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MAXMIN EXPECTED UTILITY WITH NON-UNIQUE PRIOR*

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Acts are functions from states of nature into finite-support distributions over a set of 'deterministic outcomes'. We characterize preference relations over acts which have a numerical representation by the functional $J(f)=\min\{\int u\circ f\,dP \mid P\in C\}$ where f is an act, u is a von Neumann–Morgenstern utility over outcomes, and C is a closed and convex set of finitely additive probability measures on the states of nature. In addition to the usual assumptions on the preference relation as transitivity, completeness, continuity and monotonicity, we assume uncertainty aversion and certainty-independence. The last condition is a new one and is a weakening of the classical independence axiom: It requires that an act f is preferred to an act g if and only if the mixture of f and any constant act h is preferred to the same mixture of g and h. If non-degeneracy of the preference relation is also assumed, the convex set of priors C is uniquely determined. Finally, a concept of independence in case of a non-unique prior is introduced.

1. Introduction

One of the first objections to Savage's paradigm was raised by Ellsberg (1961). He suggested the following mind experiment challenging the expected utility hypotheses: Subject is asked to preference rank four bets. (S)he is shown two urns, each containing 100 balls each one either red or black. Urn A contains 50 black balls and 50 red ones, while there is no additional information about urn B. One ball is drawn at random from each urn. Bet 1 is 'the ball drawn from urn A is black', and will be denoted by AB. Bet 2 is 'the ball drawn from urn A is red', and will be denoted by AR, and similarly we have BB and BR. Winning a bet entitles the subject \$100. The following preferences have been observed empirically: $AB \simeq AR > BB \simeq BR$. It is easy to see that there is no probability measure supporting these preferences through expected utility maximization.

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One conceivable explanation of this phenomenon which we adopt here is as follows: In case of urn B, the subject has too little information to form a prior. Hence (s)he considers a *set* of priors as possible. Being uncertainty averse, (s)he takes into account the *minimal* expected utility (over all priors in the set) while evaluating a bet.

For instance, one may consider the extreme case in which our decision maker takes into account all possible priors over urn B. In this case the minimal utility of each one of the bets AB, AR is $50, while that of bets BB and BR is $0, so that the observed preferences are compatible with the maxmin expected utility decision rule.

These ideas are not new. Hurwicz (1951) showed an example of statistical analysis where the statistician is too ignorant to have a unique 'Bayesian' prior, but 'not quite as ignorant' to apply Wald's decision rule with respect to all priors. Smith (1961) suggested considering an interval of priors in such situations. He tried to axiomatize this behavior pattern using the 'Odds' concept. Other works utilize the Choquet Integration with respect to capacities [Choquet (1955)] to deal with the problem of a non-unique prior. Huber and Strassen (1973) use the Choquet Integral in testing hypotheses regarding the choice between two disjoint sets of measures. Schmeidler (1982, 1984, 1986) axiomatizes the preferences representable via the Choquet Integral of the utility with respect to a non-additive probability measure. He used a framework including both 'Horse Lotteries' and 'Roulette Lotteries', à la Anscombe and Aumann (1963). Gilboa (1987) obtains the same representation in the original framework of Savage (1954). [See also Wakker (1986)].

In Schmeidler (1986) it has been shown, roughly speaking, that when the non-additive probability v on S is convex [i.e., $v(A \cup B)+v(A \cap B) \geqq v(A)+v(B)$], the Choquet Integral of a real-valued function, say a, with respect to v is equal to the minimum of $\{\int a \,\mathrm{d}P \mid P$ is in the core of $v\}$. The core of v, by definition, consists of all finitely additive probability measures that majorize v pointwise (i.e., event-wise). That is to say, the non-additive expected utility theory coincides with the decision rule we propose here, where the set of possible priors is the core of v.

However, when an arbitrary (closed and convex) set of priors C is given, and one defines $v(A)=\min\{P(A) \mid P \in C\}$, v need not be convex, though it is exact, i.e., pointwise minimum of additive set functions. [See examples in Schmeidler (1972) and Huber and Strassen (1973).]

