Variations in Risk and Fluctuations in Demand
– a theoretical model*

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This revision: February 21, 1996

Abstract

Can variations in risk cause non-negligible fluctuations in the flow of investment? This issue has so far only been studied in models with a fixed level of risk. Comparative statics may be inadequate, since risk varies over time. In this paper, a simple irreversible investment model is extended by including a stochastic process for the risk level. An increase in risk increases the value of waiting as in the standard model. The agent then allows larger deviations from the target level before adjusting, as long as the high risk period continues. The new result is that this effect is stronger, the shorter the high risk periods are expected to be. Comparative statics may thus underestimate the effects of fluctuations in risk.

JEL Classification Code: D11, E21, E22
Keywords: Irreversible investments, stochastic risk, durables

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*I am grateful to the anonymous referees, to Roel Beetsma, Roland Benabou, Claes Berg, Andrew Bernard, Olivier Blanchard, Ricardo Caballero, Nils Gottfries, David Laibson, Torsten Persson, Ole Risager, Lars E. O. Svensson, and to Paul Söderlind for their valuable comments and suggestions. Åsa Söderberg and Molly Åkerlund provided important editorial assistance.

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1. Introduction

To live is to take risks. This truism has its mirror image in the commonplace use of stochastic models, where people's choices are affected by risk. A direct implication is that fluctuations in risk could cause or contribute to fluctuations in economic activity. This has been stressed in recent literature on irreversible investment expenditures. A good example is Romer (1990) who suggests that increased uncertainty due to the stock market collapse of 1929 was a critical factor leading to the Great Depression. Pindyck (1991) argues that the level of risk may be more important than taxes and interest rates for aggregate investments.[1] Uncertainty is also emphasized in the policy debate. The European Commission's analysis of the 1992 project, the Cecchini Report, claims that uncertainty is the most important factor in the medium term planning of enterprises (E. C. Commission, 1988).

There is some empirical evidence of a negative relation between risk and general economic activity. See, for example, Schwert (1989). In Hassler (1993) I also show that durables demand is substantially lower during periods of large financial volatility. Theoretical work on irreversibilities has, however, treated risk as constant. Instead of modeling a fluctuating level of risk, the effects of variations in risk have been studied by comparative statics on the risk parameter (see McDonald and Siegel, 1986, Bentolila and Bertola, 1990, and Pindyck and Solimano, 1993, for example). This has a major drawback, especially in the context of business cycles; namely, one cannot examine how the expected length of a high risk period affects behavior. The purpose of this paper is to provide a model of irreversible investments where the agents know that risk fluctuates.

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[1] For firms it is not necessarily the case that increased uncertainty decreases investment. The value of waiting increases with uncertainty, which tends to delay investment. But if the first derivative of the profit function with respect to capital is convex, also the value of installed capital increases in uncertainty and this works in the opposite direction. For a discussion of this, see Caballero (1991).

In Section 2, I construct a simple model with constant risk, close in spirit to the *Ss* inventory model introduced by Arrow, Harris, and Marschak (1951). The *Ss* model has later been substantially extended and used in many different settings (menu costs and inflation in Sheshinski and Weiss, 1977, labor demand in Bentolila and Bertola, 1990, and consumer durables in Caballero, 1993, for example). Optimal behavior in these models can be described as variants of the famous *Ss* rule, proven optimal by Scarf (1959). According to the *Ss* rule the agent chooses upper and lower trigger levels for the control variable. If it reaches a trigger level, the agent adjusts it to a return point. The agent does nothing inside the band given by the triggers. Eberly (1994) has demonstrated the empirical relevance of the *Ss* model for durables demand. One of her findings is that the width of the inaction band is positively related to income variability.

The model in Section 2 with constant risk serves as a base case to which we can compare an extended model with stochastic risk in Section 3. In both models the agent wants to keep a continuously depreciating durable stock close to a target level that depends on a stochastic variable w. Without any friction, the durable stock would always be kept at the target level, which consequently is often called the frictionless level. An economic interpretation of the model is a consumer who suffers disutility if her current stock of durables deviates from a target level that depends on her wealth (w). The disutility should then be interpreted as the difference between utility at the target and actual utility. The stochastic shifts in w could also be given a more general interpretation, e.g. permanent shifts in relative prices, tastes or technology.

The agent has to pay an adjustment cost if she adjusts the stock of durables. This prevents continuous adjustment to the target. Instead, the stock is left to depreciate until a trigger is reached, when it is adjusted to the return point. This point will be above the target, since the durable is continuously depreciating.

I assume that *w* occasionally jumps up or down; the agent is infrequently hit by a relatively large wealth shock. Risk is defined as the instantaneous probability that the agent is hit by the wealth shock, i.e. the arrival rate of shocks. In Section 2, I use standard comparative statics to study how the triggers and the return point are affected by shifts in risk. I show that a shift to higher risk causes the agents to revise their *Ss* bands. In particular the lower trigger falls, as is usually found in *Ss* models (see, for example, Bentolila and Bertola, 1990). The return point also falls so the bandwidth between the lower trigger and the return point stays almost unaffected.

In Section 3, I extend the model by allowing risk to be driven by a stochastic process, known by the agents. I show that the lower trigger point falls more, the shorter the high risk period is expected to be. The tendency to delay purchases is thus larger after a temporary risk increase than after a permanent. The return point, on the other hand, falls less the shorter the expected high risk period is.

The models in Section 2 and 3 are very simple. In addition to giving transparency, this allows me to show the following useful homogeneity property of the optimal policy: The effect a risk increase has on the optimal policy, expressed in years of depreciation, is the same for all goods with the same bandwidth before the risk increase.

The simple structure also facilitates interpretation of the results. The lower trigger is determined from the condition that the value of waiting equals the temptation to adjust. We will see that the reason for the negative relation between the fall in the lower trigger and the length of the high risk period is not that the value of waiting is higher when high risk periods are short. The reason is instead that the instantaneous temptation to adjust is lower when high risk periods are expected to be short. A lower temptation to adjust implies that it is optimal to allow larger deviations before adjusting.

In Section 4, I discuss the results, the consequences of relaxing some assumptions in the model and implications for aggregate behavior.

2. A Simple Irreversible Investment Model with Constant Risk

Consider an agent who faces the problem to continuously control a stock of a durable good. The log of the stock is denoted by κ. The agent suffers a loss of instantaneous utility if the stock deviates from some target level κ^* that evolves over time, driven by a stochastic state variable w. In the case of a household controlling its stock of durables, w may be interpreted as wealth. Shifts in w will thus be called wealth shocks in what follows.

κ depreciates at a constant rate δ. When it is adjusted, a lump sum adjustment cost has to be paid. In absence of adjustment costs the agent would, of course, continuously set κ equal to the target κ^*. With adjustment costs, the agent faces the non-trivial problem of optimally balancing the cost of frequent adjustments against deviations from the target.

The solutions to problems like this generally belong to the class of *Ss* rules; if κ reaches an upper (lower) trigger it is adjusted to an upper (lower) return point. To achieve a simple form of this optimal control, I first assume that the target is proportional to wealth, i.e. κ^* equals w plus a constant. Economically this may be interpreted as a unitary long-run wealth elasticity of the stock of durables. Secondly, I assume that the utility loss of deviations from κ^* is $\frac{1}{2}(\kappa - \kappa^*)^2$. These two assumptions can be viewed as a simplification to a problem as in Grossman and Laroque (1990). In their model it is optimal to set the target to a constant fraction of wealth.

Given a suitable specification of the stochastic process for w, the trigger and return points can now be expressed as state independent relative deviations from the target. That is, the optimal control can be expressed as follows: If $\kappa - \kappa^*$ becomes smaller than some value $\underline{U}$ adjust it to $\underline{a}$; if $\kappa - \kappa^*$ becomes larger than $\overline{U}$ adjust it to $\overline{a}$; otherwise do nothing. The optimal control is then fully specified by the four numbers $\{\underline{U}, \overline{U}, \underline{a}, \overline{a}\}$ and the relevant state variable is $z \equiv \kappa - \kappa^*$. Furthermore, the assumption of a constant lump sum adjustment cost implies that $\underline{a} = \overline{a} \equiv a$.

I may now state the problem formally as

$$\min_{\{I_t, z_t\}_0^\infty} \left\{ E_0 \int_0^\infty e^{-rt} \left(\frac{z_t^2}{2} + I_t c \right) dt \right\}, \tag{1.1}$$

subject to

$$\begin{aligned} z_0 &= \bar{z}, \\ z_{t+dt} &= \begin{cases} \text{free}, & \text{if } \kappa_t \text{ is adjusted} \\ z_t - \delta dt - dw, & \text{otherwise} \end{cases} \\ I_t dt &= \begin{cases} 1, & \text{if } \kappa_t \text{ is adjusted} \\ 0, & \text{otherwise} \end{cases}. \end{aligned} \tag{1.2}$$

and the stochastic path of w. I_t is a variable with a unitary point mass at the points in time when κ is adjusted. Otherwise it is zero. r is a constant discount factor and c is the adjustment cost.[2]

To conclude the specification of the model, I assume that w follows a marked point process. With an instantaneous probability λ per unit of time, w shifts $\pm\varepsilon$. Positive and negative shocks occur with equal probability:

$$w_{t+dt} = \begin{cases} w_t + \varepsilon, & \text{with probability} \frac{\lambda}{2} dt, \\ w_t, & \text{with probability} 1 - \lambda dt, \\ w_t - \varepsilon, & \text{with probability} \frac{\lambda}{2} dt. \end{cases} \tag{1.3}$$

Lastly, I assume that ε is so large that if a shock occurs, it is always optimal to adjust κ immediately. A positive (negative) shock induces upward (downward) adjustment of κ. Positive and negative shocks thus have the same consequences for the optimal control – the relative deviation z is shifted to the return point. The relevant risk measure is thus the arrival rate of shocks, parameterized by λ. The consequences of relaxing the large shock assumption are discussed in the concluding section.

