1 a) Business cycle regularities

General idea: Some sort of recurrence: Upturns are followed by downturns.

Two approaches:

First approach:

Classical business cycle: NBER Burns and Mitchell: Try to identify common turningpoints from individual series.

Not a clear statistical foundation, a judgement by NBER, Business Cycle Dating Committee,Robert Hall, Martin Feldstein, President,Jeffrey Frankel, Robert Gordon, Christina Romer,David Romer,Victor Zarnowitz.

"A recession is a significant decline in economic activity spread across the economy, lasting more than a few months, normally visible in real GDP, real income, employment, industrial production, and wholesale-retail sales. A recession begins just after the economy reaches a peak of activity and ends as the economy reaches its trough. Between trough and peak, the economy is in an expansion. Expansion is the normal state of the economy; most recessions are brief and they have been rare in recent decades."

"The committee views real GDP as the single best measure of aggregate economic activity."

"Most of the recessions identified by our procedures do consist of two or more quarters of declining real GDP, but not all of them.

Quotes from http://www.nber.org/cycles/recessions.html

Second approach:

Idea: Non-stationary variables can be separated in a cyclical (stationary) part and a trending non-stationary. The cyclical is (approximately) covariance stationary:

$$EY_t = \mu \text{ (typically 0) } \forall t$$
$$E\left(Y_t - \mu\right)\left(Y_{t-k} - \mu\right) = \psi_k \forall t$$

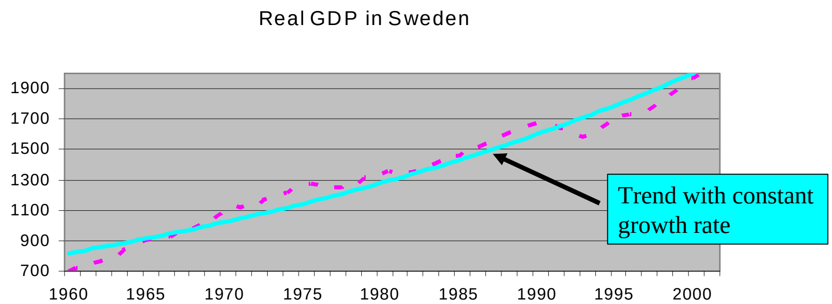
Y_t can be a vector of many variables.

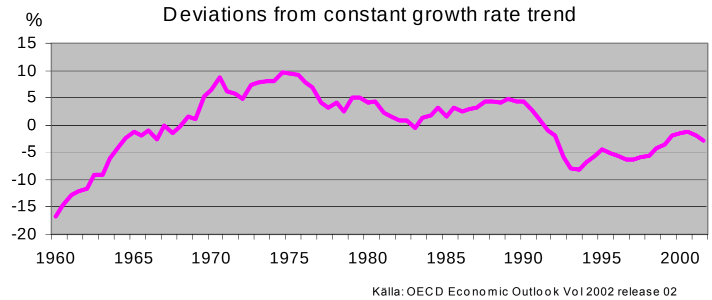
Empirical problem:

No unique way of separating cycle from trend.

A time-series - a sum of sine-waves of different frequency – like light or noise.

A lot of variation at fairly low frequencies. Should this be kept? Remains if we use log-linear trend. Show!





Most common way to separate: Hodrick-Prescott (Whittaker-Hendersson) filter. Solution to

$$\min_{\{Y_{c,t},Y_{tr,t}\}_0^T} \sum_{t=0}^{T} (Y_{c,t})^2$$

$$s.t. \sum_{t=2}^{T} ((Y_{tr,t} - Y_{tr,t-1}) - (Y_{tr,t-1} - Y_{tr,t-2}))^2 = k$$

$$Y_t = Y_{tr,t} + Y_{c,t}$$

Trading of tracking Y_t (giving small $Y_{c,t}$) against a changing the slope of the trend $Y_{tr,t}$. Lagrange multiplier on first constraint determines split. Can be *correct* given a special structure of the data generating process, e.g.,

$$(1-L)^2 Y_{tr,t} \equiv (Y_{tr,t} - Y_{tr,t-1}) - (Y_{tr,t-1} - Y_{tr,t-2}) = \varepsilon_t$$
$$Y_{c,t} = \nu_t$$

with ε_t and ν_t i.i.d.

In practice, easy to implement (a linear filter). Decide λ first, then multiply series by the matrix

$$Y_{c,t} = \left[I - (I + \lambda\kappa'\kappa)^{-1}\right] Y_t$$

.

where κ is a matrix with dimension $n-2, n$ if the samle size is n, given by

$$\kappa = \begin{bmatrix} 1 & -2 & 1 & 0 & ... & 0 \\ 0 & 1 & -2 & 1 & ... & 0 \\ 0 & 0 & 1 & -2 & ... & 0 \\ . & . & . & . & ... & . \\ . & . & . & . & ... & . \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

The matrix $\left[I - (I + \lambda\kappa'\kappa)^{-1}\right]$ doesn't contain many (any) zeros. This means that $Y_{c,t}$ is a linear combination of all previous *and future* values of Y_t. For example, take the 6 observation case for $\lambda = 10$, giving

$$\left[\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix} - \left(\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix} + \lambda \begin{bmatrix} 1&-2&1&0&0&0\\ 0&1&-2&1&0&0\\ 0&0&1&-2&1&0\\ 0&0&0&1&-2&1 \end{bmatrix}^T \begin{bmatrix} 1&-2&1&0&0&0\\ 0&1&-2&1&0&0\\ 0&0&1&-2&1&0\\ 0&0&0&1&-2&1 \end{bmatrix}\right)^{-1}\right]_{\lambda=10}$$

$$= \begin{bmatrix} 0.41 & -0.37 & -0.19 & -5.1\times10^{-2} & 0.05 & 0.14\\ -0.37 & 0.69 & -0.22 & -0.12 & -3.4\times10^{-2} & 0.05\\ -0.19 & -0.22 & 0.77 & -0.19 & -0.12 & -5.1\times10^{-2}\\ -5.1\times10^{-2} & -0.12 & -0.19 & 0.77 & -0.22 & -0.19\\ 0.05 & -3.4\times10^{-2} & -0.12 & -0.22 & 0.69 & -0.37\\ 0.14 & 0.05 & -5.1\times10^{-2} & -0.19 & -0.37 & 0.41 \end{bmatrix}$$

It has become a standard to use $\lambda = 1600$ for quarterly data.

λ should be adjusted down with lower frequency. Unclear how much, some use linear, implying ($\lambda = 1600/4 = 400$) for yearly, some quadratic (1600/16=100) some even forth power adjustment ($1600/4^4 = 6.25$). For a discussion see e.g., Marcet and Ravn, 2003. http://www.econ.upf.edu/docs/papers/downloads/588.pc

Some potential problems, but still used alot. Can keep to much low frequencies and to much high. Can be important.

Alternative: Band-pass filter. Decide a spectrum. E.g., 3-8 years.

Characteristics of different filters: Show Figure 1 in "The Swedish Business cycle".

Note also that $Y_{c,t}$ is a function of future Y_{t+s}. Dangerous to use Granger causality tests.

Definition:

Y_t Granger cause X_t if Y_t helps predict X_{t+s}, for $s > 0$ but the reverse is not true. Typically, in a regression of X_{t+1} on X_{t-s} for $s \geq 0$ and and Y_{t-s} $s \geq 0$, the coefficient on some Y_{t-s} is significant, but the reverse is not true.

1.1 Regularities

Typically look at correlation (possibly covariance) with output, $Y_{c,t}$.

X is **Procyclical** if

$corr(Y_{c,t}, X_{c,t}) > 0$.

X is **Countercyclical** if

$corr(Y_{c,t}, X_{c,t}) < 0$.

X is **Leading** if

$corr(Y_{c,t+s}, X_{c,t})$ is highest and positive for $s > 0$.

X is **Lagging** if

$corr(Y_{c,t+s}, X_{c,t})$ is highest and positive for $s < 0$.

Findings (USA):

Consumption and investments strongly procyclical.

Durables purchases very volatile and procyclical, services much less.

Some evidence consumption Granger cause output.

