The purpose of this appendix is to analyze the calibrated version our model in order to compare the macroeconomic paths under optimal policy and laissez faire.

Under our assumptions, the planning problem is

$$\max_{\{K_{t+1},R_{t+1},C_t,S_t,E_t^c,E_t^o\}_{t=0}^{\infty}} \mathbb{E}\sum_{t=0}^{\infty}\beta^t U\left(C_t\right)$$

subject to

$$\begin{aligned} C_t+K_{t+1} &= \begin{cases} \left(1-D\left(S_t\right)\right)A_t K_t^{\alpha}\left(cE_t^o+E_t^c\right)^{\nu}\left(N_t-\frac{E_t^c}{A_t^c}\right)^{1-\alpha-\nu} & \text{if } t<T^B \\ \left(1-D\left(S_t\right)\right)A_t K^t \text{ else} \end{cases}, \\ R_{t+1} &= R_t-E_t^o\geq 0, \\ E_t &= E_{ct}+cE_{ot}. \end{aligned}$$

1 Optimal allocations

We guess (and verify) that the optimal allocation can be divided into two fossil-fuel regimes, the oil regime preceding the coal regime.[1] We start the analysis from the latter when only coal is used. Conjecturing constant saving rates, the externality remains proportional to output: $\Gamma_t=\hat{\gamma}_t Y_t$. The optimal use of coal requires

$$\begin{aligned} \frac{\partial\left(\left(1-D\left(S_t\right)\right)A_t K_t^{\alpha}\left(E_t^c+cE_t^o\right)^{\nu}\left(N_t-\frac{E_t^c}{A_t^c}\right)^{1-\alpha-\nu}\right)}{\partial E_t^c} &= \Lambda_t, \\ \Rightarrow \frac{\nu}{E_t} = \frac{\left(1-\alpha-\nu\right)}{A_t^c N_t-\psi_t E_t} &+\Gamma, \end{aligned} \tag{1}$$

where ψ_t is the share of energy in period t coming from coal and

$$\Gamma=\gamma\left(\frac{\varphi_L}{1-\beta}+\frac{\left(1-\varphi_L\right)\varphi_0}{1-\beta\left(1-\varphi\right)}\right),$$

where we assume that the parameter γ in the exponential damage function is constant.[2] The left-hand side of the second line in (1) is the production value of coal. The right-hand side is the sum of the marginal extraction cost and the marginal externality cost. Given ψ_t there is a unique solution to (1) in the range $E_t\in\left[0,A_t^c L_t\right]$, denoted E_t^*, which is decreasing in Γ.[3]

Using the solution E_t^* in the net output function yields

$$Y_t=S\left(S_t,\gamma_t\right)A_t K_t^{\alpha}\left(E_t^*\right)^{\nu}\left(N_t-\frac{\psi_t E_t^*}{A_t^c}\right)^{1-\alpha-\nu},$$

which implies constant saving rates at $\alpha\beta$, as conjectured.

Consider now the oil regime. Let the optimal switching period be denoted T^*, in which possibly both oil and coal is used. In period T^*, energy use satisfies

$$\frac{\nu}{E_{T^*}}=\frac{\left(1-\alpha-\nu\right)}{A_{T^*}^c N_{T^*}-\psi_{T^*}E_{T^*}}+\Gamma.$$

[1] Due to the assumption of discrete time, there will a maximum of one period when both oil and coal is used.

[2] As shown in the main text, it would be straightforward to let this parameter vary over time.

[3] Condition (1) is a quadratic equation in E_t with the solution

$$E_t^c=\frac{1-\alpha}{2\hat{\gamma}}+\frac{A_t^c N_t}{2}-\frac{\sqrt{\left(\frac{1-\alpha}{\hat{\gamma}}+A_t^c N_t\right)^2-4\frac{\nu A_t^c N_t}{\hat{\gamma}}}}{2}.$$

Furthermore, the marginal social value of a unit of oil is equal to c times the extraction cost plus the reduction in the externality due to the fact that one unit of oil replaces can replace c units of coal, reducing the externality with $c-1$ units. Therefore, the social marginal value of oil in period T^* is

$$Y_{T^*}\left(c\left(\frac{1-\alpha-\nu}{A^c_{T^*}N_{T^*}-\psi_{T^*}E_{T^*}}\right)+(c-1)\,\Gamma\right).$$

