The Fossil Episode
APPENDIX

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First-order conditions

The associated first-order conditions with respect to $K_{t+1}, K_{2,t}, L_{2,t}, B_t, O_t$, and F_t, and the Kuhn-Tucker constraints are

$$\begin{aligned}
\frac{1}{C_t} &= \frac{\beta}{C_{t+1}} \frac{\alpha_1 Y_{t+1}}{K_{t+1} - K_{2,t+1}}, \qquad (1)\\
\frac{1}{C_t} \frac{\alpha_1 Y_t}{K_t - K_{2,t}} &= \lambda_t^F \frac{\alpha_2 F_t}{K_{2,t}},\\
\frac{1}{C_t} \frac{\varepsilon_1 Y_t}{L_t - L_{2,t}} &= \lambda_t^F \frac{\varepsilon_2 F_t}{L_{2,t}},\\
\frac{1}{C_t} \frac{\nu_1 Y_t}{O_t + B_t} + \lambda_t^B &= \frac{\theta}{F_t - B_t},\\
\frac{1}{C_t} \frac{\nu_1 Y_t}{O_t + B_t} + \lambda_t^O &= \beta^{-t} \lambda^R,\\
\frac{\theta}{F_t - B_t} &= \lambda_t^F,\\
\lambda_t^B B_t &= 0,\\
\lambda_t^O O_t &= 0.
\end{aligned}$$

Proof of proposition 1

We first show that $\lambda_t^B \lambda_t^O = 0$. To do this, suppose otherwise that both λ_t^B and $\lambda_t^O > 0$, then $B_t = O_t = 0$, implying $Y_t = 0 = C_t$ and therefore utility is minus infinity. This is not part of an optimal plan, provided $K_0 > 0$. Then, to show that $\lambda_t^O = 0$ for some t, suppose otherwise that $\lambda_t^O > 0 \;\forall t$, then $O_t = 0 \;\forall t$. Then utility could be increased by setting $O_t = R_0$ at any t.

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Proof of proposition 2

Suppose the economy is in phase 2 in period t and $t+1$. Then, the ratio of the first-order conditions for O_{t+1} and O_t can be written

$$\frac{\nu_1 Y_{t+1} E_{t+1}^{-1}}{\nu_1 Y_t E_t^{-1}} = \frac{C_{t+1}}{\beta C_t}$$

where the LHS is the growth rate of the price of energy. From the first order condition for K_{t+1} we get the standard Euler equation

$$\frac{C_{t+1}}{C_t \beta} = \frac{\alpha_1 Y_{t+1}}{K_{t+1} - K_{2,t+1}} \tag{2}$$

where the RHS is the gross interest rate. Clearly, this is simply a variant of the Solow-Stiglitz efficiency condition, which because of the assumption of competitive markets coincides with the Hotelling condition (Hotelling, 1931). Regarding the second claim made in the proposition, the first-order conditions for B_t implies

$$\frac{\theta C_t}{F_t - B_t} - \frac{\nu_1 Y_t}{O_t + B_t} = C_t \lambda_t^B$$

where the LHS is the difference between the price of biocarbon and energy and the RHS is positive by definition if the economy is in phase 2, while it is zero in phases 3 and 4.

Proof of proposition 5

Suppose that phase 2 were absorbing. In such a case, the economy would necessarily be on a balanced growth path as described in proposition **??**. Then, the growth rate of the manufacturing output is

$$\gamma_Y = \frac{\gamma_{A_1} + \varepsilon_1 \gamma_L + \nu_1 \ln \beta}{1 - \alpha_1} \tag{3}$$

and the growth rate of biocarbon production is given by

$$\gamma_F = \gamma_{A_2} + \alpha_2 \gamma_Y + \varepsilon_2 \gamma_L. \tag{4}$$

