# Sharing Ambiguity

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In a version of the Ellsberg Paradox, the decision-maker is confronted with two urns, each containing 100 balls that are either Red or Blue. She is told that there are 50 of each color in the first ("unambiguous") urn, but no further information is provided about the second ("ambiguous") urn. There is a widely exhibited preference to bet on drawing Red (or Blue) from the first urn rather than from the second. Though such rankings are intuitive, they are inconsistent with subjective expected utility theory and, more generally, with reliance on *any* single probability measure to represent beliefs. Thus, the paradox illustrates the behavioral meaning of the Knightian distinction between risk (measurable or probabilistic uncertainty) and ambiguity (unmeasurable uncertainty).

The importance of the Ellsberg Paradox is the intuition that this distinction may be important more generally. In particular, it seems at least plausible to view consumption–savings and portfolio choice decisions as being qualitatively different than the choice of which bet to accept on the outcome of a coin flip; only the latter is a choice between risky prospects. My objective in this paper is to illustrate both the tractability and potential fruitfulness (for addressing the home-bias puzzle, for example) of a macro-style model that permits aversion not only to risk, but also to ambiguity.

I employ a simple two-period heterogeneous-agent economy. The time periods are $t - 1$ ("today") and $t$ ("tomorrow"). Uncertainty is represented by the state space $\Omega$. There are two consumers and consumer $i$'s consumption process is $(c^i_{t-1}, c^i_t)$, where $c^i_{t-1}$ is deterministic, $c^i_t$ is a random variable on $\Omega$, and each is real-valued. I consider an endowment economy with aggregate endowment $(Y_{t-1}, Y_t)$, where $Y_t$ is random. The efficient allocation of this endowment is usually posed as a problem of efficient risk-sharing. In particular, it is assumed that consumers' beliefs are represented by a common probability measure. If it is assumed further that von Neumann-Morgenstern (vNM) indices are (increasing, concave, and) additive across time and that consumers have a common discount factor, then efficient allocations are such that each $c^i_t$ is an increasing function of $Y_t$. Consequently, consumption is perfectly correlated across consumers, and if preference is homothetic, then consumption growth rates are equal. These predictions are often contradicted dramatically by data, particularly in international settings where consumers represent countries and where individual country growth rates respond to idiosyncratic shocks (see Karen Lewis [1999] for a survey; she terms the observed systematic violation of efficient risk-sharing the "consumption home-bias puzzle").

The model that I outline continues to assume complete markets and hence focuses on efficient allocations. However, taking Ellsberg seriously, I drop the assumption that it is risk alone that is to be shared. I assume that consumers are not sufficiently confident to assign sharp probabilities to all future states. Rather, following Itzhak Gilboa and David Schmeidler (1989), beliefs are represented by a (nonsingleton) set of priors, and consumption prospects are ranked according to their minimum expected utility as probabilities vary over the set of priors. Thus, consumers view consumption prospects as ambiguous, and the question of interest is: What is the nature of efficient sharing of ambiguity?

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Though I will refer to consumers as countries, the international interpretation is evidently optional.[1]

## I. An Economy with Ambiguity

In order to accommodate any idiosyncratic shocks for each of two countries, the state space is taken to be the two-dimensional set $\Omega = \{-1, 1\} \times \{-1, 1\}$, and the corresponding driving state process is $W_t = (W_t^1, W_t^2)$, where $W_t^i(\omega) = \omega^i$, $i = 1, 2$. The equal-probability measure $P$ on $\Omega$ implies that each $W_t^i$ has zero mean and unit variance and that $W_t^1$ and $W_t^2$ are independently distributed.