Inventory investment procyclical. – perhaps surprising.

Exports not very cyclical.

Government spending neither.

Most sectors correlated (except mining).

Employment and hours strongly procyclical., employment with a slight lag (1 quarter). Employment more volatile than hours/employee.

Vacancy rate leads.

Capacity utilization and productivity procyclical. (Solow residual).

Real wages almost acyclical (Rubinstein Tsiddon finds that for unskilled, there is substantial procyclicality).

Business cycle not a shift in labor demand along a constant labor supply function, then wages should be procyclical.

Nominal interest rate strongly procyclical.

Real rate slightly countercyclical.

Raw inflation not cyclical while detrended is or when separating sub-periods.

Consistent with expectation augmented Phillips-curve.

Monetary policy seems to affect output, prices much later.

These characteristics are fairly stable over time and between (similar) countries.

Small economies

Sweden: similar but exports much more procyclical, also export prices.

But no evidence that foreign demand drives Swedish GDP, FY not much correlated with Y. (Special case of Sweden: **devaluation cycles**)

Continental countries, in part. Benelux, Germany, Austria highly correlated.

Fairly strong *negative* correlation between real wages and employment, positive with unemployment.

1.2 VAR

Vector-Auto regression. "Explain" data in a regression where a set of variables is regressed on lagged variables of itself. For example, let

$$x_t \equiv \begin{bmatrix} \pi_t \\ u_t \\ r_t \end{bmatrix}$$

A (reduced form) VAR is then

$$x_t = \mathbf{B}_1 x_{t-1} + \mathbf{B}_2 x_{t-2} + ... \begin{bmatrix} u_{1,t} \\ u_{2,t} \\ u_{3,t} \end{bmatrix} \tag{1}$$

$$\mathbf{B}_1 = \begin{bmatrix} b_{1,11} & b_{1,12} & b_{1,13} \\ b_{1,21} & b_{1,22} & b_{1,23} \\ b_{1,31} & b_{1,32} & b_{1,33} \end{bmatrix}$$

By including sufficiently many lags, we can always make u_t uncorrelated over time.

Purpose:

1. Data description – provide benchmark for macro models.

    (a) Impulse-responses

    (b) Granger-causality – does one variable help predict another in the VAR regression? Is, for example, $b_{.,12}$ non-zero, then unemployment helps predict inflation.

(c) Variance decomposition – which innovations are important for which variables deviations from forecast?

2. Forecasting – provide good forecasts, better than most other methods.

3. Structural interpretation and policy evaluation. What is, e.g., the impact of a increase in the fed funds rate? Can something be said about policy changes? Requires more assumptions – structure.

Non-reduced form VAR's

1. Recursive VAR's

In (1) innovations are typically correlated. What are these shocks and what accounts for their correlation? Correlation makes interpretation of a change in just one innovation a little difficult. We can make the errors orthogonal in the following way,

$$\begin{bmatrix} \pi_t \\ u_t \\ r_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ b_{0,21} & 0 & 0 \\ b_{0,31} & b_{0,32} & 0 \end{bmatrix} \begin{bmatrix} \pi_t \\ u_t \\ r_t \end{bmatrix} + \mathbf{B}_1 x_{t-1} + \mathbf{B}_2 x_{t-2} + ... \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix}$$

This ordering can in principle be done in any way – giving different estimates of parameters. Sometimes theory can make the ordering less arbitrary. Suppose, the one variable is fiscal policy, maybe political constraints makes it impossible to react to current innovations to inflation or other variables. Or conversely, suppose we know that interest rates are set based on current inflation unemployment, then it makes sense to put it last. Now more direct to interpret an innovation, e.g., $\varepsilon_{3,t}$ as an innovation to the interest rate.

Could we allow more contemporaneous interdependencies than in this triangular way? Not right away. To see why consider a simple two variable VAR with one lag. Suppose

$$u_t = b_{01} r_t + b_{11} u_{t-1} + b_{21} r_{t-1} + \varepsilon_{1,t}$$
$$r_t = b_{02} u_t + b_{12} u_{t-1} + b_{22} r_{t-1} + \varepsilon_{2,t}$$

with ε_1 and ε_2 i.i.d. over time and between each other. To consistently estimate this, we as usual need the regressors to be uncorrelated with the error term. Clearly, this is true for the lagged variables. Is is true for the r_t in the first equation and u_t in the second? Let's check

$$
\begin{aligned}
E\left(r_t \varepsilon_{1,t}\right) &= E\left(\left(b_{02}u_t + b_{12}u_{t-1} + \varepsilon_{2,t}\right)\varepsilon_{1,t}\right) \\
&= E\left(\left(b_{02}u_t\right)\varepsilon_{1,t}\right) \neq 0 \text{ if } b_{02} \neq 0. \\
E\left(u_t \varepsilon_{2,t}\right) &= E\left(\left(b_{01}r_t + b_{11}u_{t-1} + \varepsilon_{1,t}\right)\varepsilon_{2,t}\right) \\
&= E\left(\left(b_{01}r_t\right)\varepsilon_{2,t}\right) \neq 0 \text{ if } b_{01} \neq 0
\end{aligned}
$$

Therefore, we cannot estimate the model directly, need some instrumental variables, for example. On the other hand, if the recursive formulating is correct, $b_{01} = 0$ so we have

$$
\begin{aligned}
u_t &= b_{11}u_{t-1} + b_{21}r_{t-1} + \varepsilon_{1,t} \\
r_t &= b_{02}u_t + b_{12}u_{t-1} + b_{22}r_{t-1} + \varepsilon_{2,t}
\end{aligned}
$$

Then, first equation is clearly identified. What about the second? We need to check

$$
\begin{aligned}
E\left(u_t \varepsilon_{2,t}\right) &= E\left(\left(b_{11}u_{t-1} + b_{21}r_{t-1} + \varepsilon_{1,t}\right)\varepsilon_{2,t}\right) \\
&= E\left(\varepsilon_{1,t}\varepsilon_{2,t}\right) = 0.
\end{aligned}
$$

2. Structural VAR's

Suppose we know, or dare to assume, some more explicit contemporaneous relations between the variables. For example, suppose we have a theory that says that inflation is caused by deviations of u_t from the natural rate, $\pi_t = -b\left(u_t - u^*\right)$ and that interest rates follow a Taylor rule, $r_t = r^* + 1.5\left(\pi_t - \pi^*\right) - 1.25\left(u_t - u^*\right)$. We can then build in these relations in the VAR and get a structural VAR. Note that this provides both an ordering and specific parameters.

$$
\begin{bmatrix} u_t \\ \pi_t \\ r_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -b & 0 & 0 \\ -1.25 & 1.5 & 0 \end{bmatrix} \begin{bmatrix} u_t \\ \pi_t \\ r_t \end{bmatrix} + \mathbf{B}_1 \begin{bmatrix} u_{t-1} \\ \pi_{t-1} \\ r_{t-1} \end{bmatrix} + ... \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix}
$$

If all equations comes from structural relations, we can, in principle use our model for policy change evaluation. Usually difficult. We can instead use fully structural models, soon to be seen. Problem with them is that they perform much worse in terms of replicating data.

2 b) Labor/leisure trade-off – the RBC model

A reasonable model of business cycles arguably needs fluctuations in labor use. One way; search – it takes time to find a job/worker and varying amounts of job-creation - destruction – Mortensen-Pissarides – Shimer Puzzle. Can also endogeneize search effort. Various variants of labor market – random search vs. directed search. Bargaining with/without commitment, wage posting, collective bargaining.

A principally different way, assume perfect labor markets and add a labor/leisure choice. Later, we will discuss how labor market works, non-convexities, unions and other things that can cause unemployment.

Problem of planner/representative household:

$$\max_{\{C_{t+s},K_{t+1+s},L_{t+s}\}_{s\geq 0}} E\sum_{s=0}\beta^s U\left(C_{t+s},L_{t+s}\right)|\Omega_t$$

$$\text{s.t. } C_{t+s}+K_{t+1+s}=Z_tF\left(K_{t+s},1-L_{t+s}\right)+\left(1-\delta\right)K_{t+s}\forall s\geq 0$$

$$K_t \text{ given.}$$

Z_t is a productivity shock, perhaps autocorrelated and non-stationary. Why one or few shocks?