Furthermore, even if v happens to be convex C does not have to be its core. It is not hard to construct an example in which C is a proper subset of the core of v.

This paper proposes an axiomatic foundation of the maxmin expected utility decision rule. As in Schmeidler (1984), some of which notations we repeat, we use the framework of Anscombe and Aumann (1963).

The main difference among the models of Anscombe and Aumann (1963), Schmeidler (1984) and the present one lies in the phrasing of the independence axiom (Sure Thing Principle). Unlike in the other two works, we also use here an axiom of uncertainty aversion. Similarly to the non-additive expected utility theory, this model extends classical expected utility. In general, the theories differ from each other; as mentioned above, they coincide in the case of a convex v.

The straightforward interpretation of our result is an extension of the neobayesian paradigm which leads to a set of priors instead of a unique one. However, with a different interpretation, in which the set C is the set of possible probability distributions in a statistical decision problem, our result sheds light on Wald's minimax criterion and on its relation to personalistic probability. [We refer here to the minimax loss criterion, which is equivalent to maximin utility, and not to the minimax regret criterion suggested by Savage (1954, ch. 9).]

In Wald (1950, section 1.4.2), we find: 'A minimax solution seems, in general, to be a reasonable solution of the decision problem when an a priori distribution in Ω does not exist or is unknown to the experimenter.' Hence our main result can be considered as an axiomatic foundation of Wald's criterion.

The detailed exposition of the model and the main result are stated in the next section. The proof is in section 3 and section 4 is devoted to an extension and several concluding remarks. Especially, we deal there with the definition of the concept of independence in the case of a non-unique prior.

Finally we would like to note that different approaches to the phenomenon of a non-unique prior appear in Lindley, Tversky and Brown (1979), Vardennan and Meeden (1983), Agnew (1985), Genest and Schervish (1985), Bewley (1986) and others.

2. Statement of the main result

Let X be a set and let Y be the set of distributions over X with finite supports

$$Y=\left\{y\colon X\to[0,1]\,\middle|\, y(x)\neq 0 \text{ for only finitely many } x\text{'s in } X \text{ and } \sum_{x\in X} y(x)=1\right\}.$$

For notational simplicity we identify X with the subset $\{y\in Y \mid y(x)=1$ for some x in $X\}$ of Y.

Let S be a set and let Σ be an algebra of subsets of S. Both sets, X and S are assumed to be non-empty. Denote by L_0 the set of all Σ-measurable finite step functions from S to Y and denote by L_c the constant functions in L_0. Let L be a convex subset of Y^S which includes L_c. Note that Y can be considered a subset of some linear space, and Y^S, in turn, can then be considered as a subspace of the linear space of all functions from S to the first linear space. Whereas it is obvious how to perform convex combinations in Y it should be stressed that convex combinations in Y^S are performed pointwise. I.e., for f and g in Y^S and α in $[0,1]$, $\alpha f+(1-\alpha)g=h$ where $h(s)=\alpha f(s)+(1-\alpha)g(s)$ for $s\in S$.

In the neobayesian nomenclature elements of X are (deterministic) *outcomes*, elements of Y are random outcomes or (roulette) *lotteries* and elements of L are *acts* (or horse lotteries). Elements of S are *states* (of nature) and elements of Σ are *events*.

The primitive of a neobayesian decision model is a binary (preference) relation over L to be denoted by $\geqq$. Next are stated several properties (axioms) of the preference relation, which will be used in the sequel.

A.1. Weak order. (a) For all f and g in L: $f\geqq g$ or $g\geqq f$.
(b) For all f, g and h in L: If $f\geqq g$ and $g\geqq h$ then $f\geqq h$.

The relation $\geqq$ on L induces a relation also denoted by $\geqq$ on Y: $y\geqq z$ iff $y^*\geqq z^*$ where $x^*(s)=x$ for all $x\in Y$ and $s\in S$. When no confusion is likely to arise, we shall not distinguish between y^* and y. As usual, $>$ and $\simeq$ denote the asymmetric and symmetric parts, respectively, of $\geqq$.