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[2] Note that the adjustment cost is expressed in the same units as the cost function. An economic interpretation of this is that the adjustment cost represents time spent on searching when a new durable good is bought.

The optimal policy and sample paths for *w*, κ and *z* are illustrated in *Figure 1*. At t_0 *w* gets a positive shock. The relative deviation *z* then falls from $\mathbf{A_0}$ to $\mathbf{B_0}$, which is below the lower trigger, so κ is immediately adjusted from $\mathbf{D_0}$ to $\mathbf{E_0}$. This shifts *z* to the return point at $\mathbf{C_0}$. At t_1 depreciation has moved *z* to the lower trigger at $\mathbf{A_1}$. This triggers adjustment and κ is adjusted from $\mathbf{D_1}$ to $\mathbf{E_1}$, which moves the relative deviation to $\mathbf{C_1}$. At t_2 a negative shock to *w* occurs that shifts *z* from $\mathbf{A_2}$ to $\mathbf{B_2}$, which is above the upper trigger. This causes a downward adjustment of κ from $\mathbf{E_2}$ to $\mathbf{D_2}$ so that the relative deviation again is moved to the return point at $\mathbf{C_2}$.

(*Figure 1* approximately here)

2.1. Solution to the Constant Risk Model

Define the total cost function *V*(*z*) as the expected total cost when the optimal control is followed. In Appendix 1 I show that *V*(*z*) must satisfy the following differential equation[3]

$$rV(z) = \frac{z^2}{2} - \delta V_z(z) + \lambda\left(V(a) + c - V(z)\right). \tag{1.4}$$

Note that the R.H.S. of (1.4) equals the current flow of disutility plus the expected rate of change of the cost function. The set of solutions to (1.4) is found analytically in Appendix 1. Lastly we have to use numerical methods to find $\{\underline{U}, a, \overline{U}\}$ and the integration constant. These uniquely define *V*(*z*) from the set of solutions to (1.4). The total cost function has a graphical appearance like in *Figure 2.* It reaches its maxima at the two triggers and its minimum at the return point.

(*Figure 2* approximately here)

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[3] Without the large shock assumption the last term of (1.4) would read $\lambda((V(z+\varepsilon) + V(z-\varepsilon))/2 - V(z))$. This is a differential equation with both leads and lags with a set of solutions I cannot find analytically.

We should note from Figure 2 that a>0, i.e. the return point overshoots the frictionless target. This is due to the depreciation. Setting $a \le 0$ with positive depreciation would cause z to always be non-positive. This cannot not be optimal.

Another important property of the optimal control is that the agent is indifferent between adjusting and waiting an instant at the triggers. At the lower trigger we can write this condition as:

Proposition 1

$$\frac{\underline{U}^2 - a^2}{2} = rc + \lambda c \tag{1.5}$$

Proof in Appendix 2.

The L.H.S. of (1.5) can be interpreted as the instantaneous temptation to adjust and the R.H.S. as the instantaneous value of waiting.[4] This interpretation will be useful when we relax the assumption of a fixed level of risk. The temptation to adjust is the decrease in the flow of utility loss following an adjustment. The value of waiting has two positive parts. The first reflects discounting and the second reflects the possibility of a wealth shock. If the agent pays the adjustment cost and adjusts now and w shifts during the next instant, she has to adjust again. She would thus have saved one adjustment cost by waiting.

In a dynamic analysis of the effects of changes in risk, we will typically be interested in the timing of purchases. The expected value of the stochastic amount of time between successive purchases, how this changes with risk and for how long upgrades are delayed after a risk increase, are of focal interest. To be able to express the controls $\{\underline{U}, a, \overline{U}\}$ in time units I normalize them by the depreciation rate. The normalized bandwidth $(a_0 - \underline{U}_0)/\delta$, for example, equals the time it takes for z (and κ) to depreciate from the return point to the lower trigger if no shocks occur. Similarly, a shift in the normalized lower trigger equals the extra time the

[4] The phrase "Value of Waiting" was used in the seminal paper by McDonald and Siegel (1986).

stock is left to depreciate from to the previous trigger before it is upgraded. The normalized control has the following homogeneity property.

Proposition 2

Suppose that for some parameter values $\{\delta, c, r, \lambda\}$ the optimal control is given by $\{\underline{U}, a, \overline{U}\}$ and that for some other parameter values $\{\delta', (\delta'/\delta)^2 c, r, \lambda\}$ the optimal control is given by $\{\underline{U}', a', \overline{U}'\}$. Then for $\delta, \delta' > 0$,

$$\frac{\underline{U}}{\delta} = \frac{\underline{U}'}{\delta'}, \; \frac{a}{\delta} = \frac{a'}{\delta'}, \; \frac{\overline{U}}{\delta} = \frac{\overline{U}'}{\delta'}. \tag{1.6}$$

Proof in Appendix 2.

Proposition 2 states that the normalized optimal control is homogeneous of degree zero in depreciation and the square of adjustment costs.[5] Equivalently – multiplying depreciation by k and the adjustment cost by k^2 implies that the *non-normalized* optimal triggers and the return point are k times their original levels.

The homogeneity has the following implication: Take a set of goods with different and arbitrary depreciation rates but with equal normalized bandwidths, i.e. the time between adjustments is the same for both goods if no shocks occur. Proposition 2 tells us that the effect of a shift in risk on the triggers and return points, expressed in years of depreciation, is the same for both goods. This result should be of empirical importance – to predict the effects of shift in risk we only need to observe the bandwidth expressed in years of depreciation, not depreciation rates or adjustment costs.

2.2. Comparative Statics on the Level of Risk

We may now perform comparative static analysis on the model to study the effects of increased risk. This should be interpreted as a comparison between economies with different risk levels. I set r to 0.05 per year and let λ vary between 0.01 and 0.10 per year in steps of

[5] Clearly this result does not generalize to non-quadratic instantaneous loss functions.

0.0025. The adjustment cost is chosen so that with an arrival rate of 0.01 it is optimal to let the stock depreciate for 5 years before upgrading if no shocks occur. Relying on Proposition 2, I express the result normalized by the depreciation rate.

The large shock assumption and positive depreciation imply that z almost always will be in the region between the lower trigger and the *return point*. It is only at the very instant a wealth shock hits that z is outside this region. Immediately after the shock, z is adjusted back to the return point. The level and change of $\overline{U}$ is thus not important and I will focus on how the lower trigger $\underline{U}$ and the return point a respond to changes in risk.

An increase in risk increases the value of waiting as seen in (1.5). We may suspect that this tends to delay adjustment, i.e. $\underline{U}$ should fall. *Figure 3* shows that this is correct and furthermore that the fall is approximately proportional to the risk increase. An increase in the arrival rate risk with one percentage point causes the lower trigger to fall by 0.26 months.

(*Figure 3* approximately here)

Also the return point falls after an increase in risk. This can be understood as follows: Consider a case with zero risk and zero discount rate. The instantaneous utility loss due to deviations from κ* is increasing and symmetric around zero. It should then be optimal to have the band $[a, \underline{U}]$ centered around zero, i.e. to set $a = -\underline{U} > 0$. Then z spends as much time above as below 0. Now increase the arrival rate of shocks. Shocks are assumed to cause immediate adjustment back to a. A centered band in the case of high risk would thus mean that z tends to spend more time above zero than below. This cannot be optimal, and consequently, we set $a < -\underline{U}$. The intuition carries over to the case of positive interest rates as long as there is depreciation.

In Figure 3 we see that also the fall in the return point is approximately linear in the risk shift and only slightly smaller than the fall in the lower trigger. A percentage point increase in the arrival rate causes the return point to fall by 0.24 months. Since the fall in the return point is almost as large as the fall in the lower trigger, the band $[a, \underline{U}]$ is shifted downwards rather than becoming wider. The band widening is negligible also if risk is increased ten-fold; the band widens from 60 to 60.16 months. We will see in the following sections that this result depends crucially on the assumption of a constant risk level.

3. A Model with Fluctuating Risk

Let us introduce a risk state variable *s* that switches stochastically between 0 and 1. The arrival rate of shocks will now depend on the current value of the state variable. $\lambda(s)$ is the arrival rate of shocks in risk state *s* and I set $\lambda(1)>\lambda(0)$. The risk state follows a Markov process with the following instantaneous transition equation

$$s_{t+dt} = \begin{cases} s_t \text{ with probability } 1 - \gamma(s)dt \\ \bar{s}_t \text{ with probability } \gamma(s)dt \end{cases} \tag{1.7}$$

where $\bar{s}=1$ if $s=0$ and vice versa. The probability of a shift of the risk state during the next instant dt is thus $\gamma(s)dt$ and the expected length until next shift is $\gamma(s)^{-1}$. If $\gamma(1)$ is different from $\gamma(0)$, the expected lengths of high and low risk periods are different. Nothing else in the model is changed and $\lambda(s)$ and $\gamma(s)$ are the only parameters that depend on the risk state. In particular, I keep the assumption that the target κ^* is proportional to wealth. Potential shifts in precautionary savings that affect the target stock of durables are thus disregarded in the model.