Lagrange objective:

$$E_t\sum_{s=0}\beta^s\left(U\left(C_{t+s},L_{t+s}\right)+\lambda_{t+s}\left(Z_{t+s}F\left(K_{t+s},1-L_{t+s}\right)+\left(1-\delta\right)K_{t+s}-C_{t+s}-K_{t+s+1}\right)\right)$$

First-order conditions for c_t:

$$U_C\left(C_t,L_t\right)=\lambda_t$$

For L_t

$$U_L\left(C_t,L_t\right)=\lambda_tZ_tF_N\left(K_t,1-L_t\right)$$

For K_{t+1}

$$E_t\beta\lambda_{t+1}\left(Z_{t+1}F_K\left(K_{t+1},1-L_{t+1}\right)+1-\delta\right)=\lambda_t$$

Defining

$$R_{t+1}\equiv Z_{t+1}F_K\left(K_{t+1},1-L_{t+1}\right)+1-\delta$$

we get

$$U_C\left(C_t,L_t\right)=E_t\beta R_{t+1}U_C\left(C_{t+1},L_{t+1}\right) \tag{2}$$

and defining

$$w_t \equiv Z_t F_N \left(K_t, 1 - L_t\right)$$

$$\frac{U_L\left(C_t, L_t\right)}{U_C\left(C_t, L_t\right)} = w_t. \tag{3}$$

We have one *intertemporal* condition, the Euler condition (2)as before and now also a *intratemporal* decision (3).

2.1 Balanced growth

Let's now discuss under which circumstances we can have a balanced growth of the economy, i.e., a situation where wages, production and consumption grow at the same constant rate while labor supply and variables expressed per efficiency unit of labor are constant. We do this since this is an arguably reasonable description of reality and since it can have implications for business cycle behavior of the economy, being the main topic of interest in this course.

Suppose the wage increases by a factor A, and consumption also increases by the same factor A. Use intratemporal FOC. Then, if U_L/U_C *decreases* by a factor A, labor supply should be unchanged. Mathematically,

$$\frac{U_L\left(AC, L\right)}{U_C\left(AC, L\right)} = Aw \forall A \tag{4}$$

If this is the case, we can have a balanced growth path where C and w grow over time at the same rate while N is constant.

In particular, if we use $A = \frac{1}{C}$

$$\frac{U_L\left(C, L\right)}{U_C\left(C, L\right)} = w$$

$$\frac{U_L\left(1, L\right)}{U_C\left(1, L\right)} = \frac{1}{C} w$$

$$\Rightarrow \frac{U_L\left(C, L\right)}{U_C\left(C, L\right)} = C \frac{U_L\left(1, L\right)}{U_C\left(1, L\right)}$$

Clearly, this implies the utility must be of the form

$$U\left(C, L\right) = u\left(Cv\left(L\right)\right)$$

since only then is $\frac{U_L(C,L)}{U_C(C,L)}$ proportional to something only depending on L_t, with a proportionality factor

C. Therefore,

$$\frac{U_L(C,L)}{U_C(C,L)} = \frac{u'(Cv(L))\,Cv'(L)}{u'(Cv(L))\,v(L)} = C\frac{v'(L)}{v(L)}.$$

Result: Iff $U(C,L)$ can be written $u(Cv(L))$ can we have a case when wages and consumption grow while labor supply is constant.

Now, let's turn to what is required for a constant (stationary) interest rate. In a steady state with constant interest rate, and constant growth rate g of consumption the Euler equation is

$$\frac{U_C(C,L)}{U_C((1+g)C,L)} = \beta R.$$

For this to be true for all C, we need a function with constant intertemporal elasticity if substitution utility function (equivalently, CRRA). The only class of functions satisfying this are of the form

$$U(C,L) = u(Cv(L)) = \frac{\sigma}{\sigma-1}(Cv(L))^{\frac{\sigma-1}{\sigma}}, \tag{5}$$

or with $\sigma = 1$

$$U(C,L) = \ln C + \tilde{v}(L).$$

The parameter $\sigma > 0$ measures the degree of intertemporal substitution[1]

$$-\left(\frac{d\ln(U_c)}{d\ln c}\right)^{-1} = \sigma.$$

What happens with CARA?

$$U = -\sigma e^{-\sigma C}$$

$$U_C = \sigma e^{-\sigma C}$$

$$\frac{U_C(C)}{U_C(C(1+g))} = \frac{\sigma e^{-\sigma C}}{\sigma e^{-\sigma(C(1+g))}} = \frac{e^{-\sigma C}}{e^{-\sigma(C(1+g))}} = e^{\sigma C g}$$

In words, as the level of consumption increases, the rate of interest required to support a constant growth rate g *increases.* This is since with a CARA utility, the ratio of the marginal utilities between two consumption levels depends on the *difference* between them, not the ratio. Therefore, with CARA utility, constant interest rate can support growth that is constant in absolute levels, something that doesn't seem

[1] In words; the inverse of many percent decrease in the marginal utility a percent increase in consumption yields.

in accordance with empirics.

Final conclusion: if we want to have balanced growth, utility should be of the form (5).

2.2 Shocks and labor supply

A key task of the RBC model is to be able to produce variations in labor supply that is in accordance with empirics (remember the stylized facts, labor is procyclical and has quite high variability). Let's begin with a discussion on temporary vs. permanent shocks.

Consider the log-case. Then, the intertemporal Euler condition is

$$\frac{1}{C_t} = \beta E_t \frac{R_{t+1}}{C_{t+1}}$$

and the intratemporal

$$v'(L_t) = \frac{w_t}{C_t} \tag{6}$$

If a permanent technological shock changes wages and consumption by the same proportion (like along a balanced growth path), the RHS is unchanged and so should therefore labor supply be. On the other hand, a *temporary* technological should shift w_t *more* than C_t proportionally, since individuals want smooth consumption. Therefore, a temporary shock should affect labor supply more the more temporary it is. This suggests a problem with detrending. Suppose wage is a random walk, fully permanent. Then labor supply should not react. Detrend shocks using HP-filter -> series of temporary shocks. Using these shocks, we can "explain" that labor supply moves procyclically over the "cycle". Then we explain the facts (procyclical labor hours) by using an erroneous series of shocks. But in this case, are there any "cycles" at all?

We can go further by specifying

$$v(L) = \phi \ln(L).$$

Then, (6) is

$$\frac{\phi}{L_t} = \frac{w_t}{C_t}$$
$$\rightarrow C_t = \frac{w_t L_t}{\phi}$$

Using this in the Euler equation gives

$$\frac{1}{\frac{w_t L_t}{\phi}} = \beta E_t \frac{R_{t+1}}{\frac{w_{t+1} L_{t+1}}{\phi}}$$
$$1 = \beta E_t \left[R_{t+1} \frac{w_t}{w_{t+1}} \frac{L_t}{L_{t+1}} \right]$$

So, only a shock that makes w_t high *relative* to w_{t+1} should make leisure L_t low relative to L_{t+1}, for constant R_{t+1}.

If

$$w_{t+1} = w_t + \varepsilon_{t+1}$$
$$E_t w_{t+1} = w_t,$$

a shock to w_t should not in itself affect labor supply unless it affects the interest rate. We see that L_t is high when the interest rate is low. Since the real interest rate is slightly countercyclical variations in the interest rate should make leisure procyclical, and labor supply countercyclical. So this way to produce strongly procyclical labor hours seems doomed.

By changing v, we can make $\frac{L_t}{L_{t+1}}$ more or less responsive to $\frac{w_t}{w_{t+1}}$. Micro/labor evidence suggest a low elasticity, maybe 0.2, that is labor hours reacts little to wages. This is a problem since hours move much more than wages, but is the elasticity correctly measured by the labor people? Maybe idiosyncratic shocks are more permanent than aggregate?

2.3 Solving the model

2.3.1 Special case.

Again, with log consumption utility and full depreciation, we can solve the model analytically.