A.2. Certainty-Independence (C-independence for short). For all f, g in L and h in L_c and for all α in $]0,1[$: $f>g$ iff $\alpha f+(1-\alpha)h>\alpha g+(1-\alpha)h$.

A.3. Continuity. For all f, g and h in L: if $f>g$ and $g>h$ then there are α and β in $]0,1[$ such that $\alpha f+(1-\alpha)h>g$ and $g>\beta f+(1-\beta)h$.

A.4. Monotonicity. For all f and g in L: if $f(s)\geqq g(s)$ on S then $f\geqq g$.

A.5. Uncertainty Aversion. For all f, $g\in L$ and $\alpha\in]0,1[$: $f\simeq g$ implies $\alpha f+(1-\alpha)g\geqq f$.

A.6. Non-degeneracy. Not for all f and g in L, $f\geqq g$.

All the assumptions except for A.2 and A.5 are quite common. The standard independence axiom is stronger than C-independence as it allows h to be any act in L rather than restricting it to constant acts. This axiom seems heuristically more appealing: a decision maker who prefers f to g can more easily visualize the mixtures of f and g with a constant h than with an arbitrary one, hence he is less likely to reverse his preferences. An intuitive objection to the standard independence axiom is that it ignores the phenomenon of hedging. Like comonotonic independence [Schmeidler (1984)], C-independence does not exclude hedging. However, C-independence is much simpler than and implied by comonotonic independence. Uncertainty aversion [which was introduced in Schmeidler (1984)] captures the phenomenon of hedging, especially when the preference is strict. Thus this assumption complements C-independence.

Before stating the main result we mention that the topology to be used on the space of finitely additive set functions on Σ is the product topology, i.e., the weak* topology in Dunford and Schwartz (1957) terms. Recall that in this topology the set of finitely-additive probability measures on Σ is compact.

Theorem 1. *Let* $\geqq$ *be a binary relation on* L_0*. Then the following conditions are equivalent:*

(*1*) $\geqq$ *satisfies assumptions A.1–A.5 for* $L = L_0$.

(*2*) *There exist an affine function* $u: Y \to R$ *and a non-empty, closed and convex set* C *of finitely additive probability measures on* Σ *such that:*

(*) $f \geqq g$ *iff* $\min_{P \in C} \int u \circ f \, dP \geqq \min_{P \in C} \int u \circ g \, dP$ (*for all* $f, g \in L_0$).

Furthermore:

(*a*) *The function* u *in* (*2*) *is unique up to a positive linear transformation;*

(*b*) *The set* C *in* (*2*) *is unique iff assumption A.6 is added to* (*1*).

3. Proof of Theorem 1

The crucial part of the proof is that (1) implies (2). If A.6 fails to hold, then a constant function u and any closed and convex subset C will satisfy (2), hence for the next several lemmata we suppose assumptions A.1–A.6.

Lemma 3.1. *There exists an affine* $u: Y \to R$ *such that for all* $y, z \in Y: y \geqq z$ *iff* $u(y) \geqq u(z)$.

Furthermore, u *is unique up to a positive linear transformation.*

Proof. This is an immediate consequence of the von Neumann–Morgenstern theorem, since the independence assumption for L_c is implied by C-independence. [See Fishburn (1970, ch. 8)]. □

Lemma 3.2. *Given a* $u: Y \to R$ *from Lemma 3.1, there exists a unique* $J: L_0 \to R$ *such that:*

(*i*) $f \geqq g$ *iff* $J(f) \geqq J(g)$ (*for all* $f, g \in L_0$);

(*ii*) *for* $f = y^* \in L_c$, $J(f) = u(y)$.

Proof. On L_c J is uniquely determined by (ii). We extend J to L_0 as follows: Given $f \in L_0$, there are $\bar{y}, \underline{y} \in Y$ such that $\underline{y} \leqq f \leqq \bar{y}$.

