The Fossil Episode

APPENDIX

John Hassler* and Hans-Werner Sinn†

First-order conditions

The associated first-order conditions with respect to $K_{t+1}, K_{2,t}, L_{2,t}, B_t, O_t$, and $F_t$, and the Kuhn-Tucker constraints are

$$\frac{1}{C_t} = \beta \frac{\alpha_1 Y_{t+1}}{C_{t+1} K_{t+1} - K_{2,t+1}}.$$  \hspace{1cm} (1)

$$\frac{1}{C_t} \frac{\alpha_1 Y_t}{K_t - K_{2,t}} = \lambda_t^F \frac{\alpha_2 F_t}{K_{2,t}},$$

$$\frac{1}{C_t} \frac{\varepsilon_1 Y_t}{L_t - L_{2,t}} = \lambda_t^F \frac{\varepsilon_2 F_t}{L_{2,t}},$$

$$\frac{1}{C_t} \frac{\nu_1 Y_t}{O_t + B_t} + \lambda_t^B = \frac{\theta}{F_t - B_t},$$

$$\frac{1}{C_t} \frac{\nu_1 Y_t}{O_t + B_t} + \lambda_t^O = \beta^{-1} \lambda_t^R,$$

$$\frac{\theta}{F_t - B_t} = \lambda_t^F,$$

$$\lambda_t^B B_t = 0,$$

$$\lambda_t^O O_t = 0.$$

Proof of proposition 1

We first show that $\lambda_t^B \lambda_t^O = 0$. To do this, suppose otherwise that both $\lambda_t^B$ and $\lambda_t^O > 0$, then $B_t = O_t = 0$, implying $Y_t = 0 = C_t$ and therefore utility is minus infinity. This is not part of an optimal plan, provided $K_0 > 0$. Then, to show that $\lambda_t^O = 0$ for some $t$, suppose otherwise that $\lambda_t^O > 0 \forall t$, then $O_t = 0 \forall t$. Then utility could be increased by setting $O_t = R_0$ at any $t$.

*IIES, Stockholm University
†Ifo Institute – Leibniz Institute for Economic Research at the University of Munich.
Proof of proposition 2
Suppose the economy is in phase 2 in period \( t \) and \( t + 1 \). Then, the ratio of the first-order conditions for \( O_{t+1} \) and \( O_t \) can be written

\[
\frac{\nu_1 Y_{t+1} E_{t+1}}{\nu_1 Y_t E_t} = \frac{C_{t+1}}{\beta C_t}
\]

where the LHS is the growth rate of the price of energy. From the first order condition for \( K_{t+1} \) we get the standard Euler equation

\[
\frac{C_{t+1}}{C_t \beta} = \frac{\alpha_1 Y_{t+1}}{K_{t+1} - K_{2,t+1}}
\]

where the RHS is the gross interest rate. Clearly, this is simply a variant of the Solow-Stiglitz efficiency condition, which because of the assumption of competitive markets coincides with the Hotelling condition (Hotelling, 1931). Regarding the second claim made in the proposition, the first-order conditions for \( B_t \) implies

\[
\frac{\theta C_t}{F_t - B_t} - \frac{\nu_1 Y_t}{O_t + B_t} = C_t \lambda_t^B
\]

where the LHS is the difference between the price of biocarbon and energy and the RHS is positive by definition if the economy is in phase 2, while it is zero in phases 3 and 4.

Proof of proposition 5
Suppose that phase 2 were absorbing. In such a case, the economy would necessarily be on a balanced growth path as described in proposition ???. Then, the growth rate of the manufacturing output is

\[
\gamma_Y = \frac{\gamma_A_1 + \varepsilon_1 \gamma_L + \nu_1 \ln \beta}{1 - \alpha_1}
\]

and the growth rate of biocarbon production is given by

\[
\gamma_F = \gamma_A_2 + \alpha_2 \gamma_Y + \varepsilon_2 \gamma_L.
\]

