Topics in Dynamic Public Finance

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1 Topic 1 Optimal taxation – the Ramsey approach

1.1 Optimal taxation under commitment – the Ramsey problem

One of the more famous results in Public Finance is the Chamely-Judd result (Chamley 1986, Judd 1985).This results states that in a steady state of an economy with an infinite horizon, there should be no wedge between the marginal rate of transformation (market interest rate) and the intertemporal rate of substitution also if there is a wedge between the marginal rate of transformation between leisure and labor and the corresponding rate of substitution. In the decentralized equilibrium, this means that there should be no taxes on capital income also if labor taxes are need to provide income to the government.

To provide some intuition already before stating the result more formally, consider the following simple model.

Preferences

The representative agent has an additively separable utility function in consumption and leisure,

$$U = \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1 - n_t\right). \tag{1}$$

satisfying the usual Inada conditions so first-order conditions are sufficient.

Technology

Output is produced by labor only on a competitive labor market. One unit of labor produces w units of the consumption good. Individuals have one unit of labor each period to split between work and leisure $l = 1 - n$. There is a perfect market for government bonds.

Budget constraints

The government needs to finance an exogenous stream of consumption by tax revenues. For simplicity, we have already assumed that its consumption does not interfere with the individuals private problem. We will assumed that the government cannot finance its consumption by lump-sum taxation. We do this without providing an explicit reason within the model. Instead, the government has at its disposal, a linear labor income tax $\tau_{n,t}$, a consumption tax $\tau_{c,t}$ and a capital income tax, τ_k. Wages are for simplicity exogenous at rate w_t.

The representative individual's budget constraint period t is

$$c_t \left(1 + \tau_{c,t}\right) + b_{t+1} = \left(1 - \tau_{n,t}\right) w_t n_t + \left(1 - \tau_{k,t}\right) \left(1 + r_t\right) b_t$$

where b is government bonds yielding an interest rate r_t.

Substituting forward yields,

$$\sum_{t=0}^{\infty} c_t \prod_{s=0}^{t} \left(\frac{1}{1+r_s}\right) \prod_{s=0}^{t} W_{i,s} + \lim_{t\to\infty} \prod_{s=0}^{t} \left(\frac{1}{1+r_s}\right) \prod_{s=0}^{t} W_{i,s} b_t \tag{2}$$
$$= \sum_{t=0}^{\infty} w_t n_t W_{n,t} \prod_{s=0}^{t} \left(\frac{1}{1+r_s}\right) \prod_{s=0}^{t} W_{i,s} + \frac{(1-\tau_{k,0})}{(1+\tau_{c,0})} (1+r_0)\, b_0$$

where

$$W_{i,t} \equiv \frac{1+\tau_{c,t}}{(1+\tau_{c,t-1})(1-\tau_{k,t})},$$
$$W_{i,0} = 1,$$

and

$$W_{n,t} \equiv \frac{1-\tau_{n,t}}{1+\tau_{c,t}}.$$

We call $W_{i,t}$ the intertemporal wedge between $t-1$ and t and $W_{n,t}$ the intratemporal wedge. In addition, there is an aggregate resource constraint,

$$g_t + c_t = w_t n_t. \tag{3}$$

We now make the following definitions:

Definition 1 *A* feasible allocation *is a sequence* $\{c_t, n_t, g_t\}_{t=0}^{\infty}$ *that satisfies the aggregate resource constraint (3).*

Definition 2 *A* price system *is a sequence of interest rates* $\{r_t\}_{t=1}^{\infty}$ *that is bounded and such that* $1 + r_t \geq 0 \forall t$.

Definition 3 *A* government policy *is a sequence* $\{\tau_{c,t}, \tau_{k,t}, \tau_{n,t}, b_t\}_{t=0}^{\infty}$ *.*

Definition 4 *A* competitive equilibrium *is a feasible allocation, a price system and a government policy such that*

1. *Given the price system and the government policy, the allocation solves the maximization problem of the individual.*

2. *The aggregate resource constraint is satisfied.*

The problem of the consumer is to maximize (1) subject to (2).

First order conditions are

$$\beta^t u_c(c_t, 1-n_t) = \lambda \prod_{s=0}^{t} \left(\frac{1}{1+r_s}\right) \prod_{s=0}^{t} W_{i,s}$$
$$\beta^t u_l(c_t, 1-n_t) = \lambda w_t W_{n,t} \prod_{s=0}^{t} \left(\frac{1}{1+r_s}\right) \prod_{s=0}^{t} W_{i,s}$$

which we can write

$$\frac{u_l(c_t, 1-n_t)}{u_c(c_t, 1-n_t)\, w_t} = W_{n,t} \tag{4}$$
$$\frac{u_c(c_t, 1-n_t)}{u_c(c_0, 1-n_0)} \beta^t \prod_{s=0}^{t} (1+r_s) = \prod_{s=0}^{t} W_{i,s}$$

Result A sequence of consumption and labor supply satisfying (4), the budget constraint (2) and the resource constraint (3) is a competitive equilibrium.

Note: the governments budget constraint is redundant.

The Ramsey Problem

Maximizing (1) over the set of allocations that can be implemented as a competitive equilibrium is called the *Ramsey problem.*

Result

As we see, in the equations determining the competitive equilibrium, no other policy instruments than the wedges and $\tau_{k,0}$ and $\tau_{c,0}$ appear. The latter two affects the value of the initial government debt. Therefore, the government has an over-supply of instruments in the sense that many sequences of taxes $\{\tau_{c,t}, \tau_{k,t}, \tau_{n,t}\}_{t=0}^{\infty}$ can imply the same allocation.

Result Over any $t+1$ periods starting from period 0, there are $3\,(t+1)$ independent tax rates but the budget constraint of individual is determined by $2t+3$ instruments given by t intertemporal wedges $(W_{i,1}, ...W_{i,t})$, $t+1$ intratemporal wedges $(W_{l,0}, ...W_{n,t})$ and two initial tax-rates $\tau_{s,0}$ and $\tau_{c,0}$.

Result Any sequence of taxes can be replicated using only labor and consumption taxes plus an initial capital income tax.

Proof: Using labor and consumption taxes gives $2\,(t+1)$ independent instruments that together with an initial capital income tax can construct any sequence of wedges.

Result Consider a sequence of taxes such that consumption taxes are constant and capital income tax rates are constant at τ_{κ}. Given an initial consumption tax $\tau_{c,0}$,an identical intertemporal wedge can be constructed with zero capital income taxes and a sequence of consumption taxes satisfying

$$\frac{1+\tau_{c,1}}{1+\tau_{c,0}} = \frac{1}{1-\tau_k}$$
$$\frac{1+\tau_{c,t}}{1+\tau_{c,t-1}} = \frac{1}{1-\tau_k},$$

implying

$$1+\tau_{c,t} = \frac{1+\tau_{c,0}}{(1-\tau_k)^t}$$

Note that if $\tau_k > 0$, this sequence is increasing geometrically without bounds. It is perhaps intuitive that a sequence of consumption taxes that increases geometrically without bounds is suboptimal. Similarly, $\tau_k < 0$,the consumption tax approaches -100%. That neither of this is optimal is really the Chamley-Judd result.

Before proceeding, we note that using the (4) in the private budget constraint, we get

$$\sum_{t=0}^{\infty} \beta^t \left(c_t u_c \left(c_t, 1-n_t\right) - n_t u_l \left(c_t, 1-n_t\right)\right) = u_c \left(c_0, 1-n_0\right) \frac{(1-\tau_{k,0})}{(1+\tau_{c,0})} \left(1+r_0\right) b_0 \tag{5}$$

An allocation that satisfies (5) the private budget constraint and is privately optimal for some sequences of taxes. If it also satisfies the aggregate budget constraint it is also implementable as a competitive equilibrium. Note, that there is no taxes or prices here except the two initial taxes on pre-existing capital.

Nevertheless we can reformulate the Ramsey problem as max

$$\max U = \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1-n_t\right)$$

$$s.t. \sum_{t=0}^{\infty} \beta^t \left(c_t u_c\left(c_t, 1-n_t\right) - n_t u_l\left(c_t, 1-n_t\right)\right) = u_c\left(c_0, 1-n_0\right) \frac{\left(1-\tau_{k,0}\right)}{\left(1+\tau_{c,0}\right)} \left(1+r_0\right) b_0$$

$$g_t + c_t = w_t n_t$$

Sometimes, it is easier to work with this *direct or primal approach.* Here, it is then straightforward to construct the wedges and then taxes that implement the Ramsey optimal allocation.