Set

$$U(C_t, L_t) = \ln C_t + v(L_t)$$
$$K_{t+1} + C_t = Z_t K_t^{\alpha} (1 - L_t)^{1-\alpha}$$

Defining the savings ratio s_t, we have $C_t = (1 - s_t) Z_t K_t^{\alpha} (1 - L_t)^{1-\alpha}$. The Euler equation is then

$$
\begin{aligned}
1 &= \beta E_t \left(\frac{C_t}{C_{t+1}} \right) R_{t+1} \\
&= \beta E_t \left(\frac{(1 - s_t) Z_t K_t^{\alpha} (1 - L_t)^{1-\alpha}}{(1 - s_{t+1}) Z_{t+1} K_{t+1}^{\alpha} (1 - L_{t+1})^{1-\alpha}} \right) Z_{t+1} \alpha K_{t+1}^{\alpha-1} (1 - L_{t+1})^{1-\alpha} \\
&= \beta E_t \left(\frac{1 - s_t}{1 - s_{t+1}} Z_t K_t^{\alpha} (1 - L_t)^{1-\alpha} \right) \alpha \frac{1}{K_{t+1}} \\
&= \beta E_t \left(\frac{1 - s_t}{1 - s_{t+1}} Z_t K_t^{\alpha} (1 - L_t)^{1-\alpha} \right) \alpha \frac{1}{s_t Z_t K_t^{\alpha} (1 - L_t)^{1-\alpha}} \\
&= \beta E_t \left(\frac{1 - s_t}{1 - s_{t+1}} \right) \alpha \frac{1}{s_t}
\end{aligned}
$$

This is an independent of Z_t and thus a non-stochastic non-linear difference equation. It has a steady state at $\alpha\beta$ with a linearized explosive root $\frac{1}{\alpha\beta} > 1$. There is no initial condition for s_t so the only solution is to jump immediately to the steady state. Thus, we can conclude that only $s_t = \alpha\beta \forall t$ is consistent with the Euler equation.

The intratemporal FOC says

$$
\begin{aligned}
\frac{U_L (C_t, L_t)}{U_C (C_t, L_t)} &= w_t \equiv Z_t F_N (K_t, 1 - L_t) \\
v' (L_t) &= \frac{Z_t (1 - \alpha) K_t^{\alpha} (1 - L_t)^{-\alpha}}{C_t} \\
v' (L_t) &= \frac{Z_t (1 - \alpha) K_t^{\alpha} (1 - L_t)^{-\alpha}}{s Z_t K_t^{\alpha} (1 - L_t)^{1-\alpha}} \\
&= \frac{1 - \alpha}{(1 - L_t) s}
\end{aligned}
$$

So, labor supply is constant. If, for example,$v (L_t) = \phi \ln L_t$,we get $L_t = \frac{\phi\alpha\beta}{\phi\alpha\beta + 1 - \alpha} \in (0, 1)$. This is not particularly useful, right?

What is happening here?

1. Log consumption utility implies that a shock tomorrow change returns and marginal utility in opposite directions so they exactly cancel. Future does not matter for current consumption. Z_{t+1} cancels.

2. A shock today, Z_t increases the wage and consumption proportionally. The ratio of wages and marginal utility is thus not affected and marginal utility of leisure does not need to be changed.

2.3.2 Linearization

Let's specify a Markov process for the productivity shock.

$$\hat{Z}_t = \rho \hat{Z}_{t-1} + \varepsilon_t$$

where

$$\hat{Z}_t \equiv \frac{Z_t - EZ}{EZ},$$
$$\varepsilon_t, \text{i.i.d.}$$

Why first-order Markov? So that expectations about future realizations only depend on current shock. At some cost of complication, this could be relaxed, but it doesn't seem necessary empirically.

We first have to find the non-stochastic steady states. The Euler equation is $1 = \beta E_t \left(\frac{C_t}{C_{t+1}} \right) R_{t+1}$ so in a non-stochastic steady state

$$1 = \beta R = \beta \left(F_k \left(K_s, 1 - L_s \right) + 1 - \delta \right).$$

With with Cobb-Douglas production

$$\beta \left(\alpha \left(\frac{1 - L_s}{K_s} \right)^{1-\alpha} + 1 - \delta \right) = 1$$

and since in steady state

$$C_s = F \left(K_s, 1 - L_s \right) - \delta K_s$$

The intratemporal condition yields

$$\frac{U_L(F \left(K_s, 1 - L_s \right) - \delta K_s, L_s)}{U_C \left(F \left(K_s, 1 - L_s \right) - \delta K_s \right)} = F_N \left(K_s, 1 - L_s \right).$$

With log-log utility, $U = \ln C + \phi \ln L$, this is

$$\begin{aligned}
\frac{\phi C_s}{L_s} &= (1-\alpha)\left(\frac{K_s}{1-L_s}\right)^{\alpha} \\
\frac{\phi\left(K_s^{\alpha}(1-L_s)^{1-\alpha} - \delta K_s\right)}{L_s} &= (1-\alpha)\left(\frac{K_s}{1-L_s}\right)^{\alpha}
\end{aligned}$$

Giving

$$\begin{aligned}
L_s &= \phi\frac{-1+\beta-\beta\delta+\beta\alpha\delta}{-1-\phi+\beta-\beta\alpha+\beta\alpha\delta+\alpha-\beta\delta+\phi\beta-\phi\beta\delta+\phi\delta\beta\alpha} \\
K_s &= \\
&\left(\frac{(1+\phi)(1-\beta)+\beta\alpha(1-\delta)-\alpha+\beta\delta+\phi\beta\delta(1-\alpha)}{(1-\alpha)\beta\alpha}\right)^{\frac{-1}{1-\alpha}} \\
&\bullet\left(\frac{\alpha+\beta-\beta\delta-1+\beta\alpha\delta-\beta\alpha}{-1-\phi+\beta-\beta\alpha+\beta\alpha\delta+\alpha-\beta\delta+\phi\beta-\phi\beta\delta+\phi\delta\beta\alpha}\right)^{\frac{-\alpha}{1-\alpha}}
\end{aligned}$$

Now write the Euler equation (relaxing separability)

$$0 = \beta E_t U_C(C_{t+1}, L_{t+1}) R_{t+1} - U_C(C_t, L_t)$$

Using the fact that

$$\begin{aligned}
C_t &= Z_t F(K_{t,} 1-L_t) + (1-\delta) K_t - K_{t+1} \\
C_{t+1} &= Z_{t+1} F(K_{t+1,} 1-L_{t+1}) + (1-\delta) K_{t+1} - K_{t+2}
\end{aligned}$$

we see that the the Euler equation depends on $K_{t+2}, K_{t+1}, K_t, Z_{t+1}$ and Z_t as without labor, but now also on L_{t+1} and L_t. WHY?

Writing it in an abstract way, we get

$$E_t v_K(K_{t+2}, K_{t+1}, K_t, L_{t+1}, L_t, Z_{t+1}, Z_t) = 0,$$

and using a linear approximation around the steady state

$$
\begin{aligned}
& E_t v_K \left(K_{t+2}, K_{t+1}, K_t, L_{t+1}, L_t, Z_{t+1}, Z_t\right) \\
& \approx v_{K,1} E_t \left(K_{t+2} - K_S\right) + v_{K,2} \left(K_{t+1} - K_S\right) + v_{K,3} \left(K_t - K_S\right) \\
& + v_{K,4} E_t \left(L_{t+1} - L_S\right) + v_{K,5} \left(L_t - L_S\right) \\
& + v_{K,6} E_t \left(Z_{t+1} - 1\right) + v_{K,7} \left(Z_t - 1\right)
\end{aligned}
$$

The intratemporal condition is

$$
0 = Z_t F_N \left(K_t, 1 - L_t\right) U_C \left(C_t, L_t\right) - U_L \left(C_t, L_t\right).
$$

which depends on C_t, K_t and L_t but since we know that C_t can be written in terms of Z_t, K_t and K_{t+1} we can write this condition as

$$
0 = v_L \left(K_{t+1}, K_t, L_t, Z_t\right)
$$

where we need no expectations operator. Linearizing yields

$$
\begin{aligned}
& v_{L,1} \left(K_{t+1} - K_S\right) + v_{L,2} \left(K_t - K_S\right) \\
& + v_{L,3} \left(L_t - L_S\right) + v_{L,4} \left(Z_t - 1\right)
\end{aligned}
$$