I assume that $i$ is more familiar with her own (domestic) component process $W_t^i$ than with the other (foreign) component $W_t^j$.[2] In extreme form this leads to no ambiguity for $i$ about $W_t^i$, though $W_t^j$ is ambiguous for her. In particular, while $i$ assigns equal probabilities to the two possible outcomes of $W_t^i$, for the foreign process she is confident only that the probability of each possible outcome lies in the interval

$$\left[\frac{1-\kappa^i}{2}, \frac{1+\kappa^i}{2}\right]$$

where $0 \leq \kappa^i < 1$ is a parameter representing the extent of her ambiguity about $W_t^j$. Domestic and foreign shocks are viewed as independent. Accordingly, 1's set of priors on $\Omega$, $\mathcal{P}^1$, consists of all products of the $(\frac{1}{2}, \frac{1}{2})$ measure on the first component space (for $W_t^1$) with measures on the second component space lying in the appropriate interval. Define $\mathcal{P}^2$ similarly. Thus, each country faces an analogue of the two-urn Ellsberg setting, though the identity of the ambiguous urn differs between countries, consistent with the subjective nature of ambiguity. Note that each set of priors contains $P$.

The description of the economy is completed by specifying utilities and the endowment. Country $i$'s utility function is

$$(1) \quad V^i(c_{t-1}^i, c_t^i) = \log c_{t-1}^i + \beta \min_{Q^i \in \mathcal{P}^i} E_{Q^i} \log c_t^i$$

where $0 < \beta < 1$. The standard logarithmic model is obtained in the special case $\kappa^i = 0$. The aggregate endowment is $Y_{t-1}$ and $Y_t$, where

$$Y_t(\omega)/Y_{t-1} = \exp(\mu^Y + s^Y\omega)/E_P[\exp(s^Y W_t)].$$

Thus, $e^{\mu^Y} - 1$ is the expected growth rate of the endowment according to $P$. It is without loss of generality to assume that $s^Y \geq 0$, which normalization brands $W_t^i = 1$ as a good realization and $W_t^i = -1$ as a bad one. For concreteness, suppose that $s^Y > 0$.

## II. Efficient Allocations

Efficient allocations solve the planning problem

$$(2) \quad \max\{V^1(c_{t-1}^1, c_t^1) + \lambda V^2(c_{t-1}^2, c_t^2)\}$$

subject to

$$c_\tau^1(\cdot) + c_\tau^2(\cdot) = Y_\tau(\cdot) \quad \tau = t-1, t$$

where $\lambda > 0$ is the relative utility weight for country 2. At any allocation and resulting consumption for 1, there is a measure $Q^1$ that solves the minimization in (1). Then $Q^1$ is completely described by the probability, denoted $(1 + \theta^1)/2$, that it assigns to the event $W_t^2 = 1$; similarly for country 2. Thus, an envelope theorem implies the same first-order conditions as would apply for a planning problem in which sets of priors are replaced by the single prior $Q^i$ for $i = 1, 2$. That is, $c_{t-1}^2 = \lambda c_{t-1}^1$ and $c_t^2 = \lambda \rho_t c_t^1$, where

$$(3) \quad \rho_t(\omega) = (1 + \theta^2\omega^1)/(1 + \theta^1\omega^2).$$

[1] For a related paper dealing with the characterization of efficient allocations see, for example, Alain Chateauneuf et al. (2000). The two-period model that follows differs in that more concrete results are delivered as a result of strong functional-form assumptions. More importantly, the model and its essential predictions may be extended to a multiperiod dynamic setting as I describe in what follows.

[2] See Gur Huberman (2000) for recent market evidence of the preference to bet on the familiar.

Deduce that

$$c_{t-1}^1 = \frac{1}{1+\lambda} Y_{t-1} \qquad c_{t-1}^2 = \frac{\lambda}{1+\lambda} Y_{t-1}$$
$$c_t^1 = \frac{1}{1+\lambda\rho_t} Y_t \qquad c_t^2 = \frac{\lambda\rho_t}{1+\lambda\rho_t} Y_t. \tag{4}$$

These expressions do not fully describe efficient allocations because the $\theta^i$'s and hence $\rho_t$ are endogenous. Since $\theta^i$ corresponds to a subjectively worst measure in $\mathcal{P}^i$, one might expect that it equals an extreme point $\pm\kappa^i$. In fact, that is not necessarily true, as indicated in the complete description of efficient allocations that follows.