By the continuity assumption and other assumptions, there exists a unique $\alpha \in [0,1]$ such that $f \simeq \alpha \underline{y} + (1-\alpha)\bar{y}$. Define $J(f) = J(\alpha \underline{y} + (1-\alpha)\bar{y})$. By construction. J satisfies (i), hence it is also unique. □

We shall henceforth choose a specific $u: Y \to R$ such that there are $y_1, y_2 \in Y$ for which $u(y_1) < -1$ and $u(y_2) > 1$. (Such a choice of a utility u is possible in view of the non-degeneracy assumption.) We denote by B the space of all bounded Σ-measurable real valued functions on S [which is denoted $B(S, \Sigma)$ in Dunford and Schwartz (1957)]. B_0 will denote the space of functions in B which assume finitely many values. Let $K = u(Y)$, and let $B_0(K)$ be the subset of functions in B_0 with values in K. For $\gamma \in R$, let $\gamma^* \in B_0$ be the constant function on S the value of which is γ.

Lemma 3.3. There exists a functional $I: B_0 \to R$ *such that*:

(*i*) *For all* $f \in L_0$, $I(u \circ f) = J(f)$ (*hence* $I(1^*) = 1$).

(*ii*) I *is monotonic* (*i.e., for* $a, b \in B_0$: $a \geqq b \Rightarrow I(a) \geqq I(b)$).

(*iii*) I *is superlinear* (*that is, superadditive and homogeneous of degree* 1).

(*iv*) I *is C-independent: for any* $a \in B_0$ *and* $\gamma \in R$, $I(a + \gamma^*) = I(a) + I(\gamma^*)$.

Proof. We first define I on $B_0(K)$ by condition (i). (Lemma 3.2 and the monotonicity assumption assure that I is thus well-defined). We now show that I is homogeneous on $B_0(K)$.

Assume $a = \alpha b$ where a, $b \in B_0(K)$ and $0 < \alpha \leqq 1$. We have to show that $I(a) = \alpha I(b)$. (This will imply the equality for $\alpha > 1$.) Let $g \in L_0$ satisfy $u \circ g = b$. Let $z \in Y$ satisfy $J(z) = 0$ and define $f = \alpha g + (1-\alpha)z$. Hence $u \circ f = \alpha u \circ g + (1-\alpha) u \circ z = \alpha b = a$, so $I(a) = J(f)$. Let $y \in Y$ satisfy $y \simeq g$ (hence $J(y) = J(g) = I(b)$). By C-independence, $\alpha y + (1-\alpha)z \simeq \alpha g + (1-\alpha)z = f$. Hence $J(f) = J(\alpha y + (1-\alpha)z) = \alpha J(y) + (1-\alpha)J(z) = \alpha J(y)$. Whence $I(a) = J(f) = \alpha J(y) = \alpha I(b)$.

We now extend I by homogeneity to all of B_0. Note that I is monotone and homogeneous of degree 1 on B_0.

Next we show that I is C-independent [part (iv) of the lemma]. Let there be given $a \in B_0$ and $\gamma \in R$. By homogeneity we may assume without loss of generality that $2a$, $2\gamma^* \in B_0(K)$. Now define $\beta = I(2a) = 2I(a)$. Let $f \in L_0$ satisfy

$u\circ f=2a$ and let y, $z\in Y$ satisfy $u\circ y=\beta^*$ and $u\circ z=2\gamma^*$. Since $f\simeq y$, C-independence of $\geqq$ implies that $\frac{1}{2}f+\frac{1}{2}z\simeq\frac{1}{2}y+\frac{1}{2}z$. Hence

$$I(a+\gamma^*)=I(\tfrac{1}{2}\beta^*+\gamma^*)=\tfrac{1}{2}\beta+\gamma=I(a)+\gamma,$$

and I is C-independent.