1.2 The Chamley-Judd result

Now, we only add a production technology using capital.[1] There is an infinitely lived representative agent with preferences

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t, l_t\right).$$

The household has one unit of labor per period, to be split between leisure l and work n.The aggregate resource constraint is

$$c_t + g_t + k_{t+1} = F\left(k_t, n_t\right) + (1-\delta) k_t \tag{6}$$

The production function is constant returns to scale and factor markets are competitive.

[1] The exposition follows Ljungqvist & Sargent, *Recursive Macroeconomic Theory.*

Profit maximization of the representative firm implies

$$w_t = F_n(k_t, n_t)$$

$$r_t = F_k(k_t, n_t)$$

The government needs to finance an exogenous stream of expenditures $\{g_t\}_t^\infty$ using taxes on labor and capital and can smooth taxes by using a bond. Following the literature, we let the interest rate on bonds be tax-free.[2] Thus,

$$\begin{aligned} g_t + b_t &= \tau_{k,t} r_t k_t + \tau_{n,t} w_t n_t + \frac{b_{t+1}}{R_t} \\ &= F(k_t, n_t) - (1-\tau_{k,t}) r_t k_t - (1-\tau_{n,t}) w_t n_t + \frac{b_{t+1}}{R_t} \end{aligned}$$

where b_t is government borrowing and R_t is the interest rate on government bonds.

Households have budget constraints

$$c_t + k_{t+1} + \frac{b_{t+1}}{R_t} = (1-\tau_{n,t}) w_t n_t + (1-\tau_{k,t}) k_t r_t + (1-\delta) k_t + b_t$$

[2] Furthermore, we now let the value of bonds at maturity be unity, rather than as above let one bond be worth $1 + r_t$. This is only a change of the notation.

First order conditions are:

$$c_t; u_c(c_t, l_t) = \lambda_t$$

$$l_t; u_l(c_t, l_t) = \lambda_t (1 - \tau_{n,t}) w_t$$

$$k_{t+1}; \lambda_t = \beta \lambda_{t+1} ((1 - \tau_{k,t}) r_{t+1} + (1 - \delta))$$

$$b_{t+1}; \lambda_t \frac{1}{R_t} = \beta \lambda_{t+1}$$

Clearly, the first three implies

$$\frac{u_l(c_t, l_t)}{u_c(c_t, l_t)} = (1 - \tau_{n,t}) w_t$$

$$u_c(c_t, l_t) = \beta u_c(c_{t+1}, l_{t+1}) ((1 - \tau_{k,t}) r_{t+1} + (1 - \delta))$$

and the last two the no arbitrage condition

$$R_t = (1 - \tau_{k,t}) r_{t+1} + (1 - \delta)$$

Transversality conditions are

$$\lim_{T \to \infty} \left(\prod_{i=0}^{T-1} R_i^{-1} \right) k_{T+1} = 0$$

$$\lim_{T \to \infty} \left(\prod_{i=0}^{T-1} R_i^{-1} \right) \frac{b_{T+1}}{R_T} = 0$$

We can now make the following definitions:

Definition 5 *A* feasible allocation *is a sequence* $\{k_t, c_t, l_t, g_t\}_{t=0}^{\infty}$ *that satisfies the aggregate* *resource constraint (6).*

Definition 6 *A* price system *is a sequence of prices* $\{w_t, r_t, R_t\}_{t=0}^{\infty}$ *that is bounded and non-negative.*

Definition 7 *A* government policy *is a sequence* $\{\tau_{n,t}, \tau_{k,t}, b_t\}_{t=0}^{\infty}$ *and perhaps* $\{g_t\}_{t=0}^{\infty}$ *if that can be chosen.*

Definition 8 *A* competitive equilibrium *is a feasible allocation, a price system and a government policy such that*

1. *Given the price system and the government policy, the allocation solves the maximization problem of the individual and of the firm.*

2. *The government budget constraints are satisfied.*

Definition 9 *The* Ramsey problem *is to choose a competitive equilibrium (i.e.,pick a particular government policy) that maximizes the welfare of the representative individual.*

Let's now use the formulation of Chamley, and without loss of generality let the government choose after tax returns on capital and after tax wages. Therefore, we define

$$\tilde{r}_t \equiv (1 - \tau_{k,t})\, r_t,$$

$$\tilde{w}_t = (1 - \tau_{n,t})\, w_t.$$

Rather than chosing taxes, we let the government choose the net rates $\tilde{r}_t$ and $\tilde{w}_t$ given the market determined values r_t and w_t.

The Lagrangian of the Ramsey problem can be written

$$L = \sum_{t=0}^{\infty} \beta^t \{ u\left(c_t, 1 - n_t\right)$$

$$+ \psi_t \left(F\left(k_t, n_t\right) - \tilde{r}_t k_t - \tilde{w}_t n_t - b_t - g_t + b_{t+1}/R_t\right)$$

$$+ \theta_t \left(F\left(k_t, n_t\right) + (1-\delta) k_t - c_t - g_t - k_{t+1}\right)$$

$$+ \mu_{1,t} \left(u_l\left(c_t, l_t\right) - u_c\left(c_t, l_t\right) \tilde{w}_t\right)$$

$$+ \mu_{2,t} \left(u_c\left(c_t, l_t\right) - \beta u_c\left(c_{t+1}, l_{t+1}\right) \tilde{r}_{t+1} + (1-\delta)\right) \}$$

Now, the first order condition for k_{t+1} is[3]

$$\theta_t = \beta \psi_{t+1} \left(F_k\left(k_{t+1}, n_{t+1}\right) - \tilde{r}_{t+1}\right) + \beta \theta_{t+1} \left(F_k\left(k_{t+1}, n_{t+1}\right) + (1-\delta)\right).$$

The interpretation is that the RHS is the discounted value of investing one more unit at t. This value comes from having $F_k\left(k_{t+1}, n_{t+1}\right) + (1-\delta)$ more aggregate resources available next periods with a value per unit of θ_{t+1} and that government revenues are larger by an amount $F_k\left(k_{t+1}, n_{t+1}\right) - \tilde{r}_{t+1}$ with a value per unit of ψ_{t+1}. All is discounted with a factor β.

Suppose there is a steady state of the model, then

$$\theta = \beta \psi \left(r - \tilde{r}\right) + \beta \theta \left(r + (1-\delta)\right)$$

[3] Although a change in k_{t+1} affects r_{t+1} and w_{t+1} the effects on these variables need to to be taken into account since $\tilde{r}_{t+1}$ and $\tilde{w}_{t+1}$ are choice variables.

where we used thet $F_k = r$. Private optimality (the Euler equation), implies in steady state

$$\begin{aligned} u_c &= \beta u_c \left(\tilde{r} + (1-\delta)\right) \\ 1 &= \beta \left(\tilde{r} + (1-\delta)\right) \\ \frac{1-\beta\tilde{r}}{\beta} &= 1-\delta \end{aligned}$$

giving

$$\begin{aligned} \theta &= \beta\psi \left(r - \tilde{r}\right) + \beta\theta \left(r + \frac{1-\beta\tilde{r}}{\beta}\right) \\ &= \beta\psi \left(r - \tilde{r}\right) + \beta\theta \left(r - \tilde{r}\right) + \theta \\ 0 &= \beta \left(\psi + \theta\right) \left(r - \tilde{r}\right) \end{aligned}$$

requiring $r - \tilde{r} = 0$ and thus $\tau_k = 0$.

1.3 Discussion

We have shown that also in this simple economy, tax smoothing implies that the intertemporal margin should not be distorted. We have also found an equivalence between constant consumption taxes and an investment tax. In an infinite horizon model, a positive investment tax in steady state has implications identical to ever increasing consumption taxes. This can thus provide some intuition for Chamley & Judd's result that investment taxes should not be used in the long run. These results have led to a large amount of research, showing that the results are robust to important modifications of the model.

Judd (1982, 1985) studied a similar problem but more from the point of view of tax incidence and redistribution. As in the Ramsey problem, saving is done privately optimally and individuals live for ever. Judd allows two types of agents (either workers, who do not save, and capitalists) or in the more influential variant of the model, two type of agents that in a long-run steady state of the model may have different shares of the capital stock. The latter can be due to, e.g., different subjective discount rates. The government´s problem is to maximize a weighted sum of the welfare of two groups. Judd shows that if there is a steady state in this model, the optimal tax rate on capital income is zero.