Let us write the two choice variables (in log deviations) at t in vector form

$$
X_t \equiv \begin{bmatrix} \hat{K}_{t+1} \\ \hat{L}_t \end{bmatrix}.
$$

We can now write the two optimality conditions as

$$
E_t \left[\alpha_0 X_{t+1} + \alpha_1 X_t + \alpha_2 X_{t-1} + \beta_0 \hat{Z}_{t+1} + \beta_1 \hat{Z}_t\right] = 0 \tag{7}
$$

where

$$\alpha_0 = \begin{bmatrix} v_{K,1}K_s & v_{K,4}L_s \\ 0 & 0 \end{bmatrix}$$
$$\alpha_1 = \begin{bmatrix} v_{K,2}K_s & v_{K,5}L_s \\ v_{L,1}K_s & v_{L,3}L_s \end{bmatrix}$$
$$\alpha_2 = \begin{bmatrix} v_{K,3}K_s & 0 \\ v_{L,2}K_s & 0 \end{bmatrix}$$
$$\beta_0 = \begin{bmatrix} v_{K,6}Z \\ 0 \end{bmatrix}$$
$$\beta_1 = \begin{bmatrix} v_{K,7}Z \\ v_{L,4}Z \end{bmatrix}$$

We postulate (approximate) a linear decision rule

$$X_t = AX_{t-1} + B\hat{Z}_t$$
$$X_{t+1} = A^2X_{t-1} + AB\hat{Z}_t + B\hat{Z}_{t+1}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
$$B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

so

$$\begin{bmatrix} \hat{K}_{t+1} \\ \hat{L}_t \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \hat{K}_t \\ \hat{L}_{t-1} \end{bmatrix} + B\hat{Z}_t$$

Given this, and $\hat{Z}_{t+1} = \rho\hat{Z}_{t+1} + \varepsilon_{t1}$, (7) becomes

$$E_t\left[\alpha_0\left(A^2X_{t-1} + AB\hat{Z}_t + B\hat{Z}_{t+1}\right) + \alpha_1\left(AX_{t-1} + B\hat{Z}_t\right) + \alpha_2X_{t-1} + \beta_0\hat{Z}_{t+1} + \beta_1\hat{Z}_t\right] = 0$$
$$\alpha_0\left(A^2X_{t-1} + AB\hat{Z}_t + B\rho\hat{Z}_t\right) + \alpha_1\left(AX_{t-1} + B\hat{Z}_t\right) + \alpha_2X_{t-1} + \beta_0\rho\hat{Z}_t + \beta_1\hat{Z}_t = 0$$

Collecting terms,

$$\left(\alpha_0 A^2 + \alpha_1 A + \alpha_2\right) X_{t-1} + \left(\alpha_0 AB + \alpha_0 B\rho + \alpha_1 B + \beta_0 \rho + \beta_1\right) \hat{Z}_t = 0,$$

which must be true for all X_{t-1} and $\hat{Z}_t$

So,

$$\begin{aligned} \alpha_0 A^2 + \alpha_1 A + \alpha_2 &= 0 \\ \alpha_0 AB + \alpha_0 B\rho + \alpha_1 B + \beta_0 \rho + \beta_1 &= 0 \end{aligned}$$

which pins down the unknown matrices A, B.

Writing out the first equation yields,

$$\begin{aligned} &\begin{bmatrix} v_{K,1}K_s & v_{K,4}L_s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{21}a_{11} + a_{22}a_{21} & a_{12}a_{21} + a_{22}^2 \end{bmatrix} \\ &+ \begin{bmatrix} v_{K,2}K_s & v_{K,5}L_s \\ v_{L,1}K_s & v_{L,3}L_s \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} v_{K,3}K_s & 0 \\ v_{L,2}K_s & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

and changing notation

$$\begin{aligned} &\begin{bmatrix} x_1 & x_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{21}a_{11} + a_{22}a_{21} & a_{12}a_{21} + a_{22}^2 \end{bmatrix} \\ &+ \begin{bmatrix} x_3 & x_4 \\ x_5 & x_6 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} x_7 & 0 \\ x_8 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Four equations and four unknowns.

$$x_1 a_{11}^2 + x_1 a_{12} a_{21} + x_2 a_{21} a_{11} + x_2 a_{22} a_{21} + x_3 a_{11} + x_4 a_{21} + x_7 = 0$$
$$x_1 a_{11} a_{12} + x_1 a_{12} a_{22} + x_2 a_{12} a_{21} + x_2 a_{22}^2 + x_3 a_{12} + x_4 a_{22} = 0$$
$$x_5 a_{11} + x_6 a_{21} + x_8 = 0$$
$$x_5 a_{12} + x_6 a_{22} = 0$$

Solution is: $\left\{a_{22} = 0, a_{21} = \rho_1, a_{12} = 0, a_{11} = -\frac{x_6\rho_1 + x_8}{x_5}\right\}$,and
$\left\{\begin{array}{c} a_{22} = x_5 \frac{-x_7 x_6 + x_4 x_8}{x_8(-x_2 x_5 + x_1 x_6)}, a_{11} = -\frac{x_6 x_3 x_8 - x_5 x_6 x_7 - x_8^2 x_2}{x_8(-x_2 x_5 + x_1 x_6)}, \\ a_{12} = -\left(-x_7 x_6 + x_4 x_8\right) \frac{x_6}{x_8(-x_2 x_5 + x_1 x_6)}, a_{21} = -\frac{x_7 x_5^2 + x_1 x_8^2 - x_3 x_5 x_8}{x_8(-x_2 x_5 + x_1 x_6)} \end{array}\right\}$
where ρ_1 is a root of $\left(x_1 x_6^2 - x_2 x_5 x_6\right) \hat{Z}^2 + \left(2x_1 x_6 x_8 + x_5^2 x_4 - x_2 x_5 x_8 - x_3 x_6 x_5\right) \hat{Z} + x_7 x_5^2 + x_1 x_8^2 - x_3 x_5 x_8$.

Given parameters, we now know all the four solutions to A. Note that since L_{t-1} is not a state variable, we expect $a_{22} = a_{12} = 0$, so it should be the first set of roots that are relevant. We choose the stable solution (hopefully there is one).

We must then solve for B in the same way. Given parameter values with now know that the stochastic difference equation that K_{t+1} and L_t must follow satisfies

$$\begin{bmatrix} \hat{K}_{t+1} \\ \hat{L}_t \end{bmatrix} = A \begin{bmatrix} \hat{K}_t \\ \hat{L}_{t-1} \end{bmatrix} + B\hat{Z}_t$$

Clearly, what we have done extends to larger systems with more choices and state variables, but then one might not be able to solve things analytically.

2.4 Simulating the model

To test the model we need to calibrate it and provide some input shocks. The standard way of doing this is (was) to use the production function

$$Y_t = Z_t F\left(K_t, 1 - L_t\right)$$

taking logs of the Cobb-Douglas specification, we have

$$\ln Y_t = \ln Z_t + \alpha \ln K_t + (1-\alpha)\ln(1-L_t)$$
$$\ln Z_t = \ln Y_t - \alpha \ln K_t - (1-\alpha)\ln(1-L_t)$$

This is the Solow-residual.