For $s>0$ periods before T^*, the marginal social value of oil is

$$Y_{T^*-s}\left(\frac{c\nu}{E_{T^*-s}}-\Gamma\right).$$

The marginal social value of oil must satisfy the Hotelling equation, implying

$$\Longrightarrow E_{T^*-s-1}=\frac{E_{T^*-s}}{\beta+\frac{E_{T^*-s}}{c\nu}\Gamma\,(1-\beta)} \tag{2}$$

with E_{T^*-1} satisfying

$$\beta\left(c\left(\frac{1-\alpha-\nu}{A^c_{T^*}N_{T^*}-\psi_{T^*}E_{T^*}}\right)+(c-1)\,\Gamma\right)\leq\frac{c\nu}{E_{T^*-1}}-\Gamma, \tag{3}$$

which is satisfied with equality if $\psi_{T^*}<1$, so that some strictly positive amount of oil is used in period T^*. The case $\psi_{T^*}=1$ turns out not be a knife-edge case since, due to the assumption of discrete time, coal extraction technology develops in discrete steps.[4]

Together with the condition

$$\psi_{T^*}E^*_{T^*}+\sum_{s=0}^{T^*-1}E^*_s=cR_0, \tag{4}$$

where R_0 is the initial stock of oil (measured in coal equivalents), equations (2) and (3) and the requirement $\psi_{T^{lf}}\in[0,1]$ determine the sequence of values for E^*_t for $t\in[0,T^*]$.

2 *Laissez faire*

Under *laissez faire*, fossil-fuel use is determined by the private marginal values. In period T^{lf}, energy use is determined by the condition that the marginal private value of energy equal the extraction cost. Thus,

$$\frac{\nu}{E_{T^{lf}}}=\frac{(1-\alpha-\nu)}{A^c_{T^{lf}}N_{T^{lf}}-\psi_{T^{lf}}E_{T^{lf}}}\Rightarrow E_{T^{lf}}=\frac{\nu A^c_{T^{lf}}N_{T^{lf}}}{1-\alpha-\nu\,(1-\psi_{T^{lf}})}. \tag{5}$$

The private marginal value of oil is $c\frac{\nu Y_t}{E_t}$ and this value must satisfy the Hotelling equation for adjacent periods in which oil is used, implying that for $s>0$,

$$\Longrightarrow E_{T^{lf}-s-1}=\frac{E_{T^{lf}-s}}{\beta}, \tag{6}$$

with $E_{T^{lf}-1}$ satisfying

$$E_{T^{lf}-1}\geq\frac{E_{T^{lf}}}{\beta}, \tag{7}$$

which is satisfied with equality if $\psi_{T^{lf}}<1$ so that some strictly positive amount of oil is used in period T^{lf}.

Equations (7) and (6) together with the resource constraint

$$cR_0 \;=\; (1-\psi_{T^{lf}})\,E_{T^{lf}}+\sum_{s=0}^{T^{lf}-1}E^{lf}_s \tag{8}$$

and the requirement $\psi_{T^{lf}}\in[0,1]$ define the sequence of oil use.

[4]Intuitively, technological developments in the extraction technology may make the marginal value of coal strictly *higher* than the marginal value of oil in period T^* but strictly lower in period T^*-1. Even a non marginal increase in the amount of oil remaining in period T^*-1 may then be optimally used in period T^*-1.

3 Calibration and results

We use standard values for α, β and ν given by 0.3, 0.985^{10} and 0.03 respectively. The time period is set to 10 years in order not to make too much violence on the assumption of full depreciation. The marginal coal extraction cost is given by $\frac{w_t}{A_t^c}$, where w_t is the wage. Normalizing labor supply to unity, the wage if no labor is used for coal extraction is $w_t = (1-\alpha-\nu)\,Y_t$.[5] The five-year average of coal prices between 2005 and 2009 is \$74/ton.[6] Under the assumption that taxes currently are negligible and that the coal market is competitive, the price reflects the marginal carbon extraction cost. Using the yearly world (PPP-adjusted) GDP of 75 trillion US\$ and a period length of a decade, we obtain A_0^c by dividing $(1-\alpha-\nu)\,Y_t$ by the carbon extraction cost per Gigaton as

$$A_t^c = \frac{(1-0.3-0.03)\cdot 750\cdot 10^{12}}{74\cdot 10^9} = 6791/GtC.$$

Thus, a share $\frac{1}{6791}$ of a decade's labor supply is required for each Gt of coal extracted.

As noted above, we assume that coal and oil are perfect substitutes and interpret the fact that the price per ton of oil is several times higher than coal as representing higher general efficiency of oil. To estimate these efficiency differences, we calculate the average price per ton of carbon content for oil relative to the coal price over the period 1990-2009.[7] The average price per unit of coal is on average 5.0 times higher for oil than for coal and natural gas. The latter two will be priced so that oil is compensated for its higher efficiency, providing rents to extractors due to its lower extraction cost per unit of efficiency.