The optimal policy for this problem also belongs to the class of *Ss* rules. The return point and the triggers will, however, now depend on the current risk level, i.e. on the risk state. A subscript *s* will thus be attached to the optimal control values $\underline{U}_s$, a_S and $\overline{U}_s$. In Appendix 1 I show that we can derive the following system of differential equations from the Bellman equation.

$$rV(z,s) = \frac{z^2}{2} - \delta V_z(z,s) + \lambda(s)\big(V(a_s,s) + c - V(z,s)\big) + \gamma(s)\big(V(z,\bar{s}) - V(z,s)\big) \tag{1.8}$$

The interpretation is the same as that of (1.4): The value of the cost function multiplied by the discount rate equals the current flow of costs plus the expected change in the value of the cost function. We should, however, note the difference between (1.4) and (1.8). The R.H.S.

of (1.8) has a fourth term. This is due to the possibility of risk state shift, which would cause expected total costs to shift from $V(z,s)$ to $V(z,\bar{s})$.

In Appendix 1, I show how to analytically derive the set of solutions to (1.8). To find the trigger points, return points and the integration constants I have to use numerical methods. The total cost function is depicted in *Figure 4*. For each of the two values of *s* the function looks approximately like in the constant risk case. $V(z,1)$ is everywhere above $V(z,0)$, so expected costs always go up when risk shifts to the higher level.

(*Figure 4* approximately here)

We can now state the analogue to Proposition 1 for the present model with fluctuating risk.

Proposition 3

$$\frac{\underline{U}_s^2 - a_s^2}{2} = rc + \lambda(s)c + \gamma(s)\left(c + V(a_s,\bar{s}) - V(\underline{U}_s,\bar{s})\right) \tag{1.9}$$

Proof in Appendix 2

The L.H.S. of (1.9) can be interpreted as the instantaneous temptation to adjust and the R.H.S. as the instantaneous value of waiting, both computed at the lower trigger. The value of waiting consists of three non-negative terms.

In Proposition 1, we found that the value of waiting is due to the discount rate and the possibility of a shock to *w*. This is true also here, but there is also a third term in (1.9). This term represents the possibility that the risk state shifts during the next instant. This possibility also creates a non-negative option value of waiting. We will see that (1.9) is useful when we want to understand what happens to the optimal policy if risk is increased or if the expected length of a high risk period is changed.

The option value created by the potential risk state shift can be split up in two conceptually different parts. The first is the option to wait further to adjust if the risk state should shift. The other is due to that the information that a risk state shift has occurred can be used when choosing the return point. These two parts can be shown explicitly by using the value matching condition in Appendix 1 to rewrite the third term of (1.9) as

$$\gamma(s)\left(\overbrace{\left(V(\underline{U}_{\bar{s}},\bar{s})-V(\underline{U}_{s},\bar{s})\right)}^{1}+\overbrace{\left(V(a_{s},\bar{s})-V(a_{\bar{s}},\bar{s})\right)}^{2}\right) \tag{1.10}$$

The first term in (1.10) is the value of waiting further in case of a state shift. The second is the value of using the information that the risk state has changed when choosing the return point.

Proposition 2 showed that the normalized optimal control in the constant risk case is homogeneous of degree zero in depreciation and the square of adjustment costs. The following proposition shows that this is true also when risk follows the stochastic process described above.

Proposition 4

Suppose that for some parameter values $\{\delta, c, r, \lambda(0), \lambda(1), \gamma(0), \gamma(1)\}$ the optimal control is given by $\{\underline{U}_0, \underline{U}_1, a_0, a_1, \overline{U}_0, \overline{U}_1\}$ and that for some other parameter values $\{\delta', (\delta'/\delta)^2 c, r, \lambda(0), \lambda(1), \gamma(0), \gamma(1)\}$ the optimal control is given by $\{\underline{U'}_0, \underline{U'}_1, a_0', a_1', \overline{U}_0', \overline{U}_1'\}$. Then for $\delta, \delta' > 0$ and s=1,2

$$\frac{\underline{U}_s}{\delta}=\frac{\underline{U'}_s}{\delta'},\ \frac{a_s}{\delta}=\frac{a_s'}{\delta'},\ \frac{\overline{U}_s}{\delta}=\frac{\overline{U}_s'}{\delta'}. \tag{1.11}$$

Proof in Appendix 2

Proposition 4 has the following implication: Take a set of goods with different depreciation rates and adjustment costs but which have the same normalized bandwidth in the low risk state, i.e. $(a_0 - \underline{U}_0)/\delta$ is the same. These goods will have the same normalized bandwidth also in the high risk state. The shifts in the normalized controls associated with a shift to higher risk is thus the same for all these goods. In empirical work we thus only have to consider the bandwidth in years of depreciation, not the depreciation rate and the adjustment cost, to find how the optimal policy is affected by risk shifts.

3.1. Quantitative Effects of Shifts in the Level of Risk

I will now focus on how triggers and return points change when the risk state shifts for different values of $\lambda(1)$-$\lambda(0)$. In contrast to the previous section, I will now be able to relate the effects to the expected length of the high risk periods. $\lambda(1)$ and $\gamma(1)$ will be varied and $\gamma(0)$, $\lambda(0)$ and r held constant at 0.25, 0.01 and 0.05 per year. The value chosen for $\gamma(0)$ implies that low risk periods are on average 4 years long.

To evaluate the size of shifts of the controls associated with risk level shifts I need a benchmark. By keeping the normalized bandwidth in risk state 0 constant at 5 years of depreciation we may directly compare the results with the comparative statics in Section 2. A constant bandwidth in risk state 0 is achieved by using a slightly different adjustment cost for each combination of $\gamma(1)$ and $\lambda(1)$ such that $(a_0 - \underline{U}_0)/\delta = 5$. I, of course, keep c the same in the two risk states.[6,7] All controls are normalized by the rate of depreciation, as in the previous section.

Figure 5 depicts the change in normalized bandwidth, $\left((a_1 - \underline{U}_1) - (a_0 - \underline{U}_0)\right)/\delta$, for different values of $\lambda(1)$ and $\gamma(1)$. I let $\lambda(1)$ vary between 0.02 and 0.10 per year and $\gamma(1)$ between 0 and 2 per year. The values for $\gamma(1)$ imply that the average high risk periods are between infinity and 0.5 years. I also include the comparative statics results from Section 2 in the figure.

(*Figure 5* approximately here)

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[6] Since the bandwidth in state zero is relatively insensitive to $\gamma(1)$ and $\lambda(1)$, c does not have to be changed much to achieve this. The difference between the largest and the smallest adjustment cost is less than 5%.

[7] I could alternatively keep c constant and report the shifts in the controls normalized by the low risk state bandwidth. This produces almost identical results.

Figure 5 shows that the band widening increases in both $\lambda(1)$ and $\gamma(1)$. If the expected length of the high risk period is short, the band widens more for a given increase in risk. For $\lambda(1)$=0.10 per year, the band widening is between 1.2 and 3.7 months, i.e. between 2.0% and 6.1% of the bandwidth in risk state 0. This may be compared with the comparative statics result. In the constant risk model an increase in risk from 0.01 to 0.10 per year only caused the band to widen by 0.16 months or 0.3% of the bandwidth.

shows the fall in the lower trigger point after a shift to the high risk state. The fall increases in both $\lambda(1)$ and $\gamma(1)$. As in the constant risk case, the fall is almost linear in $\lambda(1)$. For $\lambda(1)$ = 0.10 the fall is between 2.8 and 3.8 months, i.e. 4.7% to 6.4% of the bandwidth in risk state 0. The lower trigger falls by 2.3 months for an equal increase in risk in the constant risk model.

(*Figure 6* approximately here)

I have shown that the fall in the lower trigger and the increase in bandwidth is smaller the longer the expected high risk period is. In Figures 5 and 6 we see that the bandwidth is more sensitive to the length of the high risk periods than the lower trigger is. This implies that the fall in the return point is *larger* for long expected high risk periods. We will see that this is of key importance for the interpretation of the results.

3.2. Some Interpretations of the Sensitivity to the Length of High Risk Periods

In the previous section I showed that how much the lower trigger falls after a risk increase depends on the expected length of the high risk period. It may seem intuitive to explain this finding by the assertion that the value of waiting is higher when the high risk period is expected to be short. This assertion, however, turns out to be incorrect. In this section, I will show that the value of waiting is relatively insensitive to the expected length of the high risk period. Instead, it is the instantaneous temptation to adjust, i.e. the difference in the flow of costs at the lower trigger and the return point, that decreases when high risk periods are expected to be short.

Let us examine how a risk state shift changes the value of waiting and the temptation to adjust when the current value of z equals $\underline{U}_0$. If the current risk state is zero the value of waiting equals

$$rc + \lambda(0)c + \gamma(0)\left(c + V(a_0,1) - V(\underline{U}_0,1)\right). \tag{1.12}$$

Now let the risk state shift to 1. The value of waiting then shifts and the change is given by

$$\overbrace{\left(\lambda(1) - \lambda(0)\right)c}^{1} + \overbrace{\left(\gamma(1)\left(c + V(a_1,0) - V(\underline{U}_0,0)\right) - \gamma(0)\left(c + V(a_0,1) - V(\underline{U}_0,1)\right)\right)}^{2}. \tag{1.13}$$

The first term in (1.13) is the increase in the value of waiting for wealth shocks. This term only depends on the change in the arrival rate of shocks. The second term, the increase in the value of waiting for a shift of the risk state, on the other hand, depends on $\gamma(1)$ in a non-trivial way. The value of waiting is, however, in both states dominated by the first term of (1.13) which is independent of $\gamma(1)$.[8] This is shown in *Figure 7* where we see that the increase in the value of waiting is almost independent of $\gamma(1)$.[9]

(*Figure 7* approximately here)

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[8] The value of waiting for a state shift, given by the third term in (1.13), actually falls after the state shift. This can be understood by using (1.10). There the option value of waiting is split up into two parts; the option to wait further before adjusting and the option to use the new information about the state when choosing the return point in case of a state shift. Both these options have strictly positive values at $\underline{U}_0$ if the current state is 0. But if the current state is 1, the first option is of zero value. This since in this case it is optimal to adjust immediately if the state shifts back to 0. The second option value is not necessarily the same in the two states, but the difference in the value of waiting for a state shift is dominated by the disappearance of the value of the first option.