With a more general production function, $Y_t = Z_t F(K_t, N_t)$, we can write

$$dY_t = F dZ_t + Z_t F_K dK_t + Z_t F_N dN_t$$
$$\frac{dY_t}{Y_t} = \frac{Z_t F}{Y_t}\frac{dZ_t}{Z_t} + \frac{Z_t F_K K_t}{Y_t}\frac{dK_t}{K_t} + \frac{Z_t F_N N_t}{Y_t}\frac{dN_t}{N_t}$$

Assuming competitive factor markets, factor prices are equal to marginal products,

$$Z_t F_N = W_t$$
$$Z_t F_K = R_t.$$

If in addition, product markets are competitive,

$$\frac{Z_t F_K K_t}{Y_t} = \frac{R_t K_t}{Y_t} \equiv \alpha_{K,t},$$
$$\frac{Z_t F_N N_t}{Y_t} = \frac{W_t N_t}{Y_t} \equiv \alpha_{N,t},$$

are then the time varying capital and labor shares of value added. So

$$\frac{dZ_t}{Z_t} = \frac{dY_t}{Y_t} - \alpha_{K,t}\frac{dK_t}{K_t} - \alpha_{N,t}\frac{dN_t}{N_t}$$

From this we can estimate the process for the technology shocks, in particular ρ. For the other parameters, we use $\alpha =$ income share of capital $\approx \frac{1}{3}$, δ we can take from other sources and β we calibrate to get some reasonable rate of return in steady state. θ we can calibrate from the share of hours spent working, something like 20%. Then, we are done and have a numerical model and we can run it to see how well it replicates business cycles. That is, we use the representation

$$\begin{bmatrix} \hat{K}_{t+1} \\ \hat{L}_t \end{bmatrix} = A \begin{bmatrix} \hat{K}_t \\ \hat{L}_{t-1} \end{bmatrix} + B\hat{Z}_t$$

and plug in our shocks.

The model does reasonably well, perhaps surprisingly well. It can account for a large share of the variation in Y and I. But there is much too little volatility in C and N in the model. Wages and productivity much too procyclical (but remember, we have only one shock.)

To drive the fluctuations in the model, we need very strong productivity shocks. They are also required to be negative often in order to cause recessions. Many argue that these negative shocks are unreasonable. One way out is to have capacity utilization and effort. Another way is to have noncompetitive markets. In this case, shocks with smaller amplitudes are required and shocks of negative values can be very rare.

The model in itself, does not produce much of a transmission mechanism. Shocks have fairly short lasting effects. To generate business cycles, the technology shocks Z, must have a high degree of auto-correlation.

Problems

1. Noncompetitive labor markets.

2. Adjustment costs.

3. Noncompetitive product markets.

4. Unobserved movements in N and K, capacity utilization and effort.

3 c) Asset pricing

The Euler equation is an equilibrium relation between prices and consumption. We can use it to derive optimal consumption and investment decisions given a prices. We may, however, also use the Euler relation in the other direction. Taking the path consumption as given and derive what the prices have to be. In very simple economies, like endowment economies without storage, we might actually know the path of what consumption. In that environment we may introduce markets for bonds and capital and study equilibrium prices. This is the setup in the seminal Lucas (Econometrica, 1978) paper.

Assumptions:

- Large number of identical agents, maximizing $\sum_{t=0}^{\infty} \beta^t u(c_t)$ s.t. $a_{t+1} = (a_t - r_r) R_{t+1}$
- Output comes from an apple tree with stochastic crop d_t, .with a distribution that is Markov. $F(d_t|d_{t-1})$ known to all agents.
- Purpose: find p_t – the price of the tree (after current period harvest) as a function of the state of the economy (d_t).
- Perfect market in ownership of (shares in) the tree. All equal so no trade in equilibrium.
- No storage or foreign trade so consumption $c_t = d_t$.
- No bonds.

The Euler equation is

$$u'(c_t) = E_t \left(\beta u'(c_{t+1}) R_{t+1} \right). \tag{8}$$

The return on investing in the tree is

$$R_{t+1} = \frac{p_{t+1} + d_{t+1}}{p_t}.$$

Substituting into (8) gives

$$\begin{aligned} p_t &= E_t\left(\beta\frac{u'(d_{t+1})}{u'(d_t)}(p_{t+1}+d_{t+1})\right) \\ &= E_t\left(\beta\frac{u'(d_{t+1})}{u'(d_t)}(p_{t+1}+d_{t+1})\right) \\ p_t = p(d_t) &= \int_0^\infty \beta\frac{p(d_{t+1})}{u'(d_t)}(p_{t+1}+d_{t+1})\,dF(d_{t+1}|d_t) \end{aligned} \tag{9}$$

This is a functional equation that we sometimes can solve analytically if we specify u and F. Before trying that, let's substitute forward

$$\begin{aligned} p_t &= E_t\left(\beta\frac{u'(d_{t+1})}{u'(d_t)}(p_{t+1}+d_{t+1})\right) \\ &= E_t\left(\beta\frac{u'(d_{t+1})}{u'(d_t)}\left(E_{t+1}\left(\beta\frac{u'(d_{t+2})}{u'(d_{t+1})}(p_{t+2}+d_{t+2})\right)+d_{t+1}\right)\right) \\ &= E_t\left(\beta\frac{u'(d_{t+1})}{u'(d_t)}d_{t+1}+\beta^2\frac{u'(d_{t+2})}{u'(d_t)}d_{t+2}+\beta^2\frac{u'(d_{t+2})}{u'(d_t)}p_{t+2}\right) \end{aligned}$$

Continuing we find that

$$p_t = E_t\sum_{s=1}^{\infty}\beta^s\frac{u'(d_{t+s})}{u'(d_t)}d_{t+s} + E_t\lim_{s\to\infty}\beta^s\frac{u'(d_{t+s})}{u'(d_t)}p_{t+s}$$

If the last term is zero, p_t is the discounted value of the dividend – the fundamental solution. Consider the log utility case.$u'(d_t) = \frac{1}{d_t}$

$$\begin{aligned} p_t &= E_t\left(\beta\frac{d_t}{d_{t+1}}(p_{t+1}+d_{t+1})\right) \\ \frac{p_t}{d_t} &= E_t\left(\beta\left(\frac{p_{t+1}}{d_{t+1}}+1\right)\right) \end{aligned}$$

Denote the price-dividend ratio $\frac{p_t}{d_t}$ by ω_t, then we have

$$\begin{aligned} \omega_t &= \beta + \beta E_t\omega_{t+1} \\ E_t\omega_{t+1} &= \frac{1}{\beta}\omega_t - 1 \end{aligned}$$

This is an explosive difference equation in the price earnings ratio provided $\beta < 1$. It has a steady state at

$$\omega_t = \frac{\beta}{1-\beta} \forall t$$

This is the solution that is consistent with the fundamental value. Why is it a constant? Other solutions are bubbles. Can we rule out them out?

Using the fundamental value with a general utility but assuming *i.i.d.* output, then $E_t u'(d_{t+s}) d_{t+s} = Eu'(d) d$ is a constant for all s. Thus,

$$p_t = E_t \sum_{s=1}^{\infty} \beta^s \frac{u'(d_{t+s})}{u'(d_t)} d_{t+s} \tag{10}$$

$$p_t u'(d_t) = \frac{\beta}{1-\beta} Eu'(d) d$$

where we note that the RHS is a constant.

Suppose u is CRRA, then $u'(c) = c^{-\sigma}$ and

$$p_t = d_t^{\sigma} \frac{\beta}{1-\beta} E d_{t+1}^{1-\sigma}$$

Since $\sigma > 0$ for risk averse individuals, the price increases in d_t. Suppose $\ln d_t$ is normal with mean μ and standard deviation ξ. Then $E d_t^{1-\sigma} = e^{(1-\sigma)\mu + \frac{\xi^2(1-\sigma)^2}{2}}$,then

$$p_t = d_t^{\sigma} \frac{\beta}{1-\beta} e^{(1-\sigma)\mu + \frac{\xi^2(1-\sigma)^2}{2}}$$

Doing comparative statics on this, we see that the price increases in risk unless we have log utility. It increases in μ iff $\sigma < 1$ (high elasticity if intertemporal substitution).

Suppose there is a positive autorcorrelation in d_t. Then a high value of d_t implies high values of d_{t+s}. However, the value $u'(d_{t+s}) d_{t+s}$ depends on on substitution (u') and income effects (d). With log utility, we have seen that they cancel. With a general CRRA, $u'(d_{t+s}) d_{t+s} = d_{t+s}^{1-\sigma}$ which is increasing in d_{t+s} if σ is smaller than 1 and decreasing otherwise. What is the explanation?