THEOREM 1:
*Write $\theta = (\theta^1, \theta^2)$ and $\kappa = (\kappa^1, \kappa^2)$. Define the functions $\Phi^i(\theta)$ for $-\kappa \leq \theta \leq \kappa$ by*

$$\Phi(\theta) = \begin{bmatrix} \Phi^1(\theta) \\ \Phi^2(\theta) \end{bmatrix} = \begin{bmatrix} \int \log[1 + (\lambda\rho_t)^{-1}]\omega^1 dP \\ \int \log[1 + \lambda\rho_t]\omega^2 dP \end{bmatrix}$$

*where $\rho_t$ is defined in (3). Then an allocation solves (2) if and only if it has the form (4) where $\theta$ is the unique solution to*

$$\Phi(\theta) \leq s^Y \text{ and } [\Phi(\theta) - s^Y](\theta + \kappa) = 0. \tag{5}$$

*The latter solution satisfies $-\kappa \leq \theta \leq 0$.*

The formalism surrounding (5) suggests an interpretation whereby $\Phi(\cdot)$ represents the demand for volatility, $s^Y$ is the supply, and these are equilibrated by adjustment of $\theta$, the relevant (constrained) "price." Consistent with this interpretation, we have a complementary slackness condition, whereby for each $i$, either

$$[\Phi^i(\theta) < s_i^Y \qquad \theta^i = -\kappa^i]$$

or

$$\Phi^i(\theta) = s_i^Y.$$

In the special case $\Phi(-\kappa) < s^Y$, then $\theta = -\kappa$, which means that each country acts as though it attaches the smallest possible probability to good realizations of the foreign shock. Consequently, (4) represents a closed-form solution. Because $\Phi(0) = 0$, this case applies for small $\kappa$ or large $s^Y$. But Theorem 1 also covers the case of large $\kappa$ (small $s^Y$), where efficiency implies

$$\Phi(-\kappa) > s^Y \tag{6}$$

and hence

$$\theta > -\kappa.$$

That is, countries do *not* act as though an extreme point in their set of priors applies. Here, beliefs (in the sense of the shadow singleton prior for each agent) are selected endogenously in equilibrium.

Uniqueness of the solution $\theta$ implies uniqueness of the Arrow-Debreu prices that support the efficient allocation corresponding to any given $\lambda$. This is in contrast to my paper with Tan Wang (1994) which emphasizes the potential of ambiguity for generating price indeterminacy. It contrasts also with Truman Bewley (1998), who in his closely related model points to the fact that preferred sets in a two-state Edgeworth box typically have corners, supporting his claim that Knightian uncertainty inhibits trade. The difference here is the special asymmetric structure of ambiguity whereby $i$ is ambiguous only about $W_t^j$. Thus $i$'s probabilistic beliefs about her own process $W_t^i$ pin down prices for consumption contingent on $W_t^i$ states.

A number of qualitative properties of efficient allocations can be derived from Theorem 1.[3] Consumption in each country is nonnegatively correlated with shocks in *both* countries. For the domestic shock, this is evident from (4); for the foreign shock, (5) implies

$$E_P[W_t^j \log c_t^i] \geq 0 \qquad i \neq j.$$

The extreme of equality with zero occurs precisely in allocations corresponding to (6); for example, it occurs if the ambiguity parameter $\kappa$ is large. Thus even when $i$'s ambiguity about

[3] These properties rely on the facts: (i) $\rho_t$ is decreasing in $W_t^1$ and increasing in $W_t^2$; (ii) Cov$[f(X), X] \geq 0$ for any random variable $X$ and increasing function $f$; and (iii) various properties of $\Phi$ that are available from the author upon request. The key property is described after the proof of the theorem.

$W_t^j$ is large, it is not efficient for her to "short" the foreign shock.

A second implication is that consumption growth rates are not perfectly correlated across countries. In fact, idiosyncratic consumption growth rates are positively correlated with idiosyncratic shocks in the sense that (if $\kappa^j > 0$)

$$\text{(7)} \qquad \mathrm{Cov}_P[\log(c_t^i/c_{t-1}^i) - \log(Y_t/Y_{t-1}),\ W_t^i] > 0.$$

Comparative-statics analysis of (5) yields that (i) each $\theta^i$ is (weakly) decreasing in each $s_j^Y$ and (ii) $\theta^2$ is decreasing and $\theta^1$ increasing in $\lambda$. Recall that $(1 + \theta^i)/2$ can be interpreted as the ambiguity-adjusted probability that $i$ assigns, in equilibrium, to the good outcome $W_t^j = 1$. Accordingly, optimism in both countries declines with an increase in the volatility of aggregate consumption (due to an increase in $s_i^Y$ or $s_j^Y$) and a redistribution toward country 2 (increase in $\lambda$) makes 2 more optimistic and 1 more pessimistic. Finally, if one measures the size of home-bias in each country by the covariance in (7), then redistribution toward country 2 reduces home bias there and increases it in country 1.