[9] The parameters are as above, $\lambda(1)$ is set to 0.10 per year and the values are normalized by the adjustment cost.

Now turn to the temptation to adjust. This shifts from $(\underline{U}_0^2 - a_0^2)/2$ to $(\underline{U}_0^2 - a_1^2)/2$, i.e. by $(a_0^2 - a_1^2)/2$ – the shift in the temptation to adjust equals the shift in the square of the return point. This shift is also depicted in Figure 7. We see that the value of waiting increases more than the temptation to adjust. It is therefore optimal to wait further before adjusting. More important, while the increase in the value of waiting is *insensitive* to the expected length of the high risk period, we see that this is not true for the increase in the temptation to adjust. It decreases in $\gamma(1)$ and it is almost zero for the shortest expected duration of the high risk period.

We may use the same intuition as in the constant risk case to understand the relation between the return point and the expected length of the high risk period. Recall the example with zero discounting and zero risk. We then wanted a band symmetric around zero. Higher risk caused the return point to fall so that z continued to spend as much time above as below zero.

Using this intuition we can understand that the fall in the return point depends on the probability of a wealth shock over the *whole period* until the next adjustment. This probability is less affected by a shift to the high risk state, the shorter the high risk period is expected to be. This explains why the return point shifts less with the state when high risk periods are short.

To summarize the findings in this section; a shift to high risk increases the value of waiting, i.e. the R.H.S. of (1.9) rises and this increase is insensitive to the length of the high risk period. To restore the equality of (1.9), the temptation to adjust, i.e. the L.H.S., must increase equally much. However, the way this is achieved depends crucially on the expected length of the high risk period. When high risk periods are long the return point falls a lot. This helps increase the temptation to adjust so the lower trigger does not have to fall as much as otherwise. When high risk periods are short, on the other hand, the return point is largely unaffected so the whole increase in the temptation to adjust must come from a fall of the lower trigger. Consequently, the lower trigger falls more after a risk increase, the shorter the high risk period is expected to be.

4. Discussion

By using a simple model and giving some numerical examples, I have shown the effect of a risk shift on optimal investment behavior. Given that the agents anticipate a cyclical behavior of risk, investment behavior is sensitive to the expected length of the high risk period. Specifically, when the expected high risk period is short, it is optimal to wait longer before acting than when the expected high risk period is long. The conventional analysis of doing comparative statics on the risk parameter misses this effect and underestimates the delay effect of a risk shift.

The mechanism at work in this model should be operative in many types of irreversible investment models where the return point overshoots the target because of an expected drift of the control variable. The exact specification of the process for wealth shocks and the instantaneous cost function will, of course, affect the quantitative effects of increased risk. A steeper cost function at the triggers, for example, would produce smaller trigger shifts.

The large shock assumption, i.e. that a wealth shock always induces immediate adjustment is unrealistic, of course. Relaxing it renders the model difficult to solve.[10] Inspection of how equation (1.9) changes when the large shock assumption is relaxed may, however, give some insights. Let $F(\varepsilon)$ represent the wealth shock's distribution function. We can then rewrite equation (1.9) as

$$\frac{\underline{U}_s^2 - a_s^2}{2} = rc + \lambda(s)\left(c + \int_{-\infty}^{\infty} \left(V(a_s + \varepsilon, s) - V(\underline{U}_s + \varepsilon, s) \right) dF(\varepsilon) \right) + \gamma(s)\left(c + V(a_s, \bar{s}) - V(\underline{U}_s, \bar{s}) \right) \qquad (1.14)$$

[10] In my dissertation (Hassler, 1994), I present an algorithm for solving the model numerically for more general assumptions about the shock. Numerical solutions for the case when ε has a normal distribution is supplied there. The qualitative results appear to be insensitive to variations in the form of the distribution of ε although good numerical accuracy was hard to achieve.

The functional form of the L.H.S. of (1.14) is identical to that of (1.9). The discussion in the previous section of why the return point is sensitive to the length of the high risk period appears to be equally applicable here. It thus seems safe to conjecture that the temptation to adjust from $\underline{U}_0$ after a shift to the high risk state increases in the expected length of the high risk state also for more general assumptions about the shock's distribution.

Now turn to the R.H.S. of (1.14). The difference between (1.14) and (1.9) is the integral in (1.14). It represents the difference in how costly a shock is if it hits the agent when *z* equals the return point relative to if *z* equals the lower trigger. The integral may be non-zero if shocks sometimes not cause immediate adjustment. Since $V(a_s, s) < V(\underline{U}_s, s)$, we expect it to be negative for sufficiently small shocks. There is, however, a more important thing to note here, i.e., how much the second term of (1.14) changes for a given increase in $\lambda(s)$ may become sensitive to $\gamma(1)$ if the integral is non-zero. The findings in Section 3 could in principle be reversed if the value of waiting increases substantially more when high risk periods are expected to be long. Such an effect, however, appears to be less general than the positive relation between the increase in the temptation to adjust and the expected length of the high risk period.

The model in this paper makes the stylized assumption that risk follows a two-state stochastic process. In principle the model could handle any finite number of risk states. It is, however, technically complicated to solve the model with many risk states, since each extra risk state will generate an extra dimension in the system of differential equations (1.8). It seems reasonable that addition of risk states with intermediate levels of risk has a simple consequence for optimal behavior – intermediate *Ss* bands. This should have small or no consequences for how optimal behavior depends on expected lengths of high risk periods.

My conjecture is that the mechanism behind the sensitivity of investment behavior to the length of the high risk period should survive modifications like the ones discussed above. The fall in the return point and thus the increase in the temptation to adjust for a given risk increase should generally be smaller, the shorter the high risk period is expected to be. A smaller temptation to adjust makes it optimal to accept larger deviations before adjusting.

In the introduction, I motivated my interest in the effects of fluctuating risk by its potential importance for aggregate behavior and, in particular, recessions and depressions. A rigorous theoretical analysis should consider general equilibrium effects, instead of assuming a fixed price as in the present model. This is, however, beyond the scope of this paper. Furthermore, the present model is concerned with individual behavior. Aggregate implications can therefore not be drawn immediately. I will, however, conclude by making some comments about aggregation.

First it should be noted that the aggregate effects of a widening of individual *Ss* bands, at least partially, are short run in nature. High risk periods may end, reversing the effects on demand. Furthermore, also if the agents allow their durable stocks to deviate more from the targets, eventually the new triggers will be hit. Of crucial importance is, thus, *how* persistent the short run effects are. This issue should be addressed by simulating aggregated *Ss*-models with fluctuating risk.

It is obvious that aggregate behavior is less lumpy and discontinuous than individual behavior in models as the ones presented in this paper. The smoothing effect aggregation may have is illustrated in Caplin and Spulber (1987), who show that individual inertia can disappear after aggregation. In *Ss* models, an individual only acts when the deviation of her control variable from the target reaches a limit of the *Ss* band. The flow of purchases and its sensitivity to aggregate shocks is thus determined by the cross section distribution of individual deviations, particularly the density close to the limits. Caballero and Engel (1991) generalize the result in Caplin and Spulber (1987) and show that under some circumstances the cross-section distribution of deviations can be invariant under aggregate and idiosyncratic shocks. In this case individual inertia is irrelevant to aggregate behavior.

A simultaneous shift in individual risk can, however, hardly wash out in the aggregate. If risk increases and the *Ss* bands widen at the same time for many agents in the economy, the cross-section density at the band limits must be affected. This will in general have a short-run effect on the flow of purchases and on its sensitivity to aggregate shocks.

Even if risk shifts thus should have an initial effect on aggregate demand, the sign is not unambiguous. An increase in risk in *Ss* models generally has both a negative and a positive immediate effect on demand. The negative effect is that a widening of the *Ss* band means that agents delay purchases, which reduces aggregate demand. This is the effect discussed in Section 3. On the other hand, an increase in risk also means that more agents are hit by wealth shocks. This tends to increase purchases. The overall initial effect is thus ambiguous in direction, although it appears more likely to be negative. Due to the large shock assumption in the present model, the two counteracting effects are here easily disentangled. The flow of agents hitting the lower trigger equals the cross-section density at the lower trigger multiplied by the depreciation rate. After a widening of the individual *Ss* bands the densities at the new triggers are initially zero. The density at the lower trigger is zero for a period equal to the fall in the normalized lower trigger, unless risk shifts back again. After that demand increases as depreciation moves the whole distribution towards the lower trigger. The second effect is that $\lambda(s)$ agents per unit of time are hit by wealth shocks and adjust.

Obviously, the model does not generate a constant cross-section density and the unconditional density is hard to compute. It is even more complicated to compute the average demand in the two risk states algebraically. By simulation this is, however, easily done. Aggregation of the model in Section 3 with equal parameters for all agents is certainly too stylized to give a realistic description of the exact path of demand after a risk shift.[11] The purpose of the simulation is rather aimed at illustrating how average demand in the two states may be affected by the length of the high risk period. reports the result from a simulation where I have used the results from Section 3 for $\lambda(1)$=0.10 and $\gamma(1)$=0.25 and 2.[12]

(*Table 1* approximately here)

In Table 1 we find that demand is substantially lower in the high risk states than in the low risk state, when high risk periods are short. Average demand in the two states is, on the other hand, equal up to two decimals when high and low risk periods are equal in length. The fact that the difference is larger in the case of short high risk periods is due to two things. First, in Section 3 we saw that the lower trigger falls more when high risk periods are expected to be short. The agents thus delay upgrades more. Second, short high risk periods imply that average demand in the high risk periods is more dependent on the short-run effect of a state shift. Short high risk periods, of course, tend to end soon and then the negative effect on demand is reversed. The initial negative effect on demand also diminish over time, as noted above. The long-run effect of a shift to higher risk was actually slightly positive in the simulations.