CAPM

Now, let us add a market for a safe asset. Individuals maximize utility subject to

$$a_{t+1} = (a_t - c_t)(r_{t+1}(1-\omega_t) + R_{t+1}\omega_t)$$

where ω_t is the period t share of wealth held in the risky asset and r_{t+1} is the return on the safe bond. Recall that R_{t+1} is a risky return.

The Bellman equation is

$$V_t(a_t) = \max_{c_t,\omega_t} \{u(c_t) + E_t\beta V_{t+1}((a_t - c_t)(r_{t+1}(1-\omega_t) + R_{t+1}\omega_t))\}.$$

Using envelope conditions to substitute for $V\prime$, FOC's are

$$\begin{aligned}
c_t; u'(c_t) &= E_t[\beta u'(c_{t+1})((r_{t+1}(1-\omega_t) + R_{t+1}\omega_t))] \\
&= E_t[\beta u'(c_{t+1})((r_{t+1} + \omega_t(R_{t+1} - r_{t+1})))] \\
\omega_t; &\, E_t\beta u'(c_{t+1})(R_{t+1} - r_{t+1})(a_t - c_t) \\
&\Rightarrow E_t u'(c_{t+1})(R_{t+1} - r_{t+1}) = 0 \\
\text{or } E_t u'(c_{t+1}) r_{t+1} &= E_t u'(c_{t+1}) R_{t+1}
\end{aligned}$$

Substituting this into the FOC for c_t gives

$$u'(c_t) = E_t\beta u'(c_{t+1}) r_{t+1} = E_t\beta u'(c_{t+1}) R_{t+1}$$

where we note that

$$E_t u'(c_{t+1}) R_{t+1} \neq E_t u'(c_{t+1}) E_t R_{t+1}$$

but

$$E_t u'(c_{t+1}) r_{t+1} = r_{t+1} E_t u'(c_{t+1}).$$

With n different risky assets, with returns R_{it}, it is straightforward to show that (provided the FOC holds),

$$r_{t+1} E_t u'(c_{t+1}) = E_t u'(c_{t+1}) R_{it+1}$$

for all i. By using the definition of a covariance, we then have that

$$r_{t+1} E_t u'(c_{t+1}) = E_t u'(c_{t+1}) E_t R_{it+1} + cov_t(u'(c_{t+1}), R_{it+1}).$$

Implying

$$E_t R_{it+1} = r_{t+1} - \frac{cov_t(u'(c_{t+1}), R_{it+1})}{E_t u'(c_{t+1})} \tag{11}$$

This a variant of the Consumption CAPM, providing the equilibrium expected return on all assets held by agents not constrained by portfolio constraints.

Furtmermore, using CRRA, we have

$$E_t R_{it+1} = r_{t+1} - \frac{cov_t\left(\left(c_{t+1}/c_t\right)^{-\sigma}, R_{it+1}\right)}{E_t u'\left(c_{t+1}^{-\sigma}/c_t^{-\sigma}\right)}.$$

Using the linear approximation $f(x) \approx f(\bar{x}) + f'(\bar{x})(x - \bar{x})$, we have $cov(f(x), y) \approx f'(\bar{x})\, cov(x, y)$, . this can be further simplified. Making the approximation around $c_{t+1}/c_t = 1$ we have

$$E_t R_{it+1} = r_{t+1} + \sigma \frac{cov_t\left(c_{t+1}/c_t, R_{it+1}\right)}{E_t u'\left(c_{t+1}^{-\sigma}/c_t^{-\sigma}\right)} \tag{12}$$

Finally, assume there is an asset m that has a return that is perfectly correlated with consumption growth. Using (12) for asset m,

$$E_t R_{mt+1} = r_{t+1} + \sigma \frac{cov_t\left(R_{mt+1}, R_{mt+1}\right)}{E_t u'\left(c_{t+1}^{-\sigma}/c_t^{-\sigma}\right)} = r_{t+1} + \sigma \frac{var_t\left(R_{mt+1}\right)}{E_t u'\left(c_{t+1}^{-\sigma}/c_t^{-\sigma}\right)}$$

$$E_t u'\left(c_{t+1}^{-\sigma}/c_t^{-\sigma}\right) = \sigma \frac{var_t\left(R_{mt+1}\right)}{E_t R_{mt+1} - r_{t+1}}$$

Using this in (12) we get

$$\begin{aligned} E_t R_{it+1} &= r_{t+1} + \sigma \frac{cov_t\left(R_{mt+1}, R_{it+1}\right)}{\sigma \frac{var_t\left(R_{mt+1}\right)}{E_t R_{mt+1} - r_{t+1}}} \\ &= r_{t+1} + \frac{cov_t\left(R_{mt+1}, R_{it+1}\right)}{var_t\left(R_{mt+1}\right)}\left(E_t R_{mt+1} - r_{t+1}\right) \end{aligned}$$

Where we note that $\frac{cov_t(R_{mt+1}, R_{it+1})}{var_t(R_{mt+1})} \equiv \beta_{i,t}$ can be interpreted as a regression coefficient in a regression of the returns of asset i on asset m provided returns are stationary.

The Mehra-Prescott Puzzle

One of the most important puzzles in macroeconomics is the Mehra-Prescott equity premium puzzle. Within a strikingly simple model, Mehra and Prescott argues that the excess return on equity over a safe return should be much smaller than what we see in the data. This result has turned out to be surprisingly robust and a large literature has followed trying to explain the puzzle.

Let's derive the original result in a way very close to the original paper.

Consider a representative household living on a Lucas type island with preferences

$$U_t = E_t \sum_{s=0}^{\infty} \beta^s \frac{c_{t+1}^{1-\sigma}}{1-\sigma}.$$

Suppose output growth λ_t follows a Markov chain with n different possible realizations $\{\lambda_1, ..., \lambda_n\}$. The Markov assumption implies that

$$\Pr\{\lambda_t = \lambda_j | \lambda_{t-1} = \lambda_i\} = \phi_{ij},$$

Assume there is a Lucas tree-type asset being a claim to aggregate (non-starable) output d_t. Using (9), the price of this is

$$p_t = E_t\left(\beta \frac{u'(d_{t+1})}{u'(d_t)}(p_{t+1} + d_{t+1})\right)$$

Conjecturing that the utility function and the Markov assumption implies that price-dividend ratios ω take n different values depending on the current realization of λ_t, denoted ω_i

$$\begin{aligned}
\frac{p_t}{d_t} &= E_t\left(\beta \frac{u'(d_{t+1})}{u'(d_t)}\left(\frac{p_{t+1}}{d_t} + \frac{d_{t+1}}{d_t}\right)\right) \\
\frac{p_t}{d_t} &= E_t\left(\beta \frac{u'(d_{t+1})}{u'(d_t)}\left(\frac{p_{t+1}}{d_{t+1}}\frac{d_{t+1}}{d_t} + \frac{d_{t+1}}{d_t}\right)\right) \\
\omega_i &= E_t\left(\beta \lambda_{t+1}^{1-\sigma}(\omega_j + 1)\right) \\
&= \beta \sum_{j=1}^{n} \phi_{ij}\left(\lambda_j^{1-\sigma}(\omega_j + 1)\right)
\end{aligned}$$

This is a linear equation in n unknowns that we can solve for the price-dividend ratio in all states. The return going from state i to j is

$$R_{ij} = \frac{\omega_j \lambda_j d_t + \lambda_j d_t}{\omega_i d_t} = \frac{\lambda_j(\omega_j + 1)}{\omega_i}$$

And expected returns

$$R_i^e = \sum_{j=1}^{n} \phi_{ij} \frac{\lambda_j(\omega_j + 1)}{\omega_j}$$

If we introduce a market for a safe one-period bond in zero aggregate supply, the price of this bond is

$$\begin{aligned} p_t^f &= E_t \beta \frac{u'(d_{t+1})}{u'(d_t)} \\ &= \beta E_t \lambda_{t+1}^{-\sigma} = \beta \sum_{j=1}^{n} \phi_{ij} \lambda_j^{-\sigma} \end{aligned}$$

