness. In section 4 we showed that average actuarial fairness is not in general consistent with budget balance.

The intuition behind this result is the same as in the non-stochastic case: only when the lump-sum transfer is zero does the social and private value of an extra hour's work coincide when the size of the pension system is taken as given. Zero lump-sum pensions, i.e., $\tilde{\alpha} = 1$, then yield the welfare maximizing value of the marginal rate of return on pension fees.

5.3 The optimal pension fee

Now consider the optimal level of the pension fee τ, i.e., the optimal size of the pension system. We use the stochastic setup from the previous subsection and ask under what conditions the expected utility of a representative individual is maximized, as viewed from the first period of his life. We want to find the conditions that determine the size of the pension system in which period-two risk is shared optimally.[24] Let us to rewrite (51) since τ is now endogenous

$$\begin{aligned} dc_2 &= (1+r)(w(1-\tau)dl_1 - dc_1 - wl_1 d\tau) + (1+r^p)(twdl_1 + wl_1 d\tau) \\ &= (1+r)(w(1-\tau)dl_1 - dc_1) + (1+r^p)w\tau dl_1 + (r^p - r)wl_1 d\tau. \end{aligned} \tag{53}$$

The derivative of U with respect to τ is

$$\frac{\partial U}{\partial \tau} = u_{c_1}\frac{\partial c_1}{\partial \tau} - u_{l_1}\frac{\partial l_1}{\partial \tau} + \frac{1}{1+\theta}Eu_{c_2}\frac{\partial c_2}{\partial \tau}. \tag{54}$$

Substituting (53) in (54) yields

$$\begin{aligned} \frac{\partial U}{\partial \tau} &= u_{c_1}\frac{\partial c_1}{\partial \tau} - u_{l_1}\frac{\partial l_1}{\partial \tau} \\ &\quad + \frac{1}{1+\theta}Eu_{c_2}\left((1+r)(w(1-\tau)\frac{\partial l_1}{\partial \tau} - \frac{\partial c_1}{\partial \tau}) + (1+r^p)w\tau\frac{\partial l_1}{\partial \tau} + (r^p - r)wl_1\right) \\ &= u_{c_1}\frac{\partial c_1}{\partial \tau} - u_{l_1}\frac{\partial l_1}{\partial \tau} + \frac{1}{1+\theta}\left(w(1-\tau)\frac{\partial l_1}{\partial \tau} - \frac{\partial c_1}{\partial \tau}\right)Eu_{c_2}(1+r) + \frac{1}{1+\theta}w\tau\frac{\partial l_1}{\partial \tau}Eu_{c_2}(1+r^p) \\ &\quad + \frac{1}{1+\theta}wl_1 Eu_{c_2}(r^p - r). \end{aligned} \tag{55}$$

Now substitute from the individual's first order conditions in (48) and collect terms

$$\begin{aligned} \frac{\partial U}{\partial \tau} &= (1-\tilde{\alpha})w\tau\frac{1}{1+\theta}\frac{\partial l_1}{\partial \tau}E\left[u_{c_2}(1+r^p)\right] \\ &\quad + \frac{1}{1+\theta}wl_1 Eu_{c_2}(r^p - r). \end{aligned} \tag{56}$$

The first term in the RHS of (56) is identical to the RHS of (52) except that the derivative is now expressed with respect to τ rather than to $\tilde{\alpha}$. We have already established that $\tilde{\alpha} = 1$ in a welfare optimum, which implies that the first term of (56) is zero.[25] The remainder of (56) is identical to the first-order condition for an optimal portfolio decision when the agents have access to two assets with stochastic returns r^p and r. When this condition is

[24] Another issue would be to find conditions for optimal sharing of both period one and period two risk.