Using panel data, Eberly (1994) finds that households with higher year-to-year income variance have wider *Ss* bands. A one standard deviation higher income volatility implied 11% wider band. Judging from the simulation results, this may be of small or no importance for long-run demand. A simultaneous widening of all agents *Ss* bands of that order could, however, have short-run effects extending over several months or a few quarters.

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[11] More elaborate simulations are done in Hassler (1995). There it is shown that the negative effect of a band-widening may be substantial also a year after a risk increase.

[12] The cross-section density, $f(z)$ is approximated by dividing the interval between the state 1 lower trigger and state 0 return point in 241 points. Time is made discrete in units Δt that equal the time it takes to depreciate between two points in the cross-section approximation. This means that the time unit is approximately equal to 1/4 month. For each time unit net demand is then calculated as the sum of net purchases from those hit by wealth shocks and those hitting the lower trigger, i.e. $\lambda(s)\Delta t \sum_z (a_s - z) f(z) + (1-\lambda(s))\Delta t \sum_{z<\underline{U}_s} (a_s - z) f(z)$.

The general conclusion is thus not that the model supports the idea that a higher level of risk permanently reduces the demand for durables or investment goods.[13] The short-run, however, may be long enough to be of importance in a business cycle context. Risk fluctuations of moderately high frequency could potentially have substantial effects on the timing and volatility of demand. To study such effects we need investment models where risk is stochastic.

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[13] Remember, however, that in the presented model the target stock is unaffected by the level of risk. If precautionary savings is important also the target may respond to the level of risk. Here we would expect the effect to increase in the expected length of the high risk period. Another issue is substitution between durables and non-durables. Higher risk makes it more costly on average to consume durables which could give risk to a substitution of non-durables for durables when risk increases. Some progress on the latter issue has been made by Beaulieu (1991).

Appendix 1 Model Solutions

The Constant Risk Model

The Bellman equation for the problem is

$$\begin{aligned} V(z_t) &= \frac{z_t^2}{2}dt + (1 - rdt)\mathbf{E}_t V(z_{t+dt}) \\ &= \frac{z_t^2}{2}dt + (1 - rdt)\big(V(z_t) - \delta dt V_z(z_t) + \lambda dt(V(a) + c - V(z))\big). \end{aligned} \tag{A.1}$$

Setting dt^2=0, dividing by dt and collecting terms we get the following differential equation:

$$(r + \lambda)V(z) + \delta V_z(z) = \frac{z^2}{2} + \lambda(V(a) + c) \tag{A.2}$$

This is a standard differential equation and its set of solutions is given by

$$\begin{aligned} V(z) = &\frac{1}{2(r+\lambda)}z^2 - \frac{\delta}{(r+\lambda)^2}z + \frac{\delta^2}{r(r+\lambda)^2} + \frac{\lambda c}{r} + \frac{\lambda a^2}{2r(r+\lambda)} \\ &- \frac{\delta\lambda a}{r(r+\lambda)^2} + \frac{\lambda}{r}c_1 e^{-(r+\lambda)\delta^{-1}a} + c_1 e^{-(r+\lambda)\delta^{-1}z} \end{aligned} \tag{A.3}$$

where c_1 denotes the integration constant. The four unknowns, $\{\underline{U}, a, \overline{U}, c_1\}$, are found by solving the usual optimality conditions

$$\begin{aligned} &\text{Smooth pasting} \quad \begin{cases} V_z(\underline{U}) = 0 \\ V_z(a) = 0 \end{cases} \\ &\text{Value matching} \quad \begin{cases} V(a) + c = V(\underline{U}) \\ V(a) + c = V(\overline{U}). \end{cases} \end{aligned} \tag{A.4}$$

The optimality conditions can only be solved by employing numerical methods.

The Stochastic Risk Model

The Bellman equation for the problem is

$$V(z_t, s_t) = \frac{z_t^2}{2}dt + (1 - rdt)\mathbf{E}V(z_{t+dt}, s_{t+dt}) \tag{A.5}$$

For some values of z, a shift of s will lead to an immediate adjustment. As is intuitive and will be shown to be correct later, if the current risk state is high and z is close to but not equal to the lower adjustment point, a shift to low risk will reduce the value of waiting and cause immediate adjustment. For an interior region this will not happen. Inside this region the Bellman equation can be written as

$$V(z,s)=\frac{z^2}{2}dt+(1-rdt)\begin{pmatrix}V(z,s)-\delta dt V_z(z,s)+\\ \lambda(s)dt\left(V(a_s,s)+c-V(z,s)\right)+\\ \gamma(s)dt(V(z,\bar{s})-V(z,s))\end{pmatrix}. \tag{A.6}$$

Setting dt^2=0, dividing by dt we get the system of differential equations (3.2). The solutions to the system are given by

$$\begin{bmatrix}V(z,1)\\ V(z,0)\end{bmatrix}=\begin{bmatrix}\frac{\alpha_1}{2} & \beta_1\\ \frac{\alpha_0}{2} & \beta_0\end{bmatrix}\begin{bmatrix}z^2\\ z\end{bmatrix}+\begin{bmatrix}v_1 & v_2\\ 1 & 1\end{bmatrix}\begin{bmatrix}c_1e^{\rho_1 z}\\ c_2e^{\rho_2 z}\end{bmatrix}+\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}+\frac{1}{\Delta}\begin{bmatrix}\lambda(1)\psi_0 & \lambda(0)\gamma(1)\\ \lambda(1)\gamma(0) & \lambda(0)\psi_1\end{bmatrix}\begin{bmatrix}V(a_1,1)\\ V(a_0,0)\end{bmatrix}, \tag{A.7}$$

where

$$\begin{aligned}
\alpha_s &= \frac{1}{\Delta}\left(r+\lambda(\bar{s})+\gamma(\bar{s})+\gamma(s)\right), \text{ and,}\\
\beta_s &= -\frac{\delta}{\Delta}\left(\psi_{\bar{s}}\alpha_s+\gamma(s)\alpha_{\bar{s}}\right),\\
&\text{with}\\
\Delta &= \psi_0\psi_1-\gamma(0)\gamma(1), \text{ and,}\\
\psi_s &= r+\lambda(s)+\gamma(s),
\end{aligned} \tag{A.8}$$

$c_{1,2}$ are the integration constants and $[v_i,1]'$ is the eigenvector corresponding to the eigenvalue ρ_i of the matrix

$$\frac{1}{\delta}\begin{bmatrix}-\left(r+\lambda(1)+\gamma(1)\right) & \gamma(1)\\ \gamma(0) & -\left(r+\lambda(0)+\gamma(0)\right)\end{bmatrix}, \tag{A.9}$$

and,

$$\phi_s=\frac{1}{\Delta}\left(\psi_{\bar{s}}\left(\lambda(s)c-\delta\beta_s\right)+\gamma(s)\left(\lambda(\bar{s})c-\delta\beta_{\bar{s}}\right)\right). \tag{A.10}$$

Lastly,

$$V(a_1,1)=\frac{\Delta\begin{bmatrix}\left(\Delta-\lambda(0)\psi_1\right)\left(.5\alpha_1a_1^2+\beta_1a_1+v_1c_1e^{\rho_1a_1}+v_2c_2e^{\rho_2a_1}+\phi_1\right)\\ +\lambda(0)\gamma(1)\left(.5\alpha_0a_0^2+\beta_0a_0+c_1e^{\rho_1a_0}+c_2e^{\rho_2a_0}+\phi_0\right)\end{bmatrix}}{(\Delta-\lambda(1)\psi_0)(\Delta-\lambda(0)\psi_1)-\lambda(0)\lambda(1)\gamma(0)\gamma(1)}, \tag{A.11}$$

and,

$$V(a_0,0)=\frac{\Delta\begin{bmatrix}\left(\Delta-\lambda(1)\psi_0\right)\left(.5\alpha_0a_0^2+\beta_0a_0+c_1e^{\rho_1a_0}+c_2e^{\rho_2a_0}+\phi_0\right)\\ +\lambda(1)\gamma(0)\left(.5\alpha_1a_1^2+\beta_1a1_1+v_1c_1e^{\rho_1a_1}+v_2c_2e^{\rho_2a_1}+\phi_1\right)\end{bmatrix}}{(\Delta-\lambda(1)\psi_0)(\Delta-\lambda(0)\psi_1)-\lambda(0)\lambda(1)\gamma(0)\gamma(1)}. \tag{A.12}$$

Guess that $\underline{U}_1 < \underline{U}_0$.[1] If z is in the region $[\underline{U}_1,\underline{U}_0]$, a switch from risk state 1 to zero then leads to immediate adjustment. The function $V(z,0)$ is thus constant for $z\in [\underline{U}_1,\underline{U}_0]$ and the system of differential equations degenerates to

$$V(z,0) = V(\underline{U}_0,0),$$
$$V(z,1) = \frac{z^2}{2}dt + (1-rdt)\begin{pmatrix} V(z,1) - \delta dt V_z(z,1) + \\ \lambda(1)dt\left(V(a_1,1) + c - V(z,1)\right) + \\ \gamma(1)dt\left(V(\underline{U}_0,0) - V(z,1)\right) \end{pmatrix}. \tag{A.13}$$

By setting dt^2=0 and collecting terms I get for s=1,

$$\left(r + \lambda(1) + \gamma(1)\right)V(z,1) + \delta V_z(z,1) = \frac{z^2}{2} + \lambda(1)\left(V(a_1,1) + c\right) + \gamma(1)\left(V(\underline{U}_0,0)\right). \tag{A.14}$$

Using the same solution technique as above,

$$V(z,1) = \frac{z^2}{2\psi_1} - \frac{\delta z}{\psi_1{}^2} + c_0 e^{-(r+\lambda(1)+\gamma(1))\delta^{-1}z} + \psi_1{}^{-1}\left(\lambda(1)\left(c + V(a_1,1)\right) + \gamma(1)V(\underline{U}_0,0) + \delta^2\psi_1{}^{-2}\right) \tag{A.15}$$

Above the upper triggers the total cost function is horizontal. I have guessed (and verified) that for all tested parameter values

$$V(a_{\bar{s}},s) - V(a_s,s) < c \quad s = \{0,1\}. \tag{A.16}$$

The economic implication of (A.16) is that a switch in risk state never in itself induces downgrading of z. Whether (A.16) is true or not has to be checked when a preliminary full solution is found. If it is not true we have to construct a degenerate system of differential equations with one equation being a constant also in the upper region. With the parameter values I have chosen $V(a_{\bar{s}},s) - V(a_s,s)$ is never bigger than 0.103% of c, so (A.16) is far from being violated.