Some important work has increased the understanding of the zero tax result by noting its limitations. Jones et al (1997) note that it is the assumption that capital is *accumulated* using a CRS production technology such that *no profits* arise that is key behind the zero tax result. If also human capital is accumulated without profits being generated, the zero tax rate results extend to human capital. In contrast, if profits arise due to decreasing returns to scale, the result does not necessarily hold. Correia (1997) shows that if there are some factors of production that cannot be taxed, capital income taxes are positive (negative) if capital is a complement (substitute) to the untaxed factor. Zhu (1992).

Two other key assumptions for the zero tax results are that i.) markets are complete and ii.) that government can implement time inconsistent policies. Aiyagari (1995) shows that if markets are incomplete, in particular if individuals face uninsurable idiosyncratic risk, there is a tendency of over-saving for precautionary motives that should be corrected by a positive capital income tax. This paper is very famous, but perhaps more for the nice way the result is demonstrated, rather than the result in itself. Regarding time inconsistency, it is clear that if a government or a benevolent planner can re-optimize unexpectedly, this is as if the initial period occurs again, leading to a new period of high capital income taxes. Without the ability to resist the temptation to re-optimize, a benevolent planner/government would set quite high taxes permanently (Klein & Ríos Rull, 2003). We will study this result below.

The result also extends to the stochastic case, in which case *expected taxes* should be zero and not distort savings.

An interesting case is if government spendings are stochastic. With complete markets, the government should then commit to a tax system that insures them against this (Chari et al. 1994). If spending needs are large, taxes on capital should be high and vice versa.

The zero capital income tax result does not go through in some cases:

1. If there are untaxed factors of production that generate profits and these factors are strict complements to capital. Then capital should be taxed (negatively if they are substitutes).

2. If market incompleteness makes people save too much for precautionary reasons.

In the short run, capital income taxes also collect revenue from sunk investments. Then, the tax is partly lump sum, which provides an argument for such taxes early in the planning horizon. But when is that zero? Has it already occurred a long time ago? In any case, we see a time consistency problem here.

Not also that the long-run maybe quite far out and people alive today might loose by a policy that maximizes the welfare of a constructed infinitely lived.

1.4 Time consistent taxation

1.4.1 A numerical approach

Here we follow Klein and Rios-Rull 2003. Consider a stochastic economy productivity is $z(s^t)$ and government consumption is $g(s^t)$ where s^t is the history of a shock that in every period belongs to a finite element set S. The shock follows a Markov chain with transition matrix Γ. The representative individuals has utility given by

$$E\sum_{t}^{\infty}\beta^t u(c_t, h_t)$$

and the aggregate resource constraint is

$$F\left(K\left(s^{t-1}\right), H\left(s^t\right), z\left(s^t\right)\right) + (1-\delta)K\left(s^{t-1}\right) = C\left(s^t\right) + K\left(s^t\right) + g\left(s^t\right)$$

Individual budget constraints are

$$c_t + k_{t+1} = (1-\tau_t)w_t h_t + (1 + r_t(1-\theta_t))k_t$$

where lower case variables denoted individual and a balanced budget constraint is imposed on the government

$$\theta_t k_t r_t + \tau_t w_t h_t = g_t$$

If the government could set θ_t at t, this would be an *ex-post* lump-sum tax. Klein and Rios-Rull assume a limited commitment, i.e., that taxes are set for the next period. To find a time consistent solution, we require that the policy the government follows is of Markov type, i.e., it is a function of the set of state variables only. These are

$$\{g, z, K, \theta\} \equiv x$$

Using budget balance, a policy rule is then

$$\theta_{t+1} = \psi(x_t)$$

We then define a recursive competitive equilibrium in the standard way, noting that the value function v depends on the policy rule

$$v(x, k; \psi).$$

Assuming the government is benevolent, it assesses welfare according to

$$V(x; \psi) = v(x, k; \psi).$$

We can also define the competitive equilibrium and its value function in case the government decides next periods tax to θ' and following government follow $\psi(x)$. The value function is then

$$\hat{v}(x, \theta', k; \psi)$$

The associated welfare of the government is

$$\hat{V}\left(x, \theta'; \psi\right)$$

Now define the current maximizing policy as

$$\Psi\left(x; \psi\right) = \arg\max_{\theta'} \hat{V}\left(x, \theta'; \psi\right)$$

A Markov perfect optimal tax policy then satisfies the fixed-point requirement

$$\Psi\left(x; \psi\left(x\right)\right) = \psi\left(x\right),$$

i.e., if the government expects coming government to use ψ it is optimal for itself to use ψ.

Klein and Rios-Rull use log utility and assume government consumption and productivity can each take on two different values respectively. They calibrate the model to US, data. Average g is 20%, varying 1.6% points up or down and an autocorrelation of .66. Productivity has a standard deviation of 2.4% with autocorrelation .88.

Comparing the commitment and no commitment they find that in commitment expected capital income tax rates are (almost) zero but with a standard deviation of 18%. having a strong positive correlation with g and a strong negative with z. Labor income taxes are 31% and almost fixed.

With 1 years commitment only, the average capital income tax rate is 65% with a standard deviation of 11%. It is positively correlated with g, but less than with full commitment.

Labor income tax rates are 12% on average with a standard deviation of 3%. Output is 14% lower than under commitment and somewhat less volatile.

Also 4 years commitment produce high average tax rates on capital income 36%.

1.4.2 A time-consistent taxation problem with an analytical solution

The model economy is populated by a continuum one of dynasties of two-period lived agents. In the first period of their lives, agents undertake an investment in human capital. The cost of investment to each individual is e^2, and the return is spread over two periods. In particular, the individual earn labor earning equal to $e \cdot w$ in the first period of her life, and $e \cdot w \cdot z$ in the second period. $z \leq 1$ captures the fact that agents retire within the second period of their life.

Dynasties derive utility from the consumption of a private and a public good. The public good is financed with a linear age-independent tax on income, denoted τ_t.

Each period's felicity depends on the total consumption (net of the investment cost) of the dynasty's member, irrespective of the split of consumption between the old and the young agent. The preferences of the dynasty which is alive at t are described by the following linear-quadratic utility functions

$$U_t = c_t + Ag_t - e_t^2 + \beta U_{t+1},$$

where $\beta \in [0, 1)$ is the discount factor, g_t denotes the public good available at t and A is a parameter describing the marginal utility of the public good. The marginal cost of the public good is unity and we focus on the case where $A \geq 1$, that will imply that the public good is socially valuable. Furthermore, we assume that the discount rate, $(1-\beta)/\beta$, equals the market interest rate. Given our assumptions, the savings decisions can be abstracted from, and the welfare of a dynasty is simply given by the present discounted value of their income net of investment costs;

$$U_t = \Sigma_{j=0}^{\infty} \beta^j \left((1-\tau_{t+j})\, y_{t+j} + A g_{t+j} - e_{t+j}^2 \right),$$

where:

$$y_{t+j} = (z e_{t+j-1} + e_{t+j})\, w, \tag{7}$$

i.e., the gross income accruing to the dynasty at $t+j$, given by the sum of the labor incomes generated by the parent born at $t+j-1$ and her offspring born at $t+j$. The parent's human capital depends on her investment at $t+j-1$ (e_{t+j-1}) while the offspring's human capital depends on her investment at $t+j$ (e_{t+j}). Since agents live for two periods, and the effect of the human capital investment dies with them, y_t only depends on the realization of two subsequent investments.

Due to a standard free-riding problem, there is not private provision of the public good. This is instead provided by an agency that will be called "government" that has access to a technology to turn one unit of revenue into one unit of public good. The government revenue is collected by taxing agents' labor income at the flat rate τ, subject to a balanced budget constraint. More formally, the government budget constraint requires that $g_t \leq \tau_t (z e_{t-1} + e_t)\, w$, where, at time t, e_{t-1} is predetermined. e_t,instead is determined after τ_t is set and in addition depends on expectations about the future tax rate. In particular, the optimal investment of a young agent at t is given by

$$e_t^* = e(\tau_t, \tau_{t+1}) \equiv \max\left[0, \frac{1+\beta z - (\tau_t + \beta z \tau_{t+1})}{2} w\right]. \tag{8}$$

This equation shows the distortionary. effect of taxation on investment. Note that taxation at $t+j$ distorts the investment of two generations: that born at $t+j-1$, as $e_{t+j-1}^* = e(\tau_{t+j-1}, \tau_{t+j})$, and that born at $t+j$, as $e_{t+j}^* = e(\tau_{t+j}, \tau_{t+j+1})$).