[25] As far as we can tell, the first term in (56) is non-positive when the second term is zero. The reason is that is that when the second term is zero, the final term in the last row of (48) shifts sign at $\tilde{\alpha}=1$. This means that increases in τ decreases (increases) the marginal value of an hour's work when $\tilde{\alpha}$ is smaller than (larger) than unity. This should mean that raising taxes has a negative (positive) effect on labor supply when $\tilde{\alpha}$ is smaller (larger) than unity. Then, $\partial l_1/\partial\tau$ should be negative (positive) if $\tilde{\alpha}$ is smaller (larger) than unity since the wealth effect of changing τ is zero when the second term in (56) is zero. The first term in (56) would then be strictly negative for all values of $\tilde{\alpha}$ different from unity at a value of τ such that the second term is zero. This would mean that deviations from the optimal degree of marginal actuarial fairness *always* imply a lower value of the optimal τ than when $\tilde{\alpha}$ is chosen optimally . This conjecture remains to be established, however.

satisfied, the agents are indifferent between "investing" a marginal dollar in the pension system and on the capital market. This leads to

Proposition 10. *A pension system in which second-period risk is shared optimally has to be actuarially fair, i.e., it should provide a return that is valued the same as the capital market return.*

It is clear that $E(r^p\text{-}r) < 0$ does not imply that the optimal τ is zero. The optimal τ depends on the covariance terms. It is straightforward to evaluate these in the case of a small open economy with exogenous capital market return. Allowing a non-zero correlation between r^p and r, we can write

$$r^p = \mu + \rho r + \varepsilon\,, \tag{57}$$

where μ equals $E(r^p\text{-}\rho r)$. ρ is a constant and ε is the mean zero idiosyncratic component of the implicit rate of return in the pension system. Now, variations in second-period consumption depend only on variations in r when $\tau = 0$. Second period consumption is then independent of ε. Then $Eu_{c_2}\varepsilon=0$. This implies that

$$\begin{aligned} Eu_{c_2}(r^p - r) &= Eu_{c_2}(\mu + \rho r + \varepsilon - r) \\ &= \mu Eu_{c_2} + (\rho - 1)Eu_{c_2}r + Eu_{c_2}\varepsilon \\ &= \mu Eu_{c_2} + (\rho - 1)Eu_{c_2}r, \end{aligned} \tag{58}$$

when $\tau = 0$.

It is easy to show that ρ is equal to the correlation coefficient between r^p and r times the ratio of their standard deviations. Thus, the value of ρ is smaller than unity unless r^p is more volatile than r. Given that $\rho < 1$, the second term in the last line of (58) is positive, so the whole expression may be positive. For sufficiently risk-averse individuals, the last term of (56) is positive at $\tau=0$ also if $E(r^p\text{-}r) < 0$, implying that the optimal τ must be positive.

Note that since the optimal τ in a fixed-fee pension system makes it actuarially fair, there is no need to force the individual to participate when such a system is offered by the government. The government should simply provide "investment opportunities" with a real return equal to the growth of GDP, unless, of course, such instruments already exist. If each generation chooses to invest the same share of income in these instruments, the system has a balanced budget.[26]

It should be kept in mind, however, that the introduction of a PAYG pension system is likely to reduce the real capital stock in models with endogenous capital formation; this will in turn may affect welfare. This is particularly clear in a closed economy. Such an effect on the capital stock could, however, also occur in an open economy, where there may be institutional links between national savings and national investments, for instance due to special financial constraints on small firms. Necessary conditions for a PAYG system to be welfare enhancing when the negative effect on the capital stock is taken into account, are derived in Siandra (1994).

6. Summary and Conclusions

We have analyzed the risk characteristics and stability of various types of PAYG pension systems. Pension systems that operate under a balanced budget were considered in sections 2 and 3. We showed that the risk-sharing properties of these systems depend crucially on whether it is the benefits or the fees that are fixed. Only the fixed-fee system, where benefits are stochastic for the individual while fees are a fixed fraction of the wage income, could potentially contribute to what we call “true” risk sharing for the young generation, i.e., sharing of risk that is not resolved until the individual enters the second period of life.

Various PAYG pension systems can under some circumstances provide *ex-ante* risk sharing, i.e., a welfare enhancing reduction in risk that is materialized already during the period of work. In the fixed-fee case, *ex-ante* risk sharing may arise if the *valuation in period t of the stochastic benefits in period t*+1, is negatively related to wage income in period *t*. Such a negative relation may result if wage growth is negatively auto-correlated between generations.