To summarize; $V(z,0)$ is given by equation (A.7) for $z \in [\underline{U}_0, \max(a_1,a_0)]$, and is constant and equal to $V(\underline{U}_0,0)$ for $z \le \underline{U}_0$. $V(z,1)$ is given by equation (A.7) for $z \in [\underline{U}_0, \max(a_1,a_0)]$, by (A.15) for $z \in [\underline{U}_1,\underline{U}_0]$, and is constant and equal to $V(\underline{U}_1,1)$ for $z \le \underline{U}_1$. Due to the large shock assumption I do not have to solve for $V(z,s)$ above $\max(a_1,a_0)$. To do this is straight forward, however.

To find the unknown integration constants, triggers and return points we solve the following non-linear system:

---

[1] That this is true is verified when a full solution is found.

$$
\begin{array}{ll}
\text{Smooth pasting} & \begin{cases} V_z(a_1,1)=0 \\ V_z(a_0,0)=0 \\ V_z(\underline{U}_1,1)=0 \\ V_z(\underline{U}_0,0)=0 \end{cases} \\
\text{Value matching} & \begin{cases} V(a_1,1)+c=V(\underline{U}_1,1) \\ V(a_0,0)+c=V(\underline{U}_0,0) \end{cases} \\
\text{Continuity} & \left\{ V(z,1)\in C^0 \right. \\
\text{Unknowns} & \left\{ a_1,a_0,\underline{U}_1,\underline{U}_0,c_0,c_1,c_2 \right\}.
\end{array}
\qquad (A.17)
$$

The continuity equation in (A.17) is needed because the differential equation changes functional form at $\underline{U}_0$ when $V(z,1)$ becomes a constant. Continuity is assured at all points except $V(\underline{U}_0,1)$ so this equation pins down one constant.

It is not possible to find an algebraic solution to this system, so it has to be solved with numeric methods.

Appendix 2 Proofs of the Propositions

Proof of Proposition 1

Evaluate equation (A.2) at $\underline{U}$ and a. Subtract the second equation from the first to get

$$(r+\lambda)\left(V(\underline{U})-V(a)\right)+\delta\left(V_z(\underline{U})-V_z(a)\right)=\frac{\underline{U}^2-a^2}{2}. \qquad (A.18)$$

The first term on the R.H.S. equals $(r+\lambda)c$ by the value matching condition and the second is zero by the smooth pasting condition so *Proposition1* follows directly.

Proof of Proposition 2

Denote the total cost function for the parameter vector $[c,\delta,\lambda,r]$ by $V(z)$ with the solutions to the optimality conditions (A.4) given by the vector $[\underline{U},a,\overline{U},c_1]$. Substitute $[kz,ka,k^2c_1,k^2c,k\delta]$ for $[z,a,c_1,c,\delta]$ in (A.3) to get a candidate cost function for the parameters k^2c and $k\delta$ that satisfies the differential equation (A.2). Denote this function by $\tilde{V}(kz)$, By doing the substitution we immediately find that

$$\tilde{V}'(kz)=kV_Z(z), \qquad (A.19)$$

and

$$\tilde{V}(kz)=k^2V(z). \qquad (A.20)$$

It now remains to show that $[k\underline{U}, ka, k\overline{U}]$ solves the optimality conditions for $\tilde{V}(\cdot)$. By using (A.19) and the assumption that $V(z)$ satisfies the smooth pasting conditions at $[\underline{U}, a, \overline{U}]$ we can write

$$\begin{aligned} \tilde{V}'(k\underline{U}) &= kV_Z(\underline{U}) = 0, \\ \tilde{V}'(ka) &= kV_Z(a) = 0, \\ \tilde{V}'(k\overline{U}) &= kV_Z(\overline{U}) = 0, \end{aligned} \tag{A.21}$$

Lastly, I use (A.20) and the assumption that $V(\cdot)$ satisfies the value matching conditions to show that

$$\tilde{V}(k\overline{U}) - \tilde{V}(ka) = k^2 V(\overline{U}) - k^2 V(\overline{U}) = k^2 c, \tag{A.22}$$

and

$$\tilde{V}(k\underline{U}) - \tilde{V}(ka) = k^2 V(\underline{U}) - k^2 V(\overline{U}) = k^2 c. \tag{A.23}$$

(A.21)-(A.23) show that $[k\underline{U}, ka, k\overline{U}, k^2 c_1]$ solves the optimality conditions for $\tilde{V}(\cdot)$ and thus defines an optimal policy. By defining $k \equiv \delta'/\delta$ *Proposition 2* follows immediately.

Proof of Proposition 3

Set the expected cost of adjusting equal to the expected cost of waiting one instant, evaluated at the lower triggers. Use the Bellman equation (A.6), the smooth pasting condition that $V_Z(\underline{U}_s, s) = 0$ and set dt^2=0; then

$$\begin{aligned} V(a_s, s) + c = \frac{\underline{U}_s^2}{2} dt + (1 - rdt)\big((V(\underline{U}_s, s)\big) \\ + \lambda(s)dt\big(V(a_s, s) + c - V(\underline{U}_s, s)\big) \\ + \gamma(s)dt\big(V(\underline{U}_s, \bar{s}) - V(\underline{U}_s, s)\big). \end{aligned} \tag{A.24}$$

The L.H.S. represents the expected total cost of immediate adjustment and the R.H.S. the expected total cost if one waits *dt* and then does what is optimal. Expanding the L.H.S. with the use of the Bellman equation (A.6) I get

$$\begin{aligned} V(a_s, s) + c = \frac{a_s^2}{2} dt + (1 - rdt)V(a_s, s) + \lambda(s)dtc + \\ \gamma(s)dt\big(V(a_s, \bar{s}) - V(a_s, s)\big) + c. \end{aligned} \tag{A.25}$$

By collecting terms, dividing by *dt* and using (A.24), (A.25) and the value matching condition in (A.17) we have

$$\frac{\underline{U}_s^2 - a_s^2}{2} = rc + \lambda(s)\big(V(\underline{U}_s, s) - V(a_s, s)\big)$$
$$+\gamma(s)\Big(\big(V(a_s, \bar{s}) - V(a_s, s)\big) - \big(V(\underline{U}_s, \bar{s}) - V(\underline{U}_s, s)\big)\Big) \tag{A.26}$$
$$= rc + \lambda(s)c + \gamma(s)\big(c + V(a_s, \bar{s}) - V(\underline{U}_s, \bar{s})\big)$$

The first two terms of the value of waiting are obviously positive and the last can be rewritten as

$$\gamma(s)\Big(c - \big(V(\underline{U}_s, \bar{s}) - V(a_s, \bar{s})\big)\Big) \geq 0. \tag{A.27}$$

The inequality is easy to see; $V(z,s) - V(z',s) \leq c \,\forall z, z'$ with equality only for $\{z \leq \underline{U}_s \text{ or } z \geq \underline{U}_s\}$ and $z' = a_s$. In less formal words, the minimum of the expected cost function can always be reached if the adjustment cost c is paid. The expected cost can thus never be higher than this minimum plus c.

Proof of Proposition 4

Fix the parameters $\{c, \delta, \lambda(0), \lambda(1), \gamma(0), \gamma(1), r\}$. Denote the cost function by $V(\cdot, s)$ with $\{\underline{U}_0, \underline{U}_1, a_0, a_1, \overline{U}_0, \overline{U}_1, c_0, c_1, c_2\}$ solving the optimality conditions. Now change the depreciation rate to $\tilde{\delta} = k\delta$ and the adjustment cost to $\tilde{c} = k^2 c$ where k is a positive scalar. Set the proposed new solution to $\{k\underline{U}_0, k\underline{U}_1, ka_0, ka_1, k\overline{U}_0, k\overline{U}_1, k^2 c_0, k^2 c_1, k^2 c_2\}$ with associated cost function by $\tilde{V}(\cdot, s)$. I now need to show that the optimality conditions in (A.17) are satisfied for the proposed solution. I use the tilde to denote constants given by the proposed solution and the new parameters. First note that from (A.15)

$$\begin{aligned}\tilde{V}_z(k\underline{U}_1, 1) &= k\underline{U}_1 \psi_1^{-1} - k\delta\psi_1^{-2} - k^2 c_0 e^{-\psi_1 \delta^{-1} \underline{U}_1} \psi_1 (k\delta)^{-1} \\ &= kV_z(a_1, 1) = 0.\end{aligned} \tag{A.28}$$

Similarly, from (A.7) we have

$$\tilde{V}_z(ka_0, 0) = \tilde{\alpha}_0 ka_0 + \tilde{\beta}_0 + k^2 c_1 e^{\tilde{\rho}_1 ka_0} \tilde{\rho}_1 + k^2 c_2 e^{\tilde{\rho}_2 ka_0} \tilde{\rho}_2. \tag{A.29}$$