It is left to show that I is superadditive. Let there be given a, $b\in B_0$. Once again, by homogeneity we may assume without loss of generality that a, $b\in B_0(K)$. Furthermore, for the same reason it suffices to prove that $I(\frac{1}{2}a+\frac{1}{2}b)\geqq\frac{1}{2}I(a)+\frac{1}{2}I(b)$. Suppose that f, $g\in L_0$ are such that $u\circ f=a$ and $u\circ g=b$. If $I(a)=I(b)$, then $f\simeq g$ and by uncertainty aversion (assumption A.5), $\frac{1}{2}f+\frac{1}{2}g\geqq f$, which, in turn, implies $I(\frac{1}{2}a+\frac{1}{2}b)\geqq I(a)=\frac{1}{2}I(a)+\frac{1}{2}I(b)$.

Assume, then, $I(a)>I(b)$, and let $\gamma=I(a)-I(b)$. Set $c=b+\gamma^*$ and note that $I(c)=I(b)+\gamma=I(a)$ by C-independence of I. Using the C-independence of I twice more and its superadditivity for the case proven above, one obtains:

$$I(\tfrac{1}{2}a+\tfrac{1}{2}b)+\tfrac{1}{2}\gamma=I(\tfrac{1}{2}a+\tfrac{1}{2}c)\geqq\tfrac{1}{2}I(a)+I(c)=\tfrac{1}{2}I(a)+\tfrac{1}{2}I(b)+\tfrac{1}{2}\gamma,$$

which completes the proof of the lemma. □

Recall that the space B is a Banach space with the sup norm $\|\cdot\|$, and B_0 is a norm-dense subspace of B. The next lemma will also be used in an extension of the Theorem.

Lemma 3.4. There exists a unique continuous extension of I to B.

Furthermore, this extension is monotonic, superlinear and C-independent.

Proof. We first show that for each a, $b\in B_0$, $|I(a)-I(b)|\leqq\|a-b\|$. Indeed, $a=b+a-b\leqq b+\|a-b\|^*$. Monotonicity and C-independence of I imply that $I(a)\leqq I(b+\|a-b\|^*)=I(b)+\|a-b\|$ or $I(a)-I(b)\leqq\|a-b\|$. The same argument implies $I(b)-I(a)\leqq\|b-a\|$. Thus there exists a unique continuous extension of I. Obviously, it is superlinear, monotonic and C-independent. □

In the next lemma the convex set of finitely additive probability measures C of Theorem 1 will be constructed via a separation theorem.

Lemma 3.5. If I is a monotonic superlinear and C-independent functional on B with $I(1^*)=1$, there exists a closed and convex set C of finitely additive probability measures on Σ such that: for all $b\in B$, $I(b)=\min\{\int b\,\mathrm{d}P\mid P\in C\}$.

Proof. Let $b\in B$ with $I(b)>0$ be given. We will construct a finitely additive probability measure P_b such that $I(b)=\int b\,\mathrm{d}P_b$ and $I(a)\leqq\int a\,\mathrm{d}P_b$ for all $a\in B$. To this end we define

$$D_1=\{a\in B\,|\,I(a)>1\},$$

$$D_2=\operatorname{conv}(\{a\in B\,|\,a\leqq 1^*\}\cup\{a\in B\,|\,a\leqq b/I(b)\}).$$

We now show that $D_1\cap D_2=\emptyset$. Let $d_2\in D_2$ satisfy $d_2=\alpha a_1+(1-\alpha)a_2$ where $a_1\leqq 1^*$, $a_2\leqq(b/I(b))$ and $\alpha\in[0,1]$. By monotonicity, homogeneity and C-independence of I,

$$I(d_2)\leqq\alpha+(1-\alpha)I(a_2)\leqq 1.$$

Note that each of the sets D_1, D_2 has an interior point and that they are both convex. Thus, by a separation theorem [see Dunford and Schwartz (1957, V.2.8)] there exists a non-zero continuous linear functional p_b and an $\alpha\in R$ such that:

$$\text{for all } d_1\in D_1 \text{ and } d_2\in D_2,\quad p_b(d_1)\geqq\alpha\geqq p_b(d_2). \tag{1}$$

Since the unit ball of B is included in D_2, $\alpha>0$. (Otherwise p_b would have been identically zero). We may therefore assume without loss of generality that $\alpha=1$.