Let us know simplify and assume only two states, λ_1 and λ_2 with a symmetric transition probability ϕ. Furthermore, let

$$\begin{aligned} \lambda_1 &= 1 + \mu + \delta \\ \lambda_2 &= 1 + \mu - \delta \end{aligned}$$

and calibrate the model to the mean US growth rate (μ) and standard deviation (δ). The autocorrelation of output growth is $2\phi - 1$

Consider first the case when $\phi = \frac{1}{2}$, implying no autocorrelation in growth rates. Then, $\omega_i = \omega_j \equiv \omega$ and

$$\omega_i = E_t \left(\beta \lambda_{t+1}^{1-\sigma} (\omega_j + 1) \right)$$

$$\begin{aligned} \omega &= \beta \frac{\left(\lambda_1^{1-\sigma} (\omega + 1) + \lambda_2^{1-\sigma} (\omega + 1) \right)}{2} \\ \omega &= (\omega + 1) \beta \frac{(1 + \mu + \delta)^{1-\sigma} + (1 + \mu - \delta)^{1-\sigma}}{2} \end{aligned}$$

and

$$p^f = \beta \left(\frac{(1 + \mu + \delta)^{-\sigma} + (1 + \mu - \delta)^{-\sigma}}{2} \right)$$

The expected return on the risky asset is

$$\begin{aligned} R_i^e &= \sum_{j=1}^{n} \phi_{ij} \frac{\lambda_j (\omega_j + 1)}{\omega_j} = \frac{\omega + 1}{\omega} \sum_{j=1}^{n} \phi_{ij} \lambda \\ &= \frac{\omega + 1}{\omega} (1 + \mu) \\ &= \frac{2 (1 + \mu)}{\beta \left((1 + \mu + \delta)^{1-\sigma} + (1 + \mu - \delta)^{1-\sigma} \right)} \end{aligned}$$

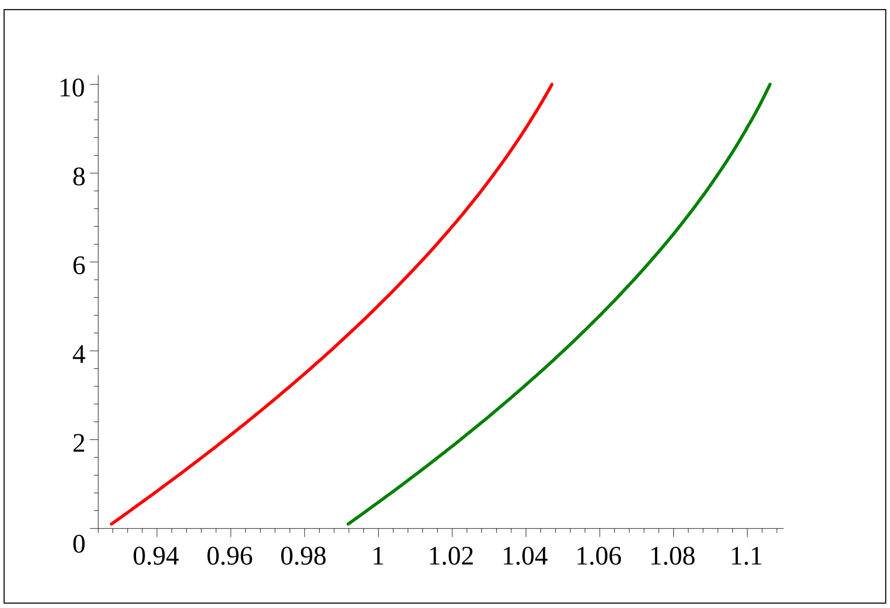
The return on the bond is

$$\frac{1}{p_f} = \frac{2}{\beta\left((1+\mu+\delta)^{-\sigma} + (1+\mu-\delta)^{-\sigma}\right)}$$

Let us now calibrate μ to the average yearly growth rate and δ to the standard deviation of output (1.8 and 3.6% respectively). Setting the average return on the stock market to 8% per year and the average return on the bondmarket to 1% per year, we get two equations in the two unknowns σ and β.

$$\left[1.08 = \frac{2(1+\mu)}{\beta\left((1+\mu+\delta)^{1-\sigma} + (1+\mu-\delta)^{1-\sigma}\right)}\right]_{\mu=0.018,\delta=0.036}$$
$$\left[1.01 = \frac{2}{\beta\left((1+\mu+\delta)^{-\sigma} + (1+\mu-\delta)^{-\sigma}\right)}\right]_{\mu=0.018,\delta=0.036}$$

In the following graph, I plot the two curves. They are both upwardsloping. The reason for this is that as we increase β, the incentive to save increases. To counteract that, a lower elasticity of substitution is required.(a higher σ).



A solution is the crossing – the problem is only that there is no crossing. Suppose for example that we set, $\beta = 0.98$. To motivate a return on equity of 8%, we then need $\sigma = 3.5$. But then, the return on bonds should be 7.6%, leaving a equity premium of of only 0.4%. This is the Mehra-Prescott Puzzle.

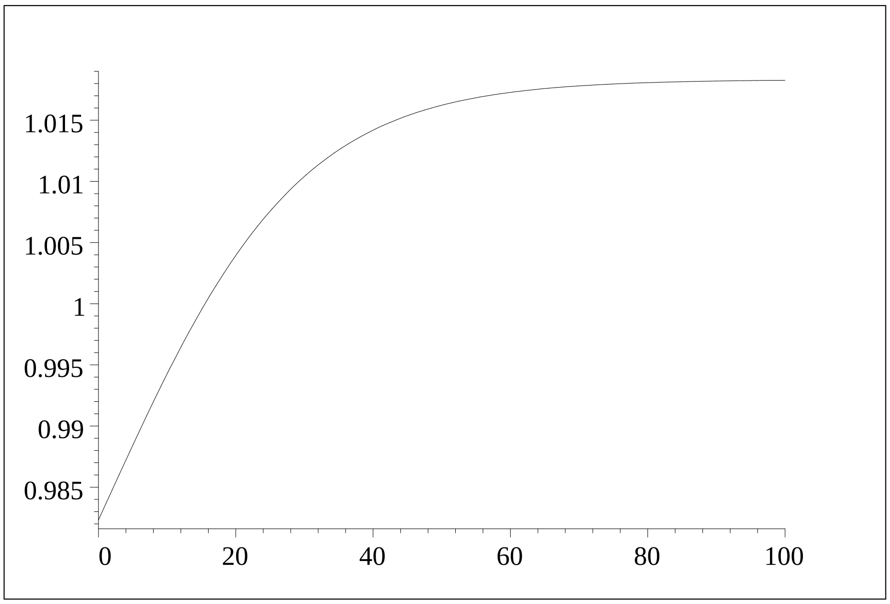
We can also look at a measure of the (adjusted) riskpremium.

$$\frac{R^e}{R_f\,(1+\mu)} = \frac{(1+\mu+\delta)^{-\sigma} + (1+\mu-\delta)^{-\sigma}}{(1+\mu+\delta)^{1-\sigma} + (1+\mu-\delta)^{1-\sigma}}$$

In reality this is

$$\frac{1.08}{1.01\,(1.018)} \approx 1.05$$

Plotting the RHS against σ we get



which as we see, increases, but is bounded. In fact,

$$\lim_{\sigma\to\infty}\left(\frac{(1+\mu+\delta)^{-\sigma} + (1+\mu-\delta)^{-\sigma}}{(1+\mu+\delta)^{1-\sigma} + (1+\mu-\delta)^{1-\sigma}}\right) = \lim_{\sigma\to\infty}\left(\frac{(1+\mu-\delta)^{-\sigma}}{(1+\mu-\delta)^{1-\sigma}}\right) = \frac{1}{1+\mu-\delta} \approx 1.0183.$$

$$\frac{1756}{40000}$$

: 0.043 9

$$(2)^{\alpha}\,(3)^{1-\alpha} = \frac{12}{5}$$

, Solution is: $\left\{\alpha = -\frac{\ln\frac{4}{5}}{\ln\frac{3}{2}} = \frac{\ln\frac{4}{5}}{\ln\frac{2}{3}}\right\}$ is true, Solution is: $\{\alpha = 0.550\,3\}$, Solution is: $\{\alpha = 0.5503\}$

: $2.\,449$