---

[26] To overcome the potentially destabilizing deficits, variations in voluntary contribution rates could be included in the implicit rate of return in the system. Let every young choose a contribution rate τ_t^i. The benefits received in the next period are then $(1+r_{t+1}^p)\tau_t^i w_t^i l_t^i$ where $1+r_{t+1}^p = \sum_i \tau_{t+1}^i w_{t+1}^i l_{t+1}^i \Big/ \sum_i \tau_t^i w_t^i l_t^i$. We thank Michael Woodford for pointing out this solution for us.

We also considered two pension systems with fixed benefit rules: the earnings-based, with benefits as a fixed fraction of previous wage income, and fixed-benefit with pensions fixed absolutely. These systems may provide *ex-ante* risk sharing if the impact of a high (low) wage on the individual budget constraint on average is mitigated by low (high) fees. Fixed-benefit pension systems are, however, less likely to be politically stable. As soon as the working generation knows its wage it can compute exactly whether it will benefit or loose from the pension system.

An earnings-based PAYG pension system can provide *ex-ante* risk sharing if the wage is positively correlated with the pension fees. If variations in the total wage bill are caused mostly by wage variation rather than by demographic factors, we expect the opposite, that fees and wages are negatively correlated. The pension system may nevertheless provide some insurance in this case. But this would require that expected income growth is sufficiently lower than the safe interest rate. The insurance value then comes at a price – the pension system yields an average implicit return lower than the safe interest rate.

The fixed-benefit system may seem somewhat more likely to provide *ex-ante* risk sharing. Suppose that high realizations of the wage are associated with low realizations of population growth, more specifically that $\mathrm{cov}(w, 1/(1+n)) > 0$. The pension system then provides insurance against uncertain wage income. A mechanism which could generate such covariance is that wage variation is driven (mostly) by changes in the size of each generation. A small generation would then earn high wages since the labor capital ratio is low.

We have also shown that the issue of actuarial fairness cannot in general be fully separated from the issue of funded versus PAYG pension systems. The link is that actuarial fairness has consequences for the budget balance of the pension system. An actuarial PAYG pension system with a stock of assets whose size differs from that of a fully funded system will in general either explode or converge to a pension system with an accumulated surplus equal to that of the fully funded system. An exploding path will be followed if the economy is dynamically efficient, while a converging path emerges in a dynamically inefficient economy.

It was shown that a balanced budget PAYG pension system of a particular size can be actuarially fair even if its expected implicit return is lower than the market return. It is important to realize that this holds only for a specific size of the system, i.e., a specific tax rate. Moreover, if a fixed-fee pension with a particular tax rate is actuarially fair for one generation of individuals, it will in general be actuarially *unfair* to other generations. Only if the insurance value of the pension system is invariant over time, will it be possible to find a tax rate that implies actuarial fairness for all generations.

We also analyzed whether the pension system should be constructed to provide a marginal return on pension fees that differs from the average return. Such a construction may seem to be an attractive option if the average implicit return in the pension system is lower than what is required for actuarial fairness. The pension system could then provide an actuarially fair return to *marginal* pension fees. This could be achieved without violating a balanced budget restriction by adjusting the infra-marginal return. In section 5, however, we showed that this is suboptimal – the marginal return should always be set equal to the average return. To understand this result, note the pension system generally should be constructed so that the individual chooses to supply labor in an amount such that the marginal utility of foregone leisure is equal to the marginal utility of the actual return to working. The actual return to working is equal to the wage net of pension fees plus the present value of the actually generated pension benefits. To be induced to choose the labor supply that is optimal *when the size of the pension system is taken as given*, the individual's return to working must include the losses generated if the pension system provides an implicit return that is lower than the market return. This only occurs if the marginal return to pension fees is equal to the average.

We also derived the optimal size of a balanced budget fixed-fee pension system. The size should be such that the value of the insurance exactly offsets the value of the difference between the expected market return and the expected implicit return on pension fees. Thus, from the point of view of intergenerational risk sharing, there is no need to force the individual to participate in a PAYG pension system of optimal size offered by the government. Since the optimal pension system is actuarially fair, individuals need not be forced to pay contributions. This contrasts with the case of intergenerational sharing of *ex-ante* risk. In this case, the pension system transfers resources from fortunate to less fortunate generations, provided, of course, that it satisfies the conditions for *ex-ante* risk sharing derived in section 3. Non-altruistic individuals belonging to a fortunate generation would not voluntarily participate in this transfer.