Letting $e_t = e(\tau_t, \tau_{t+1})$ and substituting it in into the government budget constraint, allows us to express the provision of public good at t as a function of current and future (one period ahead) taxes plus the level of investments sunk at $t-1$. More formally:

$$g_t = \tau_t (z e_{t-1} + e(\tau_t, \tau_{t+1})) w = g(\tau_t, \tau_{t+1}, e_{t-1}). \tag{9}$$

Finally, we restrict $\tau_t \in [0,1] \, \forall t$, which implies that investments, public good provision and private net income (e_t^*, g_t and $(1-\tau_t) y_t$) all are non-negative.

Before discussing the Markov equilibrium, let us state the solution to the full commitment equilibrium[4]

Proposition 10 *The optimal solution to the planner program is*

$$\tau_{t+1} = \max\{0, \tau^* - z(\tau_t - \tau^*)\} < 1, \tag{10}$$

[4] See Hassler et al (JME 2005).

for $t \geq 0$ *and*

$$\tau_0 = \begin{cases} \tau_0 = \left(1 + \frac{2ze_{-1}}{w(1-\beta z)}\right)\tau^* & \text{if } e_{-1} \leq \frac{w(1-\beta z)}{2z^2} \\ \min\left\{1, \left(1 + \beta z + \frac{2ze_{-1}}{w}\right)\tau^*\right\} & \text{else.} \end{cases},$$

where

$$\tau^* = \frac{A-1}{2A-1} \in [0, \frac{1}{2}).$$

is the steady-state tax rate. If $z < 1$, *the Ramsey tax sequence converges asymptotically in an oscillatory fashion to* τ^*. *If* $z = 1$, *the Ramsey tax sequence is a 2-period cycle such that,*

$$\tau_t = \begin{cases} \tau_0 & \text{if } t \text{ is even} \\ \max\{0, 2\tau^* - \tau_0\} & \text{if } t \text{ is odd.} \end{cases}$$

Note that if $e_{-1} = 0$,the optimal tax is at the steady state immediately. With positive e_{-1}, the planner wants to tax the pre-installed tax-base but this implies that also period 0 investments are hurt. To partly offset this, the planner promises taxes lower than steady state for period 1. But, there is then incentive to tax investments e_1 in period 1 a little higher by setting τ_2 above the steady state tax. Oscillating taxes therefore tends to smooth distortions over time.

The Markov allocation (Ramsey allocation without commitment) Let us now characterize the optimal time consistent allocation, namely, the allocation that is chosen by a benevolent planner without access to a commitment technology. Clearly, the oscillating path described above is not time-consistent.

We will use the recursive formulation of the problem, now assuming that period t taxes are set in the beginning of period t, and observed before period t investments are decided. The period t felicity of the planner is given by

$$\begin{aligned} F(e_{t-1}, \tau_t, \tau_{t+1}) &= (1-\tau_t) y_t - e(\tau_t, \tau_{t+1})^2 + A g_t \\ &= (z e_{t-1} + e(\tau_t, \tau_{t+1}))(1 + (A-1)\tau_t) w - e(\tau_t, \tau_{t+1})^2, \end{aligned}$$

where e_{t-1} is pre-determined.

Without commitment, the game between the government and the public is not degenerate. We characterize the equilibrium where e_{t-1} is the only state variable in period t and reputation is not used as a means to compensate for commitment. Thus, taxes are set according to a time-invariant function $\tau_t = T(e_{t-1})$ .Given this function, individuals rationally believe that $\tau_{t+1} = T(e_t)$ and individually rational investment choices must therefore satisfy

$$e_t = \frac{1 + \beta z - (\tau_t + \beta z T(e_t))}{2} w.$$

We can now define the equilibrium;

Definition 11 *A time-consistent (Markov) allocation without commitment is defined as a pair of functions* $\langle T, I \rangle$, *where* $T : [0, \infty) \rightarrow [0, 1]$ *is a public policy rule,* $\tau_t = T(e_{t-1})$, *and* $I : [0, 1] \rightarrow [0, \infty)$ *is a private investment rule,* $e_t = I(\tau_t)$ *such that the following functional equations are satisfied,*

1. $T(e_{t-1}) = \arg\max_{\tau_t} \{F(e_{t-1}, \tau_t, \tau_{t+1}) + \beta W(e_t)\}$ subject to $e_t = I(\tau_t)$, $\tau_{t+1} = T(I(\tau_t))$,

2. $I(\tau_t) = (1 + \beta - (\tau_t + \beta T(I(\tau_t))))\, w/2,$

3. $W(e_{t-1}) = \max_{\tau_t} \{F(e_{t-1}, \tau_t, \tau_{t+1}) + \beta W(e_t)\}$ subject to $e_t = I(\tau_t)\,,\ \tau_{t+1} = T(I(\tau_t))\,.$

The following Proposition can then be established.

Proposition 12 *Assume that either*[5] $A \leq \frac{z(z+1)}{(1+\beta)z^2-1}$ *or* $(1+\beta)\, z^2 \leq 1$. *Then, the time-consistent allocation is characterized as follows:*

$$T(e_{t-1}) = \min\{\bar{\tau} + \alpha_1 (e_{t-1} - \bar{e})\,, 1\}$$
$$I(\tau_t) = \bar{e} - \frac{w}{2 + \beta z \alpha_1 w} (\tau_t - \bar{\tau})\,,$$

where

$$\bar{e} = \frac{w(1+\beta z)(1-\alpha_0)}{2 + \alpha_1 w (1+\beta z)} \leq e^*$$
$$\bar{\tau} = \frac{2\alpha_0 + \alpha_1 w (1+\beta z)}{2 + \alpha_1 w (1+\beta z)} \geq \tau^*$$

with equalities iff $A = 1$,*and*

$$\alpha_1 = \frac{\sqrt{1 + 4A(A-1)(1-\beta z^2)} - (1 + 2(1-\beta z^2)(A-1))}{\beta z (A-1)(1-\beta z^2)\, w} \geq 0$$
$$\alpha_0 = \frac{2(A-1) - \beta z \alpha_1 w}{2 + (A-1)(4 + \beta z \alpha_1 w)} \geq 0$$
$$\frac{\partial \alpha_1}{\partial A} \geq 0, \frac{\partial \alpha_0}{\partial A} \geq 0, \frac{\partial \bar{\tau}}{\partial A} \geq 0, \frac{\partial \bar{e}}{\partial A} \leq 0.$$

[5] *This assumption ensures that the constraint* $\tau_{t+1} \leq 1$ *never binds for* $t \geq 0$. *Without this constraint, the analysis would be substantially more complicated, involving non-continuos policy functions.*

For all t*, the equilibrium law of motion is*

$$e_{t+1} = \bar{e} - z_d \left(e_t - \bar{e}\right), \tag{11}$$

$$\tau_{t+1} = \bar{\tau} - z_d \left(\tau_t - \bar{\tau}\right). \tag{12}$$

where

$$z_d \equiv \frac{\alpha_1 w}{2 + \beta \alpha_1 w} \in (0, z).$$

Given any e_{-1}*, the economy converges to a unique steady state such that* $\tau = \bar{\tau}$ *and* $e = \bar{e}$ *following an oscillating path and the constraint* $\tau_t \leq 1$ *iff t=0 and* $e_{-1} > \frac{1-\alpha_0}{\alpha_1}$*, while* $\tau_t \geq 0$ *never binds.*

The parameter restriction under which the Proposition is stated is a sufficient condition for the constraint $\tau_{t+1} \leq 1$ never to bind for $t \geq 0$. When this constraint is violated, the equilibrium policy functions may be non-continuous, making the analysis substantially more involved.

The main findings are that

1. the Markov allocation implies higher steady-state taxation ($\bar{\tau} > \tau^*$) and lower output and investment ($\bar{e} < e^*$) than the Ramsey allocation.

2. the Markov allocation implies less oscillations (i.e., a smoother tax sequence) than the Ramsey allocation: $z_d < z$.