The price of fossil fuel is determined by the ratio $\frac{\nu_1 Y_t}{O_t}$. Since by (6), the growth rate of fossil fuel use is $\ln \beta$, the growth rate of the fossil fuel price is $\gamma_Y - \ln \beta$, i.e., the growth rate of output plus the subjective discount rate. The growth rate of the food price ($\frac{\theta C_t}{F_t}$) is given by the difference $\gamma_Y - \gamma_F$, since C grows at the same rate as Y. Thus, the growth rate of fossil fuel price is larger than the growth rate of the food price if $-\ln \beta > -\gamma_F$. Thus, as long as biocarbon production is not *falling* at a rate higher than the subjective discount rate, fossil fuel prices grow faster than food prices. Since phase 2 requires that fossil fuel prices are below food prices, $-\ln\beta > -\gamma_F$ is a sufficient condition for phase 2 not being absorbing.[1]

Now, consider phase 4. The first-order condition for O_t can be written

$$\frac{\nu_1 Y_t}{B_t} = C_t\beta^{-t}\lambda^R - C_t\lambda_t^O$$

where $\lambda_t^O > 0$ by definition of phase 4. The LHS is the energy price and the first term on the RHS is the current shadow value of the resources constraint on fossil fuel ($\beta^{-t}\lambda^R$) divided by marginal utility to express it in terms of the manufacturing good. The term $C_t\beta^{-t}\lambda^R$ is to be interpreted as the *price of energy required* for it to be worthwhile to save fossil fuel for use in period t. If the actual price (LHS) is equal to (below) this, $\lambda_t^O = 0$ (> 0). In phase 4, $\lambda_t > 0$ implying that the actual energy price is below the threshold required for it to save fossil fuel until period t.

Now, let us consider what is required for us to conclude that if the economy is in phase 4 in period t, it is also in phase 4 in period $t+1$. A sufficient condition for this is that $\frac{\nu_1 Y_t}{B_t}$ grows slower than $C_t\beta^{-t}\lambda^R$, i.e., that $\frac{Y_{t+1}}{Y_t}\frac{B_t}{B_{t+1}} < \frac{C_{t+1}}{\beta C_t}$. The RHS of this inequality is equal to the marginal product of capital (the interest rate) by the Euler equation (2). From the Hotelling condition, we know that in order for it to be worthwhile to save fossil fuel, its price must grow at the interest rate. Thus, if the energy price is growing at a lower rate, it is not worthwhile to save fossil fuel. Using the fact that the savings rate and the share of biocarbon burned are constant, this simplifies to

$$-\ln\beta > -\gamma_F$$

i.e., the same condition as by which we could rule out an absorbing phase 2.[2]

Simulation

The idea of the algoritm is to guess on a sequence of savings rates which is straighforward given s_B and s_F and choose initial conditions for capital and oil use K_0 and O_0. Note, in particular, that the savings rate equals s_B from the period before phase 4 is entered. We

[1] It is immediate to express this condition in terms of exogenous parameters. Using (3) and (4) we calculate $\gamma_F = \frac{\gamma_{A_1}(1-\alpha_1)+\alpha_2\gamma_{A_2}+(\alpha_2\varepsilon_1+\varepsilon_2(1-\alpha_1))\gamma_L+\alpha_2\nu_1\ln\beta}{1-\alpha_1}$. Clearly, non-negative growth rates of technology and labor are sufficient for $\gamma_F > 0 > \ln\beta$.

[2] Again, this is easily expressed in exogenous parameters. The growth rate of biocarbon production as $\gamma_F = \frac{\gamma_{A_f}(1-\alpha_1)+\alpha_2\gamma_{A_c}+(\alpha_2\varepsilon_1+\varepsilon_2(1-\alpha_1))\gamma_L}{1-\alpha_1-\alpha_2\nu_1}$ which is larger than the growth rate calculated in the previous footnote.

can then easily solve for the endogenous allocations, i.e., for the sequences for κ_t, Λ_t O_t and Φ_t forward until any given final date. We have closed form solutions for all variables except Φ_t in phase 3, which needs to be solved numerically. After finding the sequence of Φ_t, we update the guess for the savings rate and iterate if necessary. After finding the solution, we update O_0 if $\sum O_t \neq R_0$.