A limitation of our model is that life is assumed to consist of only two periods. We believe, however, that this analytical simplification is of little importance for our results. Suppose that we instead have a multi-period model and that the pension system was evaluated from the perspective of a median voter belonging to a middle-age group. All risk-generating variables that are already realized for this individual could be aggregated into what we call "wage risk". The risk that remains to be resolved could similarly be aggregated into what we call "return risk". The general results and ideas in the paper would then still be valid.

As noted in the introduction, we have also totally disregarded individual heterogeneity and informational asymmetries. The reason is that we wanted to focus on the issue of intergenerational risk sharing. Needless to say, adding intra-generational heterogeneity may change the results, in particular those regarding optimality. We believe, however, that the issue of intergenerational risk sharing is best highlighted by not mixing it up analytically by including intra-generational heterogeneity in the analysis.

7. Appendix

7.1 Proof of proposition 1

Expanding cov(w_t,w_t(P_t-1)) yields

$$\begin{aligned}\operatorname{cov}(w_t, w_t(P_t-1)) &= E(w_t w_t(P_t-1)) - \overline{w}E(w_t(P_t-1)) \\ &= \int w_t^2 E(P_t - 1|w_t) f(w_t) dw_t - \overline{w}\left(\overline{w}(\overline{P}-1) + \operatorname{cov}(w_t,(P_t-1)\right).\end{aligned} \quad \text{(A.1)}$$

The assumptions that cov(w_t,P_t)=0 and $E\left(P_t|w_t\right) = \overline{P}$ implies that the last line becomes

$$(\overline{P}-1)\int w_t^2 f(w_t)dw_t - \overline{w}^2(\overline{P}-1) = (\overline{P}-1)Var(w_t). \quad \text{(A.2)}$$

Expression (15) can then be written

$$\operatorname{cov}\left(V'(w_t), w_t(P_t-1)\right) \approx V''(\overline{w})(\overline{P}-1)\operatorname{var}(w_t). \quad \text{(A.3)}$$

which implies that (14) becomes

$$(\overline{P}-1)\overline{w}EV'(w_t)+(\overline{P}-1)V''(\overline{w}_t)\,\mathrm{var}(w_t)\,. \tag{A.4}$$

So if $\overline{P}$ is larger (smaller) than unity, the pension system also adds (reduces) risk and provides a negative (positive) insurance value.

7.2 Proof of equation (16) and proposition 2

The assumption that $E(P_t|w_t)=\overline{P}+\phi(w_t-\overline{w})$ implies that

$$\begin{aligned}\mathrm{cov}(w_t,P_t)&=\phi\,\mathrm{var}(w_t)\\ \mathrm{cov}(w_t,w_t(P_t-1))&=(\overline{P}-1)\,\mathrm{var}(w_t)+\phi\,\mathrm{cov}(w_t,w_t^{\,2})-\phi\overline{w}\,\mathrm{var}(w_t).\end{aligned} \tag{A.5}$$

It is straightforward to show that $\mathrm{cov}(w_t,w_t^2)=\mathrm{skew}(w_t)+2\,\overline{w}\,\mathrm{var}(w_t)$, where $\mathrm{skew}(w_t)$ is the skewness of w_t, i.e., $E(w_t-\overline{w})^3$, we get $\mathrm{cov}(w_t,w_t(P_t-1))=$ $(\overline{P}-1)\,\mathrm{var}(w_t)+\phi\big(skew(w_t)+\overline{w}\,\mathrm{var}(w_t)\big)$. Using this, (A.5) and (15) in (14) yields

$$\begin{aligned}\frac{\partial}{\partial\beta|\beta=0}EV(y_t)&=EV'(w_t)\big(\overline{w}(\overline{P}-1)+\phi\,\mathrm{var}(w_t)\big)\\ &\quad+V''(\overline{w})\big((\overline{P}-1)\,\mathrm{var}(w_t)+\phi\big(\mathrm{skew}(w_t)+\overline{w}\,\mathrm{var}(w_t)\big)\big).\end{aligned} \tag{A.6}$$

The necessary conditions in proposition 1 follow directly from (A.6).