It is interesting to note that the steady-state tax rate, $\bar{\tau}$, can exceed 1/2, i.e., it can be larger than the constant value of taxes that maximizes tax revenues. Specifically, this happens if $A > 1 + \frac{2+z(1-\beta z)}{z(2+z(1+\beta))}$, as threshold that decreases in β and z. I may seem counter intuitive that a benevolent planner would choose a tax rate that in steady state is on the wrong side of the Laffer curve. The fact that it can happen is due to lack of commitment; if $\bar{\tau} > 1/2$, the planner would clearly want to reduce the steady state tax rate. However, the planner can only control current tax rate and reducing that leads to higher taxes the next period and the overall effect of this is to reduce current welfare.

Here follows a sketch of the proof. The idea of the proof is as follows; Guess that the optimal policy function is linear in the state variable

$$\tau_{t+1} = T(e_t) = \alpha_0 + \alpha_1 e_t, \tag{13}$$

for the undetermined coefficients α_0 and α_1. Use the guess to derive the investment rule. Substitute these to into the Bellman equation for period t. Derive the first-order condition for period t and verify that it is linear in e_{t-1}. Find a_0 and α_1 such that the FOC in period t is satisfied.

The planner felicity in period t is

$$F(e_{t-1}, \tau_t, \tau_{t+1}) = (ze_{t-1} + e(\tau_t, \tau_{t+1}))(1 + (A-1)\tau_t)w - e(\tau_t, \tau_{t+1})^2,$$

Given the guess, the investment decision is $e_t = (1 + \beta z - (\tau_t + \beta z(\alpha_0 + \alpha_1 e_t)))w/2$, implying

$$e_t = I(\tau_t) = \frac{(1 + \beta z(1-\alpha_0))w}{2 + \beta z \alpha_1 w} - \frac{w}{2 + \beta z \alpha_1 w}\tau_t$$

and

$$\tau_{t+1} = T\left(I\left(\tau_t\right)\right) = \bar{\tau} + z_d\left(\tau_t - \bar{\tau}\right),$$

$$e_{t+1} = I\left(T\left(I\left(e_t\right)\right)\right) = \bar{e} + z_d\left(e_t - \bar{e}\right),$$

where

$$\bar{\tau} = \frac{2\alpha_0 + \alpha_1 w\left(1+\beta z\right)}{2 + \alpha_1 w\left(1+\beta z\right)}, \tag{14}$$

$$\bar{e} = \frac{w\left(1+\beta z\right)\left(1-\alpha_0\right)}{2 + \alpha_1 w\left(1+\beta z\right)}, \tag{15}$$

$$z_d = -\frac{w\alpha_1}{2 + \beta z \alpha_1 w} \tag{16}$$

The problem then admits the following recursive formulation:

$$W\left(e_{t-1}\right) = \max_{\tau_t}\left\{F\left(e_{t-1}, \tau_t, \tau_{t+1}\right) + \beta W\left(e_t\right)\right\}, \tag{17}$$

$$\text{s.t. } \tau_{t+1} = \alpha_0 + \alpha_1 e_t,$$

$$e_t = \frac{\left(1+\beta z\left(1-\alpha_0\right)\right)w}{2+\beta z \alpha_1 w} - \frac{w}{2+\beta z\alpha_1 w}\tau_t.$$

Given the guess, the first-order condition for maximizing the RHS of the Bellman equation is

$$\frac{\partial F}{\partial \tau_t} + \frac{\partial F}{\partial \tau_{t+1}}\frac{d\tau_{t+1}}{d\tau_t} + \beta\frac{dW\left(e_t\right)}{de_t}\frac{de_t}{d\tau_t} = 0,$$

where

$$\frac{\partial F}{\partial \tau_t} = \left(ze_{t-1} + e\left(\tau_t, \tau_{t+1}\right)\right)(A-1)\,w - \left(\left(1 + (A-1)\,\tau_t\right)w - 2e\left(\tau_t, \tau_{t+1}\right)\right)\frac{w}{2},$$

$$\frac{\partial F}{\partial \tau_{t+1}} = -\beta z\left(\left(1 + (A-1)\,\tau_t\right)w - 2e\left(\tau_t, \tau_{t+1}\right)\right)\frac{w}{2}$$

where we have used the fact that

$$\frac{\partial e_t}{\partial \tau_t} = -\frac{w}{2}, \frac{\partial e_t}{\partial \tau_{t+1}} = -\beta z\frac{w}{2},$$

Using the envelope condition, we obtain

$$W'\left(e_t\right) = \frac{\partial F\left(e_t, \tau_{t+1}, \tau_{t+2}\right)}{\partial e_t} = \left(1 + (A-1)\,\tau_{t+1}\right)wz.$$

which can be expressed in terms of τ_t using the constraints in (17). We can then can write the first-order condition as

$$\begin{aligned}
0 &= \frac{\partial F}{\partial \tau_t} + \frac{\partial F}{\partial \tau_{t+1}}\frac{d\tau_{t+1}}{d\tau_t} + \beta W'\left(e_t\right)\frac{de_t}{d\tau_t} \\
0 &= \left(A - \frac{\beta z\alpha_1 w}{2 + \beta z\alpha_1 w}\right)e_t - \frac{2\,(A-1)\,w}{\left(2 + \beta z\alpha_1 w\right)^2}\tau_t + z\,(A-1)\,e_{t-1} \\
&- w\frac{(1+\beta z)\left(2 + A\beta z\alpha_1 w\right) + 2\beta z\alpha_0\,(A-1)}{\left(2 + \beta z\alpha_1 w\right)^2}
\end{aligned}$$

Using the fact that, $e_t = \frac{(1+\beta z(1-\alpha_0))w}{2+\beta z\alpha_1 w} - \frac{w}{2+\beta z\alpha_1 w}\tau_t$ and the guess $\tau_t = \alpha_0 + \alpha_1 e_{t-1}$, dividing by w and collecting terms, this yields

$$0 = \left(z\left(A-1\right) - \left(\frac{2A}{\left(2+\beta z\alpha_1 w\right)} + \left(A-1\right) \right) \frac{w\alpha_1}{\left(2+\beta z\alpha_1 w\right)} \right) e_{t-1}$$
$$+ \frac{w\left(1+\beta z\right)}{2+\beta z\alpha_1 w} \left(\frac{2A\left(1-\alpha_0\right)}{2+\beta z\alpha_1 w} - \left(1+\alpha_0\left(A-1\right)\right) \right)$$

In order for this condition to be satisfied for all e_{t-1} we need,

$$z\left(A-1\right) - \left(\frac{2A}{\left(2+\beta z\alpha_1 w\right)} + \left(A-1\right) \right) \frac{w\alpha_1}{2+\beta z\alpha_1 w} = 0 \tag{18}$$

$$\frac{2A\left(1-\alpha_0\right)}{2+\beta z\alpha_1 w} - \left(1+\alpha_0\left(A-1\right)\right) = 0 \tag{19}$$

A solution for these equations (ignoring the roots that would generate instability) is: :

$$\alpha_1 = \frac{\sqrt{1+4A\left(A-1\right)\left(1-\beta z^2\right)} - \left(1+2\left(1-\beta z^2\right)\left(A-1\right)\right)}{\beta z\left(A-1\right)\left(1-\beta z^2\right)w} \geq 0$$

$$\alpha_0 = \frac{2\left(A-1\right) - \beta z\alpha_1 w}{2+\left(A-1\right)\left(4+\beta z\alpha_1 w\right)}$$
$$= \frac{2A\left(A-1\right)\left(1-\beta z^2\right) - \left(\sqrt{1+4A\left(A-1\right)\left(1-\beta z^2\right)}-1\right)}{\left(A-1\right)\left(2A\left(1-\beta z^2\right) + \left(\sqrt{1+4A\left(A-1\right)\left(1-\beta z^2\right)}-1\right)\right)} \geq 0$$

The non-negativity of α_0 and α_1 are established by standard algebra, since, in both the expressions, the numerator and denominators are both positive.

2 Optimal unemployment insurance (UI)

There is a large literature of optimal unemployment insurance. The basic issue is how to provide the most efficient unemployment insurance when there is a moral hazard problem. This is arising from an assumption that unemployed individuals can affect the probability they find (and accept) a job offer. However, it is costly for the worker to increase this probability, e.g., because of effort costs, reduced reservation wages or opportunity costs of time.