PROOF OF THEOREM 1:

I include a sketch of the nontrivial part of the proof in order to emphasize its simplicity and because it is informative also about the nature of arguments needed in a multi-period setting. To show that every efficient allocation has the stated form, focus on the period-$t$ component of the planning problem (2), namely, on

$$\begin{aligned}
&\max_{c_t^i} \sum_i \lambda_i \min_{Q^i \in \mathcal{P}^i} E_{Q^i} u^i(c_t^i) \\
&= \max_{c_t^i} \min_{Q^i} \sum_i \lambda_i E_{Q^i} u^i(c_t^i) \\
&= \min_{Q^i} \max_{c_t^i} \sum_i \lambda_i E_{Q^i} u^i(c_t^i) \\
&= \min_{|\theta^i| \le \kappa^i} \max_{c_t^i} \sum_i \lambda_i E_P[u^i(c_t^i)(1 + \theta^i W_t^j)] \\
&\equiv \min_\theta J(\theta; \lambda)
\end{aligned}$$

where consumption levels are constrained by $\sum c_t^i \le Y_t$ and where

$$J(\theta, \lambda) \equiv \max_{c_t^i} \sum \lambda_i E_P[u^i(c_t^i)(1 + \theta^i W_t^j)].$$

Note that I have applied the minimax theorem to reverse the min and max operations and that $j \neq i$ in the last summation.

The envelope theorem implies that [using the fact that each $u^i(\cdot) = \log(\cdot)$]

$$\text{(8)}$$
$$\begin{aligned}
J_{\theta^1}(\theta, \lambda) &= E_P[W_t^2 u^1(c_t^1)] \\
&= E_P[W_t^2(\log Y_t - \log[1 + \lambda\rho_t])] \\
&= s_2^Y - E_P[W_t^2 \log(1 + \lambda\rho_t)] \\
&= s_2^Y - \Phi^2(\theta) \\
J_{\theta^2}(\theta, \lambda) &= \lambda E_P[W_t^1 u^2(c_t^2)] \\
&= \lambda E_P[W_t^1(\log Y_t - \log[1 + (\lambda\rho_t)^{-1}])] \\
&= \lambda s_1^Y - \lambda E_P[W_t^1 \log(1 + [\lambda\rho_t]^{-1})] \\
&= \lambda s_1^Y - \lambda\Phi^1(\theta).
\end{aligned}$$

Thus the Kuhn-Tucker theorem implies (5).

The optimal $\theta$ must satisfy $-\kappa \le \theta \le 0$ because $-\kappa^i < \theta^i < \kappa^i \Rightarrow s_j^Y = \Phi^j(\theta)$, $i \neq j$, $\Rightarrow \theta^i < 0$. [By elementary arguments one can show that $\Phi^2(\theta^1, \theta^2) < 0$ if $\theta^1 > 0$, and $\Phi^1(\theta^1, \theta^2) < 0$ if $\theta^2 > 0$.] Similarly one can exclude an optimum at $+\kappa_1$: that would require $s_2^Y < \Phi^2(\kappa^1, \theta^2)$, but the latter is negative. This completes the proof.

The comparative-statics analysis made use of the fact that, as a pointwise maximum of a collection of linear (in $\theta$) functions, $J(\cdot, \lambda)$ is convex for each $\lambda$. Therefore, (8) implies that

$$-D_{\theta\theta} J(\theta, \lambda) = \begin{bmatrix} \partial\Phi^2/\partial\theta^1 & \partial\Phi^2/\partial\theta^2 \\ \lambda\partial\Phi^1/\partial\theta^1 & \lambda\partial\Phi^1/\partial\theta^2 \end{bmatrix}$$

and both matrices are negative definite. In particular, $\det[D_\theta \Phi(\theta)] < 0$, and $\partial \Phi^i / \partial \theta^j < 0$ for $i \neq j$.