By (1), $p_b(1^*)\leqq 1$. Since 1^* is a limit point of D_1, $p_b(1^*)\geqq 1$ is also true, hence $p_b(1^*)=1$. We now show that p_b is non-negative, or, more specifically, that $p_b(1_E)\geqq 0$ whenever 1_E is the indicator function of some $E\in\Sigma$. Since

$$p_b(1_E)+p_b(1^*-1_E)=p_b(1^*)=1,$$

and $1^*-1_E\in D_2$, the inequality follows.

By the classical representation theorem there exists a finitely additive probability measure P_b on Σ such that $p_b(a)=\int a\,\mathrm{d}P_b$ for all $a\in B$. We will now show that $p_b(a)\geqq I(a)$ for all $a\in B$, with equality for $a=b$: First assume $I(a)>0$. It is easily seen that $a/I(a)+(1/n)^*\in D_1$, so the continuity of p_b and (1) imply $p_b(a)\geqq I(a)$. For the case $I(a)\leqq 0$ the inequality follows from C-independence. Since $b/I(b)\in D_2$, we obtain the converse inequality for b, thus $p_b(b)=I(b)$.

We now define the set C as the closure of the convex hull of $\{P_b\,|\,I(b)>0\}$ (which, of course, is convex). It is easy to see that $I(a)\leqq\min\{\int a\,\mathrm{d}P\,|\,P\in C\}$. For a such that $I(a)>0$ we have shown the converse inequality to hold as well. For a such that $I(a)\leqq 0$, it is again a simple implication of C-independence. □

Conclusion of the proof of Theorem 1

Lemmata 3.1–3.5 prove that (1) implies (2). Assuming (2) define I on B by

$I(b)=\min\{\int b\,\mathrm{d}P \mid P\in C\}$, C compact and convex. It is easy to see that I is monotonic, superlinear, C-independent and continuous. So, in turn, the preference relation defined on L_0 by (2) satisfies A.1–A.5.

We now turn to prove the uniqueness properties of u and C. The uniqueness of u up to positive linear transformation is implied by Lemma 3.1.

If assumption A.6 does not hold, the range of u, K, is a singleton, and C can be any non-empty closed and convex set. We shall now show that if assumption A.6 does hold, C is unique. Assume the contrary, i.e., that there are $C_1\neq C_2$, both non-empty, closed and convex, such that the two functions on L_0:

$$J_1(f)=\min\left\{\int u(f)\,\mathrm{d}P \,\middle|\, P\in C_1\right\},$$

$$J_2(f)=\min\left\{\int u(f)\,\mathrm{d}P \,\middle|\, P\in C_2\right\},$$

both represent $\geqq$.

Without loss of generality one may assume that there exists $P_1\in C_1\backslash C_2$. By a separation theorem [Dunford and Schwartz (1957, V.2.10)], there exists $a\in B$ such that

$$\int a\,\mathrm{d}P_1<\min\left\{\int a\,\mathrm{d}P \,\middle|\, P\in C_2\right\}.$$

Without loss of generality we may assume that $a\in B_0(K)$. Hence there exists $f\in L_0$ such that $J_1(f)<J_2(f)$. Now let $y\in Y$ satisfy $y\simeq f$. We get

$$J_1(y)=J_1(f)<J_2(f)=J_2(y),$$

a contradiction.

4. Extension and concluding remarks

A natural question arising in view of Theorem 1 is whether it holds when the set of acts L, on which the preference relation is given, is a convex superset of L_0. A partial answer is presented in the sequel. It will be shown that, for a certain superset of L_0, the preference relation on it is completely determined by its restriction to L_0, should it satisfy the assumptions introduced in section 2.