2.1 The semi-static approach to optimal UI

The basic idea in Baily and Chetty is to simplify the dynamic problem into a static one. This makes the model simple and tractable also when savings is allowed. An important lesson is that when savings is allowed, we can use the drop in consumption at unemployment as a measure of the welfare loss associated with unemployment. In a dynamic model, this does not work when there is no market for savings. Why? The trade-off faced by the planner is to balance the loss of welfare associated with unemployment against the negative effect on search induced by UI.

2.1.1 The simplest model following Baily

- In the first period, the individual works and chooses how much to consume of the income, normalized to unity, and how much to save.
- In the beginning of the second period, the individual becomes unemployed with probability $1 - \alpha$ and otherwise keeps his job.

- During the second period, the individual can determine how long it takes to find a job by choosing the reservation wage y_n and costly search effort c. A share $\beta = \beta(c, y_n)$ of the second period is spent working in the new job.

- While unemployed, the individual gets UI-benefits b.These are paid by taxes on workers.

- Agents have access to a market for precautionary (buffer stock) savings.

- Both the unemployment duration and the wage upon rehiring is non-stochastic.

Total disposable income in second period if laid off is therefore the non-stochastic value

$$(1-\beta)(b-c) + \beta y_n (1-\tau) \equiv y_l.$$

In first periods, individuals decide how much to save, s. Interest rate and subjective discount rate is normalized to zero. If an individual gets laid off, he consumes his resources, i.e., his disposable income plus savings.

[6]

$$V = u(1-\tau-s) + \alpha u(1-\tau+s) + (1-\alpha)(u(y_l+s)).$$

Government budget constraint is

$$(1+\alpha+(1-\alpha)\beta y_n)\tau = (1-\alpha)(1-\beta)b.$$

$$\Longrightarrow b = \frac{(1+\alpha+(1-\alpha)\beta y_n)}{(1-\alpha)(1-\beta)}\tau \equiv \mu\tau$$

[6] by individual

Denoting the *endogenous* total income by $Y \equiv 1 + \alpha + (1-\alpha)\beta y_n$, this implies

$$b = \frac{Y}{(1-\alpha)(1-\beta)}\tau$$
$$\equiv \mu\tau,$$

where we note that μ is *not* a constant, but depends on individual choices of y_n and c and thus indirectly on taxes and benefits. Given the budget constraint and individual choices, we can therefore write $\mu = \mu(\tau)$ (provided there is a solution, which is not necessarily true for all τ.Explain!)

Note that in first best, c should be chosen to satisfy

$$(y_n + c)\beta_c = 1 - \beta$$

since social income is

$$-(1-\beta(y_n, c))c + \beta(y_n, c)y_n$$

implying that the marginal gain of a marginal unit of effort is $\beta_c(y_n + c)$ and the cost is $1 - \beta$.

The individual instead gains,

$$\beta_c(y_n(1-\tau) + c - b)$$

so the private value of search is lower while the private and social cost is the same. Similarly, an increase in y_n has benefits β and costs $-(y_n + c)\beta_{y_n}$. While private benefits are

$(1-\tau)\beta$ and private costs $-(y_n(1-\tau)+c-b)\beta_{y_n}$. The wedges between private and social costs/benefits imply that both choices will be distorted in second best.

We can now write

$$V = u(1-\tau-s) + \alpha u(1-\tau+s) + (1-\alpha)(u((1-\beta)(\mu(\tau)\tau - c) + \beta y_n(1-\tau)+s))$$
$$V = V(c, y_n, s, \mu, \tau)$$

The optimal UI system maximizes solves

$$\max_{\tau} V(c, y_n, s, \mu(\tau), \tau)$$

Although, c, y_n, s are affected by τ, these effects need not be taken into account since by individual optimality,

$$V_c = V_{y_n} = V_s = 0.$$

This is the envelope theorem. Therefore, the first order condition for maximizing V by choosing τ is

$$\frac{dV}{d\tau} = V_\mu \frac{d\mu}{d\tau} + V_\tau = 0,$$

where

$$V_\mu = (1-\alpha)u'(c_u)(1-\beta)\tau$$
$$V_\tau = -u'(c_1) - \alpha u'(c_2) - (1-\alpha)u'(c_u)\beta y_n + (1-\alpha)u'(c_u)(1-\beta)\mu,$$

where $c_1 = 1 - \tau - s$ is first period consumption, $c_2 = 1 - \tau + s$ is second period consumption if the job is retained and $c_u = (1-\beta)(\mu\tau - c) + \beta y_n (1-\tau) + s$ is second period consumption if the individual lost his job.

Note that by individual savings optimization (the Euler equation)

$$u'(c_1) = \alpha u'(c_2) + (1-\alpha) u'(c_u)$$

$$u'(c_1) - (1-\alpha) u'(c_u) = \alpha u'(c_2)$$

implying

$$\begin{aligned} V_\tau &= -u'(c_1) - (u'(c_1) - (1-\alpha) u'(c_u)) - (1-\alpha) u'(c_u) \beta y_n + (1-\alpha) u'(c_u)(1-\beta)\mu \\ &= -2u'(c_1) + (1-\alpha)(1 - \beta y_n + (1-\beta)\mu) u'(c_u). \end{aligned}$$

Approximating

$$u'(c_1) \approx u'(c_u) + u''(c_u) \Delta c$$

where $\Delta c \equiv c_1 - c_u$ is the fall in consumption if becoming unemployed. The first order condition is then

$$
\begin{aligned}
0 &= (1-\alpha) u'(c_u)(1-\beta)\tau \frac{d\mu}{d\tau} - 2\left(u'(c_u) + u''(c_u)\Delta c\right) \\
&\quad + (1-\alpha)(1-\beta y_n + (1-\beta)\mu) u'(c_u) \\
2\left(1 + \frac{u''}{u'}\Delta c\right) &= (1-\alpha)(1-\beta)\tau\frac{d\mu}{d\tau} + (1-\alpha)(1-\beta y_n + (1-\beta)\mu) \\
2\left(1 + \frac{u''}{u'}\Delta c\right) &= (1-\alpha)(1-\beta)\tau\frac{d\mu}{d\tau} + (1-\alpha)\left(1-\beta y_n + (1-\beta)\frac{Y}{(1-\alpha)(1-\beta)}\right) \\
2\left(1 + \frac{u''}{u'}\Delta c\right) &= (1-\alpha)(1-\beta)\tau\frac{d\mu}{d\tau} + (1-\alpha)\left(1-\beta y_n + \frac{Y}{1-\alpha}\right) \\
2\left(1 + \frac{u''}{u'}\Delta c\right) &= (1-\alpha)(1-\beta)\tau\frac{d\mu}{d\tau} + (1-\alpha)\left(1-\beta y_n + \frac{1+\alpha+(1-\alpha)\beta y_n}{(1-\alpha)}\right) \\
2\left(1 + \frac{u''}{u'}\Delta c\right) &= (1-\alpha)(1-\beta)\tau\frac{d\mu}{d\tau} + 2 \\
\frac{u''}{u'}\Delta c &= \frac{(1-\alpha)(1-\beta)\tau}{2}\frac{d\mu}{d\tau}
\end{aligned}
$$

Using the definition

$$\mu \equiv \frac{b}{\tau} = \frac{Y}{(1-\alpha)(1-\beta)}$$

we get

$$
\begin{aligned}
\frac{u''}{u'}\Delta c &= \frac{\tau}{\mu}\frac{d\mu}{d\tau}\frac{Y}{2} \\
-R_r\frac{\Delta c}{c} &= E_{\mu,t}\frac{Y}{2}
\end{aligned}
$$

Where $E_{\mu,t}$ is the elasticity of μ with respect to taxes and R_r the relative risk aversion coefficient. Recall that μ is the ratio between benefits and taxes should be interpreted as the ratio between employment and unemployment.

Note, finally, that $1+\alpha+(1-\alpha)\beta y_n \approx 1$, giving

$$R_r \frac{\Delta c}{c} \approx -E_{\mu,t}$$

The interpretation is that the welfare loss (the LHS) should optimally be given by how elastic the ratio of employment to unemployment is with respect to taxes.

Without moral hazard, $\frac{d\mu}{d\tau} = 0 = E_{\mu,t}$, in which case optimality requires $\Delta c = 0$. With moral hazard, higher taxes tends to reduce μ since the employment to unemployment falls in in taxes, i.e., $\frac{\tau}{\mu}\frac{d\mu}{d\tau} = E_{\mu,t}$ is negative. Therefore, $\frac{\Delta c}{c} > 0$. We see that $\frac{\Delta c}{c}$ increases if $\frac{\tau}{\mu}\frac{d\mu}{d\tau}$ is large in absolute terms and falls if risk aversion is large. Baily claims that $E_{\mu,t}$ is in the order $0.15 - 0.4$. With log utility, this is also how much consumption should fall on entering unemployment.