### III. Concluding Remarks

The preceding model can be extended to a multi-period setting. Think of a two-dimensional state process $W_t = (W_t^1, W_t^2)$ that is a random walk under a reference probability measure. Suppose that, while country $i$ is confident that the domestic shock is a random walk, she views $W_t^j$ as an ambiguous random walk; that is, conditional on the state at time $t-1$, her beliefs are that $W_t^j - W_{t-1}^j = \pm 1$ according to the color of the ball drawn from an ambiguous Ellsberg urn. Thus, conditional one-step-ahead beliefs have the same form as in the two-period model. Using them, one can define utility recursively, essentially by replacing log $c_t^i$ in (1) by $V_t^i(c^i)$, the continuation utility for periods $t$ and beyond. The resulting model of single-agent utility admits the explicit representation

$$(9) \quad V_t^i(c^i) = \min_{Q \in \mathcal{P}^i} E_Q[\Sigma_{\tau \geq t} \beta^{\tau - t} u^i(c_\tau^i) | \mathcal{F}_t]$$

for a suitable set $\mathcal{P}^i$ of priors over possible trajectories of $W_t$.[4] This utility specification has a number of attractive features that I now describe.

First, it has a suitable continuous-time limit, as described in Epstein and Werner Ploberger (2001), where the driving state process is an ambiguous Brownian motion. Jianjun Miao and I (Epstein and Miao, 2000) have applied the resulting model of utility to a two-country setting that is the continuous-time counterpart of this paper's model. The analytical power of continuous time permits sharp results to be derived; we confirm and extend those reported above. In particular, we describe the implementation of efficient allocations as a Radner equilibrium and describe asset market implications (home-bias in equities, for example) of ambiguity.

In Epstein and Martin Schneider (2001b) a simple axiomatic basis is provided for a generalization of (9) in which $\mathcal{P}^i$ is restricted to conform to the "spirit" but not the letter of the above story about an ambiguous random walk. The essential characterizing axioms are: (i) each conditional utility $V_t^i$ satisfies the axioms described by Gilboa and Schmeidler (1989) that characterize the multiple-priors model in an atemporal or one-shot choice framework; and (ii) the collection $\{V_t\}_{t \geq 0}$ of all conditional preferences is dynamically consistent.[5]

Further, learning can be accommodated. Though the specific conditional one-step-ahead beliefs described above are the same at every node and thus do not respond to past observations, the model in its general axiomatic form permits such responsiveness to data (see Zengjing Chen and Epstein, 2000; Epstein and Schneider, 2001a). Prior-by-prior application of Bayes' Rule provides a dynamically consistent updating rule for recursive multiple-priors utility. Moreover, a rich set of learning dynamics is admitted. For example, in many environments, ambiguity can plausibly persist indefinitely.[6] In others, ambiguity may increase in response to a "surprising" observation that leads the agent to doubt her previous view (model) of the environment.

An important outstanding question is: what are reasonable values for $\kappa^i$? One possible approach is to apply Bayesian detection theory for discriminating between probability laws in order to assess how difficult it would be to discriminate between measures lying in the set of priors corresponding to a specific value for $\kappa^i$. This route has been developed by Evan Anderson et al. (2000) for their model of robust decision-making; it seems likely that the approach could be adapted to our model. Alternatively, interpret the challenge as being the difficulty of transferring ambiguity parameters across settings. There is no difficulty transferring risk-aversion parameters because any given lottery presumably represents the same prospect, regardless of the context. In contrast, ambiguity is by its very nature tied to a specific

[4] The model is a special case of that described in Epstein and Wang (1994).

[5] The reader may wish to compare these foundations with those provided in another paper in this session for the related model of utility proposed by Evan Anderson et al. (2000).

[6] Bewley's (1998) discussion of learning under "Knightian uncertainty" is very relevant here.

state space. There is a need to uncover deeper structural parameters underlying the $\kappa^i$'s that are transferable across settings.

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