Given a weak order $\geqq$ on L_c, an act $f\colon S\to Y$ is said to be *Σ-measurable* if for all $y \in Y$ the sets $\{s \mid f(s) > y\}$ and $\{s \mid f(s) \geqq y\}$ belong to Σ. It is said to be bounded (or, more precisely, $\geqq$-bounded) if there are $y_1, y_2 \in Y$ such that $y_1 \geqq f(s) \geqq y_2$ for all $s \in S$. The set of all Σ-measurable bounded acts in Y^S is denoted by $L(\geqq)$. It is obvious that $L(\geqq)$ is convex and contains L_0.

Proposition 4.1. Suppose that a preference relation $\geqq$ over L_0 satisfies assumptions A.1–A.5. Then it has a unique extension to $L(\geqq)$ which satisfies the same assumptions [over $L(\geqq)$].

Proof. Because of monotonicity, the proposition is obvious in case that assumption A.6 does not hold. Therefore we assume it does, and we may apply Lemmata 3.1–3.4. We then define the extension of $\geqq$ (also to be denoted by $\geqq$) as follows: $f \geqq g$ iff $I(u(f)) \geqq I(u(g))$. It is obvious that $\geqq$ satisfies A.1–A.5 and that $\geqq$ on $L(\geqq)$ is the unique monotonic extension of $\geqq$ on L_0. □

Remark. Suppose that $\geqq$ satisfies A.1–A.5 over L, which is convex and contains L_0. Then, in view of Proposition 4.1, $\geqq$ may be represented as in Theorem 1 on $L \cap L(\geqq)$.

We now introduce the concepts of independence of acts and products of binary relations.

Suppose that a given preference relation $\geqq$ satisfies A.1–A.6 over L_0. By Proposition 4.1 we extend it to $L = L(\geqq)$ and let u and C be as in Theorem 1. Two acts $f, g \in L$ are said to be independent if the following two conditions hold:

(1) There exists $P_0 \in C$ such that

$$\int u \circ f \, dP_0 = \min \{ \int u \circ f \, dP \mid P \in C \}, \quad \text{and}$$

$$\int u \circ g \, dP_0 = \min \{ \int u \circ g \, dP \mid P \in C \};$$

(2) $u \circ f$ and $u \circ g$ are two stochastically independent random variables with respect to any extreme point of C [for short: Ext(C)].

As expected, this notion of independence turns out to be closely related to that of product spaces, once the latter is defined. We will refer to a triple (S, Σ, C) as a non-unique probability space. Given two non-unique probability spaces (S_i, Σ_i, C_i) $i = 1, 2$, we define their product (S, Σ, C) as follows: $S = S_1 \times S_2$, $\Sigma = \Sigma_1 \otimes \Sigma_2$ and C is the closed convex hull of $\{P_1 \otimes P_2 \mid P_1 \in C_1, P_2 \in C_2\}$.

Suppose that for a given set of outcomes X, there are given two acts spaces $L_0^i \subset Y^{S_i}$, $i = 1, 2$, and two preference relations $\geqq^i$ correspondingly, such that the restrictions of $\geqq^1$ and $\geqq^2$ to Y coincide. As before, we suppose that each $\geqq^i$ satisfies A.1–A.6 and we consider its extension to $L^i = L^i(\geqq^i)$. For the product acts space $L_0 \subset Y^{S_1 \times S_2}$ we define the product preference relation $\geqq = \geqq^1 \otimes \geqq^2$ as derived from u and C. It is obvious that $\geqq$ also satisfies A.1–A.6, and it has a unique extension to $L = L(\geqq)$. Given $f^i \in L^i$, it has a unique trivial extension $\bar{f}^i \in L$.

Now we formulate the result which justifies our definition of independence:

Proposition 4.2. Given L^1, $\geqq^1$, L^2, $\geqq^2$ and L as above, $\geqq$ is the unique preference relation over L satisfying:

(*1*) *assumptions A.1–A.6;*

(*2*) *for all $f^i, g^i \in L^i$, $f^i \geqq^i g^i$ iff $\bar{f}^i \geqq \bar{g}^i$ $(i=1,2)$;*

(3) *for all $f \in L^1$ and $g \in L^2$, $\bar{f}$ and $\bar{g}$ are independent.*

Proof. It is trivial to see that $\geqq$ indeed satisfies (1)–(3). To see that it is unique, let $\geqq'$ also satisfy (1)–(3). By (1) and our main result, $\geqq'$ is representable by a utility u' and a convex and closed set of finitely additive measures C'. By 3.1 we assume without loss of generality that $u = u'$.