The price of fossil fuel is determined by the ratio \( \nu_1 Y_t / O_t \). Since by (6), the growth rate of fossil fuel use is \( \ln \beta \), the growth rate of the fossil fuel price is \( \gamma_Y - \ln \beta \), i.e., the growth rate of output plus the subjective discount rate. The growth rate of the food price (\( \theta C_t / F_t \)) is given by the difference \( \gamma_Y - \gamma_F \), since \( C \) grows at the same rate as \( Y \). Thus, the growth rate of fossil fuel price is larger than the growth rate of the food price if \( - \ln \beta > -\gamma_F \). Thus, as long as biocarbon production is not falling at a rate higher than the subjective discount
rate, fossil fuel prices grow faster than food prices. Since phase 2 requires that fossil fuel prices are below food prices, \(-\ln \beta > -\gamma_F\) is a sufficient condition for phase 2 not being absorbing.\(^1\)

Now, consider phase 4. The first-order condition for \(O_t\) can be written

\[
\frac{\nu_1 Y_t}{B_t} = C_t \beta^{-t} R - C_t \lambda_t^O
\]

where \(\lambda_t^O > 0\) by definition of phase 4. The LHS is the energy price and the first term on the RHS is the current shadow value of the resources constraint on fossil fuel \((\beta^{-t} R)\) divided by marginal utility to express it in terms of the manufacturing good. The term \(C_t \beta^{-t} R\) is to be interpreted as the \textit{price of energy required} for it to be worthwhile to save fossil fuel for use in period \(t\). If the actual price (LHS) is equal to (below) this, \(\lambda_t^O = 0\) \((> 0)\). In phase 4, \(\lambda_t > 0\) implying that the actual energy price is below the threshold required for it to save fossil fuel until period \(t\).

Now, let us consider what is required for us to conclude that if the economy is in phase 4 in period \(t\), it is also in phase 4 in period \(t + 1\). A sufficient condition for this is that \(\frac{\nu_1 Y_t}{B_t}\) grows slower than \(C_t \beta^{-t} R\), i.e., that \(\frac{Y_{t+1} B_t}{Y_t B_{t+1}} < \frac{C_{t+1}}{C_t} \frac{1}{\beta}\). The RHS of this inequality is equal to the marginal product of capital (the interest rate) by the Euler equation (2). From the Hotelling condition, we know that in order for it to be worthwhile to save fossil fuel, its price must grow at the interest rate. Thus, if the energy price is growing at a lower rate, it is not worthwhile to save fossil fuel. Using the fact that the savings rate and the share of biocarbon burned are constant, this simplifies to

\[-\ln \beta > -\gamma_F\]

i.e., the same condition as by which we could rule out an absorbing phase 2.\(^2\)

### Simulation

The idea of the algorithm is to guess on a sequence of savings rates which is straightforward given \(s_B\) and \(s_F\) and choose initial conditions for capital and oil use \(K_0\) and \(O_0\). Note, in particular, that the savings rate equals \(s_B\) from the period before phase 4 is entered. We

\(^1\)It is immediate to express this condition in terms of exogenous parameters. Using (3) and (4) we calculate \(\gamma_F = \gamma_A_1 (1-\alpha_1) + \alpha_2 \gamma_A_2 + (\alpha_2 z_1 + \varepsilon_2 (1-\alpha_1)) \gamma_L + \alpha_2 z_1 \ln \beta \). Clearly, non-negative growth rates of technology and labor are sufficient for \(\gamma_F > 0 > \ln \beta\).

\(^2\)Again, this is easily expressed in exogenous parameters. The growth rate of biocarbon production as \(\gamma_F = \gamma_A_1 (1-\alpha_1) + \alpha_2 \gamma_A_2 + (\alpha_2 z_1 + \varepsilon_2 (1-\alpha_1)) \gamma_L / (1-\alpha_1-\alpha_2 \varepsilon_1)\) which is larger than the growth rate calculated in the previous footnote.
can then easily solve for the endogenous allocations, i.e., for the sequences for \( \kappa_t, \Lambda_t, O_t \) and \( \Phi_t \) forward until any given final date. We have closed form solutions for all variables except \( \Phi_t \) in phase 3, which needs to be solved numerically. After finding the sequence of \( \Phi_t \), we update the guess for the savings rate and iterate if necessary. After finding the solution, we update \( O_0 \) if \( \sum O_t \neq R_0 \).