From (A.8) and (A.9) we find that $\tilde{\alpha}_s = \alpha_s, \tilde{\beta}_s = k\beta_s, s = 0,1,$ and $\tilde{\rho}_i = k^{-1}\rho_i, i = 1,2$. Substituting this into (A.29) we get

$$\begin{aligned}\tilde{V}_z(ka_0, 0) &= \alpha_0 ka_0 + k\beta_0 + kc_1 e^{\rho_1 a_0} \rho_1 + kc_2 e^{\rho_2 a_0} \rho_2 \\ &= kV_z(a_0, 0) = 0.\end{aligned} \tag{A.30}$$

Following the same steps we find that also the other smooth pasting conditions are satisfied. For the value matching condition we note that

$$
\begin{aligned}
&\tilde{V}(ka_0,0)-\tilde{V}(k\underline{U}_0,0)\\
&=\tilde{\alpha}_0k^2a_0^2/2+\tilde{\beta}_0ka_0+k^2c_1e^{\tilde{\rho}_1ka_0}+k^2c_2e^{\tilde{\rho}_2ka_0}-\\
&\left(\tilde{\alpha}_0k^2\underline{U}_0^2/2+\tilde{\beta}_0k\underline{U}_0+k^2c_1e^{\tilde{\rho}_1k\underline{U}_0}+k^2c_2e^{\tilde{\rho}_2k\underline{U}_0}\right)\\
&=k^2\left(\alpha_0a_0^2/2+\beta_0a_0+c_1e^{\rho_1a_0}+c_2e^{\rho_2a_0}\right)-\\
&k^2\left(\alpha_0\underline{U}_0^2/2+\beta_0\underline{U}_0+c_1e^{\rho_1\underline{U}_0}+c_2e^{\rho_2\underline{U}_0}\right)\\
&=k^2\left(V(a_0,0)-V(\underline{U}_0,0)\right)=k^2c.
\end{aligned}
\tag{A.31}
$$

Doing the same for the other value matching conditions and checking the continuity condition using the same method we find that the optimality conditions are satisfied for the proposed solution.

References

Arrow, K. J. T. Harris, and J. Marschak, 1951, Optimal Inventory Policy, Econometrica 19, 250-272.

Beaulieu, J., 1991, Optimal Durable and Nondurable Consumption with Transaction Costs, mimeo, Massachusetts Institute of Technology.

Bentolila, S. and G. Bertola, 1990, Firing costs and Labor Demand: How bad is Eurosclerosis?, Review of Economic Studies 57, 381-402.

Caballero, R., 1991, On the Sign of the Investment-Uncertainty Relationship, The American Economic Review 81, 279-287.

Caballero, R., 1993, Durable Goods: An Explanation for Their Slow Adjustment, Journal of Political Economy 101, 351-383.

Caballero, R. and E. Engel, 1991, Dynamic (*S,s*) Economies, Econometrica 59, 1659-1686.

Caplin, A. S. and D. F. Spulber, 1987, Menu Costs and the Neutrality of Money, The Quarterly Journal of Economics 102, 703-725

E.C. Commission, 1988, The Economics of 1992 (Oxford University Press).

Eberly, J., 1994, Adjustment of Consumers' Durables Stocks: Evidence from Automobile Purchases, Journal of Political Economy 102, 403-436.

Grossman, S. J. and G. Laroque, 1990, Asset Pricing and Optimal Portfolio Choice in the Presence of Illiquid Durable Consumption Goods, Econometrica 58, 25-51.

Hassler, J., 1993, Effects of Variations in Risk on Demand – the Empirics, Seminar Paper No. 554, Institute for International Economic Studies, Stockholm University.

Hassler, J., 1994, chapter III of dissertation at Massachusetts Institute of Technology, published as Effects of Variations in Risk on Demand and Measures of Business Cycle Comovements - Three essays about the business cycle, Monograph Series No. 24, Institute for International Economic Studies, Stockholm University.

Hassler, J., 1995, Fluctuating Risk in an Aggregated Ss-Model – some empirical results, mimeo, Institute for International Economic Studies, Stockholm University.

McDonald, R. and D. Siegel, 1986, The Value of Waiting to Invest, The Quarterly Journal of Economics 101, 707-727.

Pindyck, R., 1991, Irreversibility, Uncertainty, and Investment, Journal of Economic Literature 29, 1110-1148.

Pindyck, R., and A. Solimano, 1993, Economic Instability and Aggregate Investment, NBER Working Paper #4380.

Romer, C. D., 1990, The Great Crash and the Onset of the Great Depression, Quarterly Journal of Economics 105, 597-624.

Scarf, H. E., 1959, The Optimality of s,S Policies in the Dynamic Inventory Problem, in: K. J. Arrow, S. Karlin and P. Suppes, eds., Mathematical Methods in the Social Sciences (Stanford University Press).

Schwert, G. William, 1989, Why Does Stock Market Volatility Change Over Time?, The Journal of Finance 44, 1115-1153.

Sheshinski, E., and Y. Weiss, 1977, Inflation and Costs of Price Adjustment, Review of Economic Studies 44, 287-303.

Figure 1 A Sample Path of *w*, κ and *z*.