7.3 Proof of Proposition 7

Assume that for all t

$$\begin{aligned}&\frac{1+r_t}{1+g_t}\in[\mu_1,\ldots,\mu_n],\ \text{with}\\ &prob\left(\frac{1+r_t}{1+g_t}=\mu_i\right)=p[i],i=1,\ldots,n\end{aligned} \tag{A.7}$$

where $p[i]$ is an element of a vector of probabilities that sum to unity.

Now we use a result in Karlsen (1990).[27] A sufficient condition for stability of a first-order autoregressive model with state dependent AR coefficients denoted μ_i and with a state transition matrix denoted Π is that the largest eigenvalue of

$$\begin{bmatrix}\mu_1^2\Pi_{1,1} & \cdots & \mu_1^2\Pi_{n,1}\\ \vdots & & \vdots\\ \mu_n^2\Pi_{1,n} & \cdots & \mu_n^2\Pi_{n,n}\end{bmatrix} \tag{A.8}$$

[27] We are grateful to Anders Warne for showing us Karlsen's proof.

is smaller than unity. In (A.8) $\Pi_{i,j}$ is the probability of moving from state *i* to *j*. In the case of proposition 7, the Π is particularly simple since the probabilities of different states are independent of previous states. This implies that

$$\begin{bmatrix} \mu_1^2\Pi_{1,1} & \cdots & \mu_1^2\Pi_{n,1} \\ \vdots & & \vdots \\ \mu_n^2\Pi_{1,n} & \cdots & \mu_n^2\Pi_{n,n} \end{bmatrix} = \begin{bmatrix} \mu_1^2 p_1 & \cdots & \mu_1^2 p_1 \\ \vdots & & \vdots \\ \mu_n^2 p_n & \cdots & \mu_n^2 p_n \end{bmatrix}. \tag{A.9}$$

Using a result in Magnus & Neudecker (1988), it can be shown that the only non-zero eigenvalue of the matrix in (A.9) is given by [28]

$$\lambda + \left[\mu_1^2, \cdots, \mu_n^2\right] \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}. \tag{A.10}$$

which is the expected value of the square of $(1+r_t)/(1+g_t)$ as stated in the proposition.

7.4 A sufficient condition for equation (37)

The budget restriction of an individual facing a fixed-fee PAYG system implies

$$c_{2,t+1} = (w_t(1-\tau) - c_{1,t})(1 + r_{t+1}^m) + w_t\tau(1 + g_{t+1}) . \tag{A.11}$$

Now denote $\omega_t \equiv ((w_t(1\text{-}\tau)\text{-}c_{1,t})/c_{1,t}$ and $\omega_t^p \equiv w_t\tau/c_{1,t}$ and substitute into (37)

$$\begin{aligned} &\operatorname{cov}(c_{2,t+1}/c_{1,t}, r_{t+1}^m) - \operatorname{cov}(c_{2,t+1}/c_{1,t}, g_{t+1}) \\ &= \operatorname{cov}(\omega_t r_{t+1}^m + \omega_t^p g_{t+1}, r_{t+1}^m) - \operatorname{cov}(\omega_t r_{t+1}^m + \omega_t^p g_{t+1}, g_{t+1}) \\ &= \omega_t\left(\operatorname{var}(r_{t+1}^m) - \operatorname{cov}(r_{t+1}^m, g_{t+1})\right) + \omega_t^p\left(\operatorname{cov}(r_{t+1}^m, g_{t+1}) - \operatorname{var}(g_{t+1})\right) \\ &= \sigma_{g_{t+1}}\sigma_{r_{t+1}^m}\left(s_t\left(\frac{\sigma_{r_{t+1}^m}}{\sigma_{g_{t+1}}} - corr(r_{t+1}^m, g_{t+1})\right) + s_t^p\left(\operatorname{cov}(r_{t+1}^m, g_{t+1}) - \frac{\sigma_{g_{t+1}}}{\sigma_{r_{t+1}^m}}\right)\right). \end{aligned} \tag{A.12}$$