Specifically, in phase 2 \( \Phi_t = 0 \) and

\[
\Lambda_t = \frac{1}{1 + \frac{\varepsilon_t}{(1 - \delta_t)\theta}} , \quad \kappa_t = \frac{1}{1 + \frac{\alpha_1}{(1 - \delta_t)\theta}} .
\]

In addition, we have the Hotelling condition, which in phase 2 satisfies

\[
O_{t+1} = O_t \beta \frac{1 - s_t}{1 - s_{t+1}}
\]

This gives for every period except the initial, three closed form equations for the three unknown \( O_t, \Lambda_t \) and \( \kappa_t \). Phase 2 continues until the oil price were to be higher than the food price if \( \Phi_t = 0 \). Then phase 3 is entered.

Then the condition that the price of energy and biocarbon are the same. This requires

\[
\frac{1}{C_t \nu_1} \frac{Y_t}{O_t + B_t} = \frac{\theta}{D_t} , \quad \frac{1}{(1 - \hat{s}_t) \nu_1} \frac{O_t + \Phi_{t} F_t}{(1 - \Phi_t) F_t} = \frac{\theta}{\nu_1} , \quad \frac{1}{O_t + \Phi_t (\kappa_t K_t) \Lambda_t^2} \frac{\Lambda_t^2}{\nu_1} = \beta (1 - \hat{s}_t) ,
\]

(5)

Furthermore, the hotelling condition must be satisfied as long as oil is still used, i.e.,

\[
\frac{O_t + \Phi_t (\kappa_t K_t) \Lambda_t^2}{O_{t-1} + B_{t-1}} = \frac{\beta (1 - \hat{s}_{t-1})}{1 - \hat{s}_t} , \quad \frac{O_t + \Phi_t (\kappa_t K_t) \Lambda_t^2}{1 - \hat{s}_t} = \frac{O_{t-1} + B_{t-1}}{\beta (1 - \hat{s}_{t-1})} \frac{\beta (1 - \hat{s}_{t-1})}{1 - \hat{s}_t} ,
\]

(6)

Together with

\[
\Lambda_t = \frac{1}{1 + \frac{\varepsilon_t}{(1 - \delta_t)\theta \varepsilon_2}} , \quad \kappa_t = \frac{1}{1 + \frac{\alpha_1}{(1 - \delta_t)\theta \alpha_2}} ,
\]

(7)
equations (5) and (6) are four equations in the four unknowns $O_t, \Phi_t, \Lambda_t$ and $\kappa_t$ where $\Phi_t, \Lambda_t, \kappa_t \in [0, 1]$. The values of $O_t, \Lambda_t$ and $\kappa_t$ have closed for expressions in terms of $\Phi_t$ so only equation (5) needs to be solved numerically.

Eventually, phase 4 is entered. This happens when a negative $\Phi_t$ would be required to solve the equations, we are then back to three unknowns, now $\Phi_t, \Lambda_t$ and $\kappa_t$ since $O_t = 0$ which are given by $\Phi_B, \Lambda_B$ and $\kappa_B$.

These satisfy the condition that energy price and the food price are the same, namely

The last step is to use the sequence of $\Phi_t$ to update the conjectured savings rates using

$$s_t = \frac{\beta \alpha_1 (1 - \Phi_{t+1}) + \theta \alpha_2 (1 - s_{t+1})}{\beta \alpha_1 (1 - \Phi_{t+1}) + (1 - \Phi_{t+1} + \beta \theta \alpha_2) (1 - s_{t+1})}$$

and iterate until convergence. This provides a solution for given initial conditions $K_0$ and $O_0$ and results in a solution for total oil use. This is compared with the initial stock of oil $R_0$ and the initial guess for $O_0$ is updated. The algorithm is then repeated until $R_0 \approx \sum_{t=0}^{T} O_t$