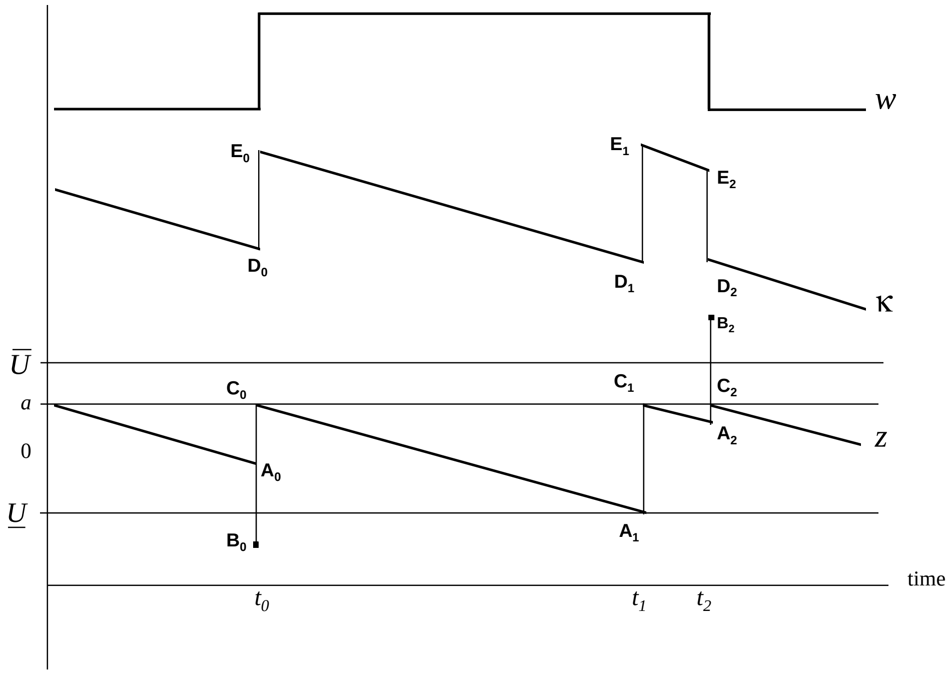


Figure 2 Total Cost Function

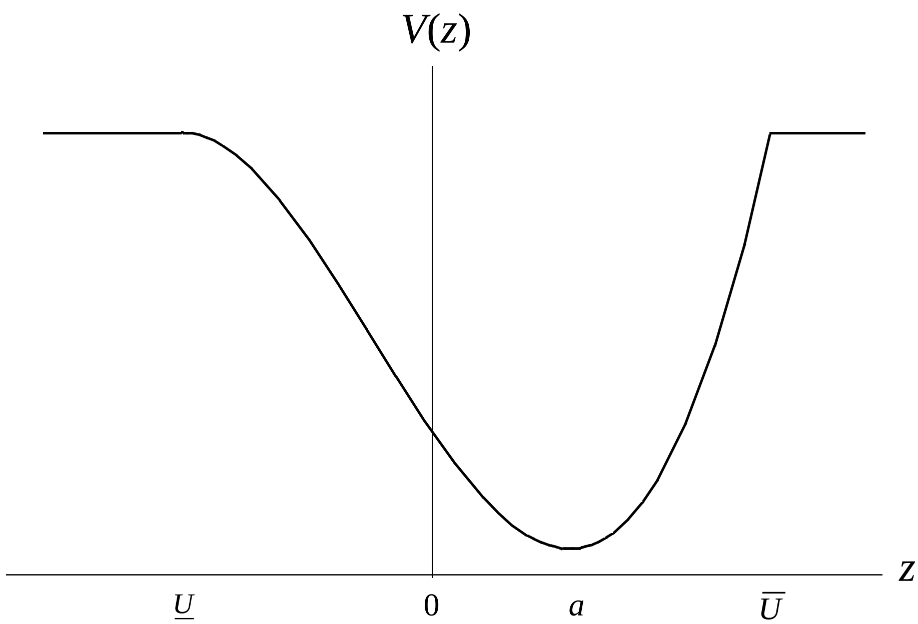


Figure 3 Effects of Risk Increase From 1% Per Year

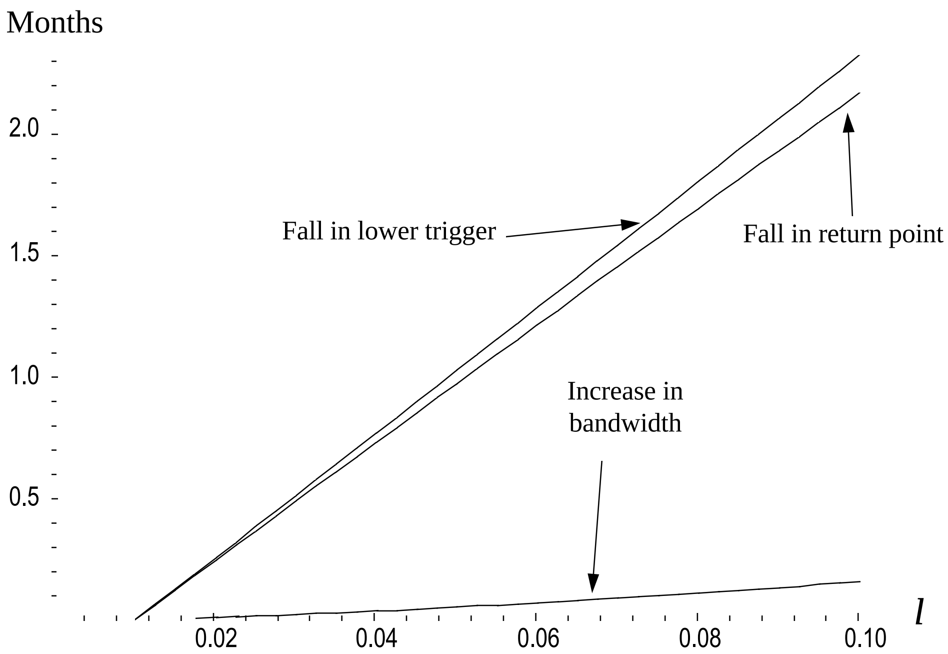


Figure 4 Total Cost Function

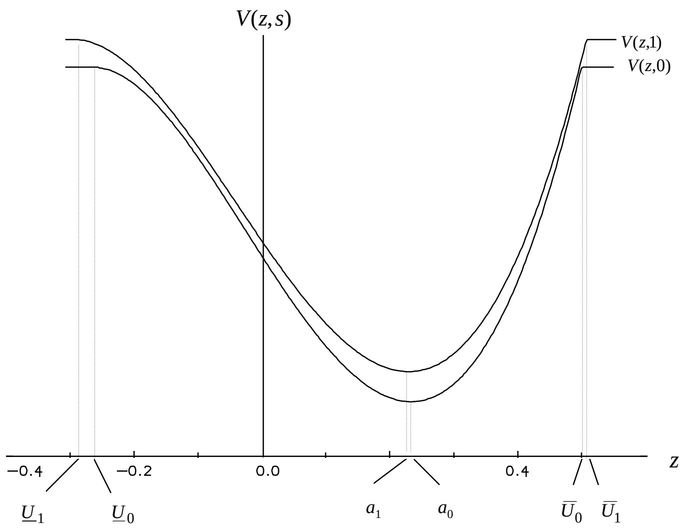


Figure 5 Change in Bandwidth After Shift to High Risk

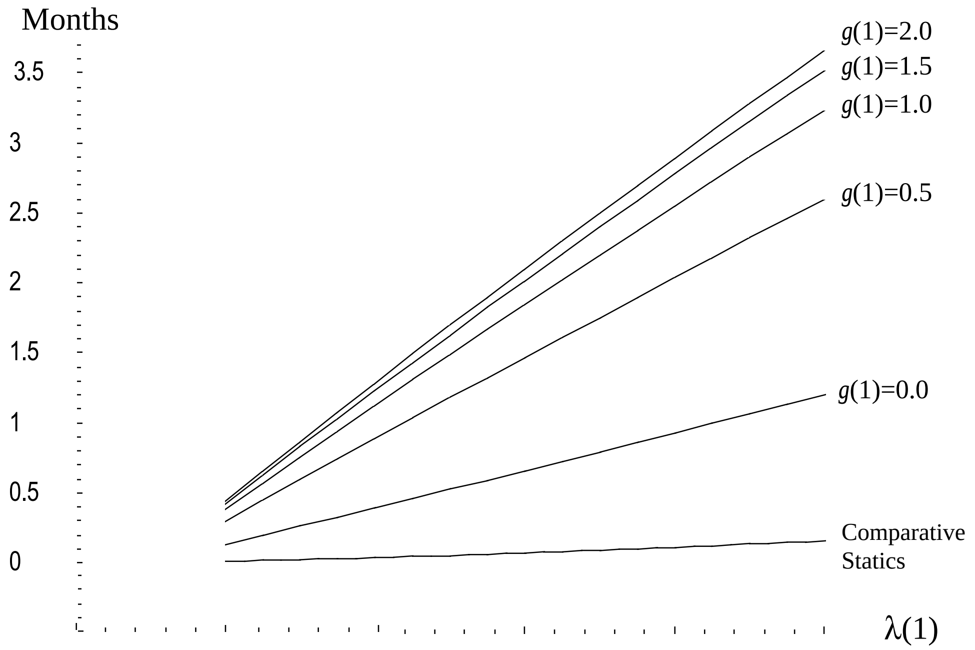


Figure 6 Fall in Lower Trigger Point after a Shift to High risk

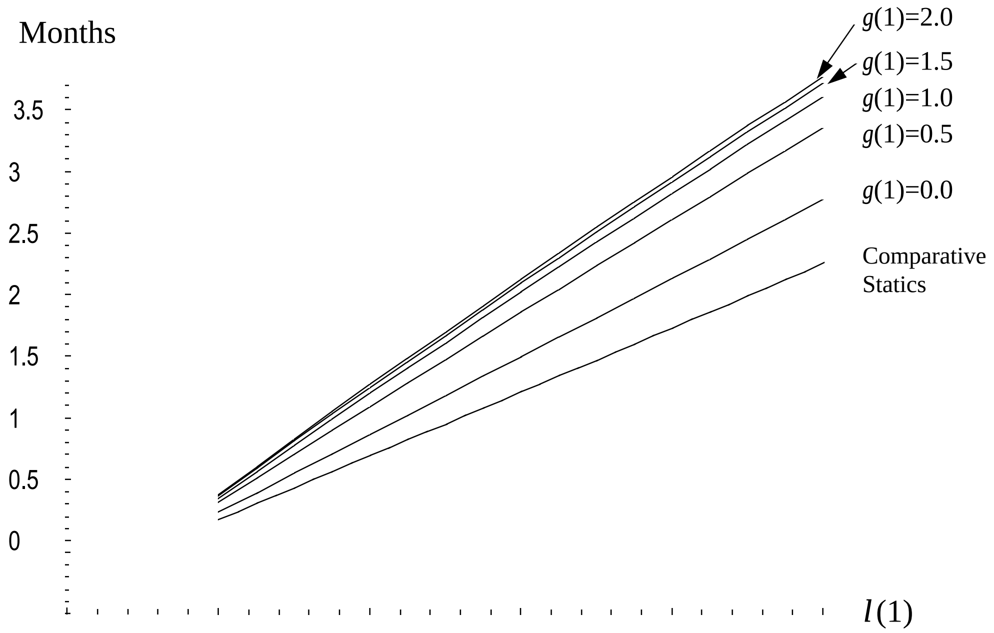
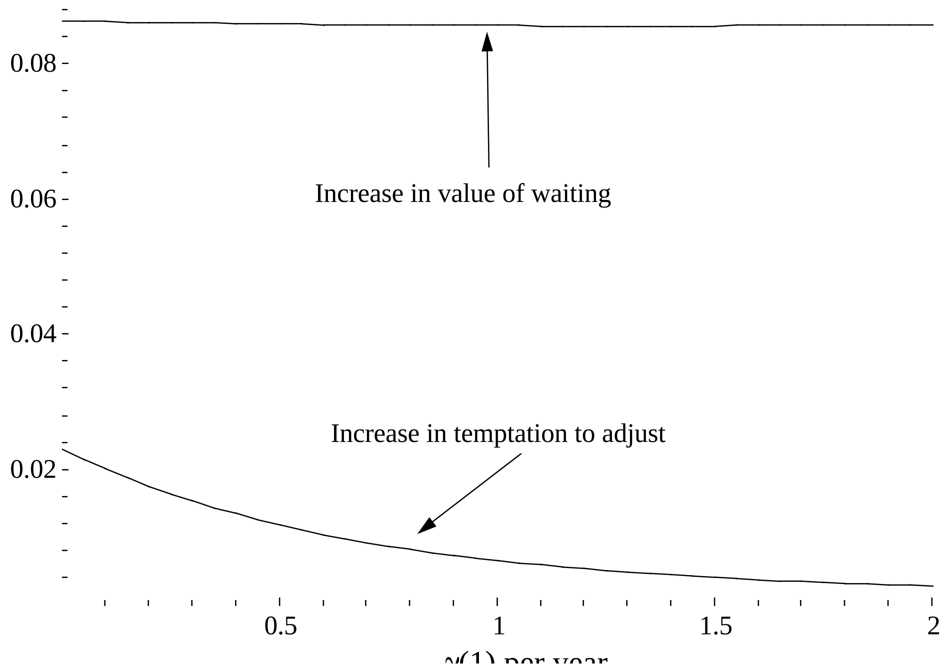


Figure 7 **Increase in Value of Waiting and Temptation to Adjust After a Risk State Shift***



* Computed at $\underline{U}_0$

Table 1 Net Demand for the Durable

	Relative[1] Average Net Demand	
	Short High Risk Periods, $\gamma(1)=2$	Long High Risk Period, $\gamma(1)=0.25$
In Low Risk State	1.03	1.00
In High Risk State	0.76	1.00
First Month After Shift to High Risk	0.25	0.24
First Month After Shift to Low Risk	3.41	3.20

1. Net demand relative to average over all time intervals.