Specifically, in phase 2 $\Phi_t = 0$ and

$$\begin{aligned}\Lambda_t &= \frac{1}{1+\frac{\varepsilon_1}{(1-\hat{s}_t)\theta\varepsilon_2}},\\ \kappa_t &= \frac{1}{1+\frac{\alpha_1}{(1-\hat{s}_t)\theta\alpha_2}}.\end{aligned}$$

In addition, we have the Hotelling condition, which in phase 2 satisfies

$$O_{t+1} = O_t\beta\frac{1-s_t}{1-s_{t+1}}$$

This gives for every period except the initial, three closed form equations for the three unknown O_t, Λ_t and κ_t.Phase 2 continues until the oil price were to be higher than the food price if $\Phi_t = 0$. Then phase 3 is entered.

Then the condition that the price of energy and biocarbon are the same. This requires

$$\begin{aligned}\frac{1}{C_t}\nu_1\frac{Y_t}{O_t+B_t} &= \frac{\theta}{D_t}\\ \frac{1}{(1-\hat{s}_t)}\nu_1\frac{1}{O_t+\Phi_t F_t} &= \frac{\theta}{(1-\Phi_t)\,F_t}\\ \frac{(1-\Phi_t)\,(\kappa_t K_t)^{\alpha_2}\,\Lambda_t^{\varepsilon_2}}{O_t+\Phi_t\,(\kappa_t K_t)^{\alpha_2}\,\Lambda_t^{\varepsilon_2}} &= \frac{\theta\,(1-\hat{s}_t)}{\nu_1}\end{aligned} \tag{5}$$

Furtermore, the hotelling condition must be satisfied as long as oil is still used, i..e.,

$$\begin{aligned}\frac{O_t+\Phi_t\,(\kappa_t K_t)^{\alpha_2}\,\Lambda_t^{\varepsilon_2}}{O_{t-1}+B_{t-1}} &= \frac{\beta\,(1-\hat{s}_{t-1})}{1-\hat{s}_t},\\ O_t+\Phi_t\,(\kappa_t K_t)^{\alpha_2}\,\Lambda_t^{\varepsilon_2} &= \frac{(O_{t-1}+B_{t-1})\,\beta\,(1-\hat{s}_{t-1})}{1-\hat{s}_t}\end{aligned} \tag{6}$$

Together with

$$\begin{aligned}\Lambda_t &= \frac{1}{1+\frac{1-\Phi_t}{1-\hat{s}_t}\frac{\varepsilon_1}{\theta\varepsilon_2}},\\ \kappa_t &= \frac{1}{1+\frac{1-\Phi_t}{1-\hat{s}_t}\frac{\alpha_1}{\theta\alpha_2}}.\end{aligned} \tag{7}$$

equations (5) and (6) are four equations in the four unknowns O_t, Φ_t, Λ_t and $\kappa_t$where $\Phi_t, \Lambda_t, \kappa_t \in [0,1]$. The values of O_t, Λ_t and κ_t have closed for expressions in terms of Φ_t so only equation (5) needs to be solved numerically.

Eventually, phase 4 is entered. This happens when a negative Φ_t would be required to solve the equations, we are then back to three unknowns, now Φ_t, Λ_t and κ_t since $O_t = 0$ which are given by Φ_B, Λ_B and κ_B.

These satisfy the condition that energy price and the food price are the same, namely The last step is to use the sequence of Φ_t to update the conjectured savings rates using

$$s_t = \beta \frac{\alpha_1 \left(1 - \Phi_{t+1}\right) + \theta \alpha_2 \left(1 - s_{t+1}\right)}{\beta \alpha_1 \left(1 - \Phi_{t+1}\right) + \left(1 - \Phi_{t+1} + \beta \theta \alpha_2\right) \left(1 - s_{t+1}\right)}$$

and iterate until convergence. This provides a solution for given inital conditions K_0 and O_0 and results in a solution for total oil use. This is compared with the initial stock of oil R_0 and the initial guess for O_0 is updated. The algoritm is then repeated until $R_0 \approx \sum_{t=0}^{T} O_t$