This approach has been generalized by Chetty showing that we can have repeated spells of unemployment, uncertain spells of unemployment, value of leisure, private insurance and borrowing constraints. The model can therefore be extended to evaluate UI reforms. With a more dynamic model, and in particular if capital markets are imperfect, it should be noted that one needs to know how the whole consumption profile is affected by unemployment. The drop at entering unemployment may not be enough. Shimer and Werning (2007), shows that the *reservation wage* can be used as a summary measure of how bad unemployment is.

In any case, this the model is not suitable to analyze

1. General equilibrium effects like impacts on wages, search spillovers and job creation.

2. Interaction with other taxes-fiscal spillovers.

3. Time varying benefits.

2.2 The dynamic approach with observable savings

The seminal paper by Shavel & Weiss (1979) focuses on the optimal time profile of benefits. It is a simple infinite horizon discrete time model where the aim is to maximize utility of a representative unemployed subject to a government budget constraint. Utility is given by

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t (u(c_t) - e_t)$$

where c_t is period t consumption and e_t is a privately chosen unobservable effort associated with job search. The subjective discount rate is r, which is assumed to coincide with an exogenous interest rate.

It is assumed that the individual has no access to capital markets so $c_t = b_t$ when the individual is unemployed. After regaining employment, the wage is w forever.

When the individual becomes employed he stays employed for ever for simplicity. Agents have no access to credit markets (or equivalently, savings is perfectly monitored and benefits can be made contingent on them) so the planner can perfectly control the consumption of the individual. The moral hazard problem is that individuals can affect the probability of finding a job. As in Baily (1978), the individual controls both the search effort (here called e_t) and the reservation wage (here w_t^*).

Given an effort level e_t, the individual receives one job offer per period with an associated wage drawn from a distribution with a time invariant probability density $f(w_t, e_t)$. The probability of finding an acceptable job in period t is thus

$$p(w_t^*, e_t) = \int_{w_t^*}^{\infty} f(w_t, e_t)\, dw_t$$

with

$$p_w(w_t^*, e_t) = -f(w_t, e_t) \leq 0 \text{ and}$$
$$p_e(w_t^*, e_t) > 0$$

where the latter is by assumption.

Let E_t be the expected utility of an unemployed individual that choose optimally a sequence $\left\{e_{t+s}, w_{t+s}^*\right\}_{s=0}^{\infty}$. Define

$$u_t = \tilde{u}(w_t^*, e_t) \equiv \frac{1+r}{r} \int_{w_t^*}^{\infty} u(w_t) \frac{f(w_t, e_t)}{p(w_t^*, e_t)} dw_t$$

This is the expected utility from next period, *conditional* on finding a job this period, which starts next period. We note that

$$\tilde{u}_w(w_t^*, e_t) \geq 0$$
$$\tilde{u}_e(w_t^*, e_t) \geq 0.$$

The first inequality follows from the fact that *conditional* on finding a job, wages are higher for higher reservation wages. The second inequality is by assumption, higher search effort leads to no worse distribution of acceptable job offers.

E_t satisfies the standard Bellman equation

$$E_t = \max_{e_t, w_t^*} u(b_t) - e_t + \frac{1}{1+r} \left(p(w_t^*, e_t) \tilde{u}(w_t^*, e_t) + (1 - p(w_t^*, e_t)) E_{t+1} \right)$$

The first-order conditions are

$$e_t; \frac{1}{1+r} \left(p_e(w_t^*, e_t) \left(\tilde{u}(w_t^*, e_t) - E_{t+1} \right) + p(w_t^*, e_t) \tilde{u}_e(w_t^*, e_t) \right) = 1$$

$$w_t^*; -p_w(w_t^*, e_t) \left(\tilde{u}(w_t^*, e_t) - E_{t+1} \right) = p(w_t^*, e_t) \tilde{u}_w(w_t^*, e_t).$$

In the first equation, the LHS is the marginal benefit of higher search effort, coming from a higher probability of finding a job and better jobs if found. These balances the cost which is 1. In the second equation, the LHS is the marginal cost of higher reservation wages, coming from a lower probability of finding a job. The RHS is the gain, coming from better jobs if accepted.

By the envelope theorem

$$\frac{dE_t}{dE_{t+1}} = \frac{\partial E_t}{\partial E_{t+1}} = \frac{1 - p(w_t^*, e_t)}{1+r}$$

Now, we will show the important results that anything that reduces next periods unemployment value E_{t+1} will reduce $1 - p(w_t^*, e_t)$, i.e., make hiring more likely. To see this, note that if E_{t+1} falls,

$$\begin{aligned} & p_e\left(w_t^*, e_t\right)\left(\tilde{u}\left(w_t^*, e_t\right)-E_{t+1}\right)+p\left(w_t^*, e_t\right) \tilde{u}_e\left(w_t^*, e_t\right), \text { and } \\ & -p_w\left(w_t^*, e_t\right)\left(u\left(w_t^*, e_t\right)-E_{t+1}\right) \end{aligned}$$

both becomes larger if choices are unchanged. In words, the marginal benefit of searching harder and the marginal cost of setting higher reservation wages both increase. Thus, a reduction in E_{t+1} increase search effort and reduce the reservation wage increasing p.

Now, we can use this to show the key result that benefits should have a decreasing profile.

Proof:

Suppose contrary that $b_t = b_{t+1}$. Then consider an infinitessimal increase in b_t financed by an actuarially fair reduction in b_{t+1}, that is

$$db_t = -\frac{1-p}{1+r} db_{t+1} > 0$$

where $p\left(w_t^*, e_t\right)$ is calculated at the initial (constant) benefit levels. The direct effect on felicitity levels (period utilities) is

$$\begin{aligned} & u'\left(b_t\right) db_t + \frac{1-p}{1+r} u'\left(b_{t+1}\right) db_{t+1} \\ & -u'\left(b_t\right) \frac{1-p}{1+r} db_{t+1} + \frac{1-p}{1+r} u'\left(b_{t+1}\right) db_{t+1} \\ & = 0 \end{aligned}$$

since $u'\left(b_t\right) = u'\left(b_{t+1}\right)$.By the envelope theorem, we need not take into account changes in endogenous variables when calculating welfare. Therefore, E_t is unchanged. Since $u(b_t)$ has increased, E_{t+1} must have fallen. When calculating the budgetary effects we need to into account the endogenous changes on p.

Let

$$B_t = b_t + \frac{1-p}{1+r} b_{t+1}$$

Then,

$$\begin{aligned} dB_t &= db_t + \frac{1-p}{1+r} db_{t+1} - \frac{dp}{1+r} b_{t+1} \\ &= -\frac{dp}{1+r} b_{t+1} \end{aligned}$$

Since E_{t+1} has fallen, $dp > 0$. Thus $dB_t < 0$. I.e., the cost of providing utility E_t has fallen. Equivalently, the insurance is more efficient than the starting point $b_t = b_{t+1}$.

2.2.1 Extensions

Hopenhayn and Nicolini extend the model by Shavel & Weiss in an important dimension – it enriches the policy space of the government by allowing taxation of workers to be contingent on their unemployment history. It is shown that the government should use this extra way of "punishing" unemployment. The intuition is that relative to the first best, which is a constant unemployment benefit, the government must "punish" unemployment. Doing this by only reducing unemployment benefits is suboptimal, by spreading the punishment of unsuccessful search over the entire future of the individual, a more efficient insurance can be achieved. I.e., lower cost of providing a given utility level. It is shown that this may be quantitatively important. Another contribution is to show that the problem can be formulated in a recursive way with the *promised utility* as state variable.

Using H&N's notation, we assume that individuals can choose an unobservable effort level a_t that positively affects the hiring probability. In H&N 1997, it is assumed that $p(a_t)$ is an concave and increasing function and hiring is an absorbing state with a wage w forever. In H&N 2005, it is instead assumed that spells are repeated, with an exogenous separation probability s and

$$p(a) = \begin{cases} p \text{ if } a = 1 \\ 0 \text{ otherwise} \end{cases}$$

which is the assumption we make here.