Then the sign of the derivative of (A.12) with respect to τ is equal to the sign of

$$\frac{d\omega_t}{d\tau}\left(\frac{\sigma_{r_{t+1}^m}}{\sigma_{g_{t+1}}} - corr(r_{t+1}^m, g_{t+1})\right) + \frac{d\omega_t^p}{d\tau}\left(corr(r_{t+1}^m, g_{t+1}) - \frac{\sigma_{g_{t+1}}}{\sigma_{r_{t+1}^m}}\right) \tag{A.13}$$

A sufficient condition for (A.13) to be negative is that $d\omega_t/d\tau = -\, d\omega_t^p/d\tau$, i.e., that private savings fall one-to-one with increased pensions fees. In this case, we can write (A.13) as

---

[28] See Warne (1996).

$$\frac{w_t}{c_{1,t}}\left(2corr(r_{t+1}^m, g_{t+1}) - \frac{\sigma_{g_{t+1}}}{\sigma_{r_{t+1}^m}} - \frac{\sigma_{r_{t+1}^m}}{\sigma_{g_{t+1}}}\right) < 0. \quad \text{(A.14)}$$

Note also that in the likely case that $\sigma_g < \sigma_g$, the reduction in private saving can be smaller than one-to-one. It is also the case that the smaller is $corr(r^m, g)$, the greater is the likelihood that (37) is satisfied.

8. References

Aaron, Henry, (1966), "The Social Insurance Paradox", *Canadian Journal of Economics*, XXXII (3), 371-377.

Attanasio, Orazio, and Steven J. Davis, (1996), "Relative Wage Movements and the Distribution of Consumption", *Journal of Political Economy,* 106 (6), 1227-62.

Barro, Robert J., (1974), "Are Government Bonds Net Wealth?", *Journal of Political Economy* 82 (6), 1095-1117.

Blanchard, Olivier and Philippe Weil, (1992), "Dynamic Efficiency, the Riskless Rate, and Debt Ponzi Games under Uncertainty", NBER Working Paper No. 3992.

Bohn, Henning, (1995), "The Sustainability of Budget Deficits in a Stochastic Economy", *Journal of Money, Credit and Banking,* 27 (1), 257-71.

Cass, D., (1972), "On Capital Overaccumulation in the Aggregative Neoclassical Model of Economic Growth: A Complete Characterization", *Journal of Economic Theory,* 4 200-223.

Enders, Walter and Harvey E. Lapan, (1982), "Social Security Taxation and Intergenerational Risk Sharing", *International Economic Review*, 23 (3), 647-58.

Feldstein, Martin, (1996), "The Missing Piece in Policy Analysis: Social Security Reform", The Richard T. Ely Lecture, meeting of the American Economic Association, January 5, 1996.

Gordon, Roger H. and Hal R. Varian, (1988), "Intergenerational Risk Sharing", *Journal of Public Economics* 37, 185-202.

Hassler, John and Assar Lindbeck, (1997) "Optimal Actuarial Fairness in Pension Systems - a Note", Economic Letters, 55 (2), 251-55.

Karlsen, Hans, (1990), *A Class of Non-Linear Time Series Models,* Ph.D. Thesis, Department of Mathematics, University of Bergen, Norway.

Lagerlöf, Nils-Petter, (1994), "The Optimality of Pay-As-You-Go Pension Schemes Under Uncertainty", mimeo, Stockholm University.

Magnus, Jan R. and Heinz Neudecker, (1988), *Matrix Differential Calculus with Applications in Statistics and Econometrics,* John Whiley, Chichester.

Samuelson, Paul, (1958), "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money", *Journal of Political Economy* 66 (6), 467-82.

Siandra, Eduardo, (1994), "Optimal Mix of Pension Systems", Economic Theory Discussion Paper No. 1016, University of Cambridge.

Smith, Alasdair, (1982), "Intergenerational Transfers as Social Insurance", *Journal of Public Economics* 19, 97-106.

Vickrey, William, (1945), "Measuring Marginal Utility by Reactions to Risk", *Econometrica* 13, October, 215-236.

Warne, Anders, (1996), "Autocovariance Functions and Maximum Likelihood in a VAR Model under Markov Switching", mimeo, IIES, Stockholm University.

Zilcha, Itzhak, (1991), "Characterizing Efficiency in Stochastic Overlapping Generation Models", *Journal of Economic Theory,* 55(1), 1-16.