Using these values, we can now find T^{lf} by finding a T^{lf} so that the solution to (8) is a number $\psi_{T^{lf}} \in [0,1]$. For $R_0 = 400$, we conjecture that $T^{lf} = 3$ and $\psi_{T^{lf}} \in (0,1)$, implying that (7) is satisfied with equality. Using (5), (6), and (7), normalizing $N_0 = 1$ in the resource constraint yields the equation

$$c\cdot R_0 = \left(\frac{\nu A_0\cdot 1.01^{10T^{lf}}}{1-\alpha-\nu\,(1-\psi_{T^{lf}})}\right)\left(1-\psi_{T^{lf}} + \frac{\frac{1}{\beta^{T^{lf}}}-1}{1-\beta}\right),$$

which is solved by $\psi_{T^{lf}} = 0.172$. Thus, oil is used exclusively for 3 decades, and in the first decade of coal use, coal represents 17.2% of the energy supply. In the first period of coal use, (5) gives energy consumption an $E_{T^{lf}}$ at 406.7. In the periods before, energy use is $E_s = \beta^{-(T^{lf}-s)}406.7$ and, expressed as yearly oil consumption, this amounts to $E_2^{o^{lf}} = 94.6, E_1^{o^{lf}} = 110.1$ and $E_0^{o^{lf}} = 128.0$. After T^{lf}, energy use increases by 1% per year due to technical progress in extraction.

Now consider the optimal allocation. Finding the optimal sequence of energy use during the oil phase amounts to finding a T^* such that the sequence implied by (2) and (4) satisfies (3) for a $\psi^* \in [0,1]$. Using the constant value $\bar{\gamma}_t = 2.3793\times 10^{-5}$ and the calibrated carbon-cycle parameters we find a corner solution. Oil lasts for exactly six periods and in period $T^* = 6$ only coal is used. The social value of oil in period T^*-1 is lower than the social cost of using coal, so no coal is used in period T^*. However, if a marginal unit of oil were to be saved until period T^*, its social value implied by the Hotelling equation would be higher than the cost of using coal, and thus no oil should be saved.

Expressed as oil consumption per decade this yields a sequence of oil consumption given by $\left\{E_s^{o^*}\right\}_0^5 = \{88.4, 78.7, 69.7, 61.5, 54.2, 47.5\}$.

In periods from $T^* = 6$, energy use (entirely made up of coal) is given by the smaller root to (1), namely

$$E_t = \frac{1}{2}\left(\frac{1-\alpha}{\Gamma} + A_t N_t\right) - \frac{1}{2}\sqrt{\left(\frac{1-\alpha}{\Gamma} + A_t N_t\right)^2 - 4\frac{\nu}{\Gamma}A_t N_t},$$

[5] We calibrate using current costs of coal extraction where a negligible share of world labor is employed in coal extraction. If instead only coal was used, the wage would be $(1-\alpha)\,Y_t$, producing only a minor difference in the quantitative results.

[6] This refers to U.S. Central Appalachian coal; source BP (2010). The 10-year average from 2000-2009 is 58.8 dollars per ton.

[7] We use spot prices of Dubai crude oil per barrel from BP (2010) and convert this to tons by multiplying by 7.33 barrels per ton and use a carbon content of 85%. For natural gas, we use the price of LNG Japan cif from BP (2010) per million BTU and use a conversion factor of 14,47 ton carbon per billion BTU. For the coal price, we use the price of U.S. Central Appalachian coal from BP (2010).

giving $\left\{E_s^{c^*}\right\}_6^{11} = \{220.6, 229.5, 238.21, 246.7, 254.9, 262.8\}$.

In the *laissez-faire* allocation, around 5000 GtC of coal has been used at 2120. We assume that fossil-fuel burning is then replaced by a backstop technology; for simplicity, we assume that this implies that the production function changes to $(1 - D(S_t))K_t^{\alpha}$. Finally, we set S_{-1} to 802 GtC and the preindustrial level to 581 GtC. Of the excess 221 GtC we assume that half will remain forever, and the rest depreciates by a factor $1-\varphi = 1-0.0228$ per decade. (Recall that our parametrization implies that approximately 50% of an emission leaves the atmosphere within a period, 20% stays forever, and 30% decays exponentially.) Having determined the paths for energy, the other variables follow immediately by noting that next period's capital stock is $\alpha\beta$ times net current output. The calculations of output, temperature, damages, and fossil-fuel use are conveniently executed in a spread-sheet program. The file is available upon request.

We then repeat the exercise for $R_0 = 600$, finding that in *laissez faire,* oil consumption is {164.1, 141.1, 121.3, 104.3, 69.2} and $E_{T^{LF}} = 448.18$. In the optimal allocation, oil is used exclusive for eight periods, with $\left\{E_s^{o^*}\right\}_0^7 =$ {108.8, 97.6, 87.1, 77.4, 68.6, 60.5, 53.2, 46.7} and for the remaining periods energy use is {238.2, 248.7, 254.9, 262.8}.