We now wish to show that $C' = C$.

Step 1. $C' \subset C$.

Proof of Step 1. As C is convex, it suffices to show that $\mathrm{Ext}(C') \subset C$. Let, then $P_0 \in \mathrm{Ext}(C')$. Define P_i to be the restriction of P_0 to Σ_i $(i=1,2)$. Choose $A \in \Sigma_1$ and $B \in \Sigma_2$, and let $f \in L^1$ and $g \in L^2$ satisfy $u \circ f = 1_A$, $u \circ g = 1_B$. Since $\bar{f}$ and $\bar{g}$ are independent, they are independent with respect to P_0. Hence $P_0(A \times B) = P_0(A \times S_2) P_0(S_1 \times B) = P_1(A) P_2(B)$. This implies $P_0 = P_1 \otimes P_2 \in C$.

Step 2. $C \subset C'$.

Proof of Step 2. We begin with

Step 2a. If Σ_1 and Σ_2 are finite, then $C \subset C'$.

Proof of Step 2a. By a theorem of Straszewicz (1935), it suffices to show that $P_1 \otimes P_2 \in C'$ for all $P_1 \in \mathrm{Exp}(C_1)$ and $P_2 \in \mathrm{Exp}(C_2)$, where $\mathrm{Exp}(C)$ denotes the set of exposed points in C, i.e., the points at which there exists a supporting hyperplane which does not pass through any other point of C. Let there be given, then, $P_1 \in \mathrm{Exp}(C_1)$ and $P_2 \in \mathrm{Exp}(C_2)$. Let $f \in L^1$ and $g \in L^2$ be such that

$$\int u \circ f \, \mathrm{d}P_1 = \min \{ \int u \circ f \, \mathrm{d}P \mid P \in C_1 \}$$

and

$$\int u \circ g \, dP_2 = \min\{\int u \circ g \, dP \mid P \in C_2\}.$$

By the independence of f and g, there exists $P_0 \in C'$ for which $\int u \circ \bar{f} \, dP$ and $\int u \circ \bar{g} \, dP$ are minimized simultaneously. By step 1, $P_0 \in C$, hence there are $P'_1 \in C_1$ and $P'_2 \in C_2$ such that $P_0 = P'_1 \otimes P'_2$. However, $\int u \circ \bar{f} \, dP_0 = \int u \circ f \, dP'_1$ and $\int u \circ \bar{g} \, dP_0 = \int u \circ g \, dP'_2$. By the uniqueness property of $\mathrm{Exp}(C_i)$ $(i = 1, 2)$, we obtain $P_1 = P'_1$ and $P_2 = P'_2$. Hence $P_1 \otimes P_2 = P_0 \in C$, and step 2a is proved.

We will now complete the proof of step 2. Assume that, by way of negation, $C' \setminus C \neq \emptyset$, i.e., $\geqq \neq \geqq'$. As in the proof of the Theorem, there exists $f \in L_0$ and $y \in Y$ such that $f > y^*$ and $y^* >' f$. Consider the finite sub-algebra, say $\tilde{\Sigma}$, of Σ generated by f. There are Σ'_i finite sub-algebras of Σ_i $(I = 1, 2)$, such that $\tilde{\Sigma} \subset \Sigma' = \Sigma'_1 \otimes \Sigma'_2$. Next consider the restrictions of $\geqq_i$ to the Σ'_i-measurable functions, and the restrictions of $\geqq$, $\geqq'$ to Σ'-measurable functions. Obviously, both $\geqq$ and $\geqq'$ satisfy requirements (1)–(3) of the proposition, although they differ on the set of Σ'-measurable functions (to which f and y^* belong.) This contradicts step 2a, and the proof of the proposition is thus completed. □

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