The individual has a utility function

$$E \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t (u(c_t) - a_t).$$

Let $\theta_t \in \{0, 1\}$ be the employment status of the individual in period t, where $\theta_t = 1$ represents employment. Let $\theta^t = (\theta_0, \theta_1, ...\theta_t)$ be the history of the agent up until period t. The history of a person that is unemployed in period t is therefore $\theta^{t-1} \times 0 = (\theta_0, \theta_1, ...\theta_t, 0) \equiv \theta^t_u$, and similarly, $\theta^{t-1} \times 1 \equiv \theta^t_e$).

An allocation is now defined as a rule that assigns consumption and effort as a function of θ^t at every point in time and for every possible history, $c_t = c(\theta^t)$. We focus on allocations where $a_t = 1$. Individuals must be induced to voluntarily choose $a_t = 1$. Allocations that satisfies this are called incentive compatible allocations.

Given an allocation we can compute the expected discounted utility at every point in time for every possible history, $V_t = V(\theta^t)$. The problem is now to choose the allocation that minimizes the cost of giving some fixed initial utility level to the representative individual. This problem can be written in a recursive way. In period zero, the planner gives a consumption level c_0, prescribes an effort level a_0 (=1) and promised continuation utilities $V_1^e \equiv V(\theta_e^1)$ and $V_1^u = V(\theta_u^1)$. The problem of the planner in period zero is to minimize costs of providing a given expected utility level V_0 subject to the incentive constraint the individual voluntarily chooses a_0. The problem is recursive and at any node, costs of providing promised utilities are minimized given incentive constraints

The problem of the unemployed individual is also recursive. – as unemployed, maximized utility is (the agent only controls a_t)

$$V\left(\theta_u^t\right) = u(c_t) - 1 + \frac{1}{1+r}\left(pV\left(\theta_u^t \times 1\right) + (1-p)\,V\left(\theta_u^t \times 0\right)\right)$$

with the incentive constraint

$$\frac{1}{1+r}p\left(V\left(\theta_e^{t+1}\right) - V\left(\theta_u^{t+1}\right)\right) \geq 1.$$

Define $W(V_t)$ as the minimum cost for the planner to provide a given amount of utility V_t to an employed. Similarly, let $C(V_t)$ denote the minimal cost of providing utility V to an unemployed (are these function changing over time?). W satisfies

$$W(V_t) = \min_{c_t, V_{t+1}^e, V_{t+1}^u} c_t - w + \frac{1}{1+r}\left((1-s)\,W\left(V_{t+1}^e\right) + sC\left(V_{t+1}^u\right)\right)$$

$$s.t. V_t = u(c_t) + \frac{1}{1+r}\left((1-s)\,V_{t+1}^e + sV_{t+1}^u\right),$$

where$V_t = V\left(\theta_e^t\right)$, $c_t = c\left(\theta_e^t\right)$, $V_{t+1}^e = V\left(\theta_e^t \times 1\right)$ and $V_{t+1}^u = V\left(\theta_e^t \times 0\right)$.

The constraint can be called promise keeping constraint and has a Lagrange multiplier δ_t^e.

C satisfies

$$\begin{gathered} C\left(V_t\right) = \min_{c_t, V_{t+1}^e, V_{t+1}^u} c_t + \frac{1}{1+r}\left(pW\left(V_{t+1}^e\right) + (1-p)\, C\left(V_{t+1}^u\right)\right) \\ \text{s.t. } \frac{1}{1+r} p\left(V_{t+1}^e - V_{t+1}^u\right) \geq 1, \\ V_t = u\left(c_t\right) - 1 + \frac{1}{1+r}\left(pV_{t+1}^e + (1-p)\, V_{t+1}^u\right). \end{gathered}$$

where $V_t = V\left(\theta_u^t\right)$, $c_t = c\left(\theta_u^t\right)$, $V_{t+1}^e = V\left(\theta_u^t \times 1\right)$ and $V_{t+1}^u = V\left(\theta_u^t \times 0\right)$.

The first constraint is the incentive constraint, with an associated Lagrange multiplier γ_t and the second is the promised utility with Lagrange multiplier δ_t^u.[7] Given that $u\left(c_t\right)$ is concave and $u^{-1}\left(V_t\right)$ therefore is convex, it is straightforward to show that C and W are convex functions.

First order conditions when the agent is employed are

$$\begin{aligned} 1 &= \delta_t^e u'\left(c_t\right) \qquad (20) \\ W'\left(V_{t+1}^e\right) &= \delta_t^e \\ C'\left(V_{t+1}^u\right) &= \delta_t^e. \end{aligned}$$

[7] Note that the Lagrange multipliers depends on the history θ_t.

The envelope condition is

$$W'(V_t) = \delta_t^e = \frac{1}{u'(c_t)} = W'\left(V_{t+1}^e\right) = C'\left(V_{t+1}^u\right).$$

The fact that $W'(V_t) = W'\left(V_{t+1}^e\right)$ implies that nothing change for the employed individual as long as his remains employed. Since $W'(V_t) = C'\left(V_{t+1}^u\right)$,marginal marginal utility does not change if the person becomes unemployed, i.e., consumption does not change upon loosing his job either. This is due to the fact that there is no moral hazard problem on the job and full insurance is therefore optimal.[8]

When the agent is unemployed, the FOC and envelope conditions are

$$\begin{aligned} 1 &= \delta_t^u u'(c_t) \\ W'\left(V_{t+1}^e\right) &= \gamma_t + \delta_t^u \\ (1-p)\, C'\left(V_{t+1}^u\right) &= -\gamma_t p + \delta_t^u (1-p) \\ C'(V_t) &= \delta_t^u. \end{aligned}$$

Giving

$$\begin{aligned} C'(V_t) &= \frac{1}{u'(c_t)} \qquad (21) \\ W'\left(V_{t+1}^e\right) &= \frac{1}{u'(c_t)} + \gamma_t \\ C'\left(V_{t+1}^u\right) &= \frac{1}{u'(c_t)} - \gamma_t \frac{p}{1-p} \end{aligned}$$

[8]From now, I will mostly skip writing out the explicit dependence on history, hopefully without creating confusion.

Results

Since the incentive constraint will bind[9], $\gamma_t > 0$ and therefore

$$W'\left(V^e_{t+1}\right) > C'\left(V_t\right) > C'\left(V^u_{t+1}\right),$$
$$\frac{1}{u'\left(c\left(\theta^t_u \times 1\right)\right)} > \frac{1}{u'\left(c\left(\theta^t_u\right)\right)} > \frac{1}{u'\left(c\left(\theta^t_u \times 1\right)\right)}$$
$$c\left(\theta^t_u \times 1\right) > c\left(\theta^t_u\right) > c\left(\theta^t_u \times 0\right)$$

The result $C'\left(V_t\right) > C'\left(V^u_{t+1}\right)$ and the convexity of C implies that the unemployed should be made successively worse off ($V^u_{t+1} < V_t$) as long as he remains unemployed. Since $C'\left(V_t\right) = \frac{1}{u'(c_t)}$ this means that consumption must fall. Furthermore, as the IC-constraint $\frac{1}{1+r}p\left(V^e_{t+1} - V^u_{t+1}\right) \geq 1$ binds, if V^u_{t+1} keeps falling as long as the unemployed remains unemployed, so must V^e_{t+1} implying that consumption when becoming employed is lower the longer the agent has been unemployed.

2.2.2 The inverse Euler equation.

Multiplying the second line of (21) by p and the third by $(1-p)$ and adding them yields,

$$\frac{1}{u'\left(c_t\right)} = pW'\left(V^e_{t+1}\right) + (1-p)\, C'\left(V^u_{t+1}\right). \tag{22}$$

Recall that V^e_{t+1} is the utility next period if the agent becomes employed, in which case, by (20), $W'\left(V^e_{t+1}\right) = \frac{1}{u'(c_{t+1})}$, where $c_{t+1} = c\left(\theta^e_{t+1}\right)$ denotes consumption in period $t+1$ conditional on the getting a job in $t+1$ (and the history that led to consumption in t

[9] Prove that it must by assuming that it doesn't and derive the implications of that.

being $c_t = c(\theta_t)$). Similarly, V_{t+1}^u is next periods utility if the agent remains unemployed. By (21), $C'\left(V_{t+1}^e\right) = \frac{1}{u'(c(\theta_u^t \times 0))}$, where $c(\theta_u^t \times 0)$ denotes consumption if the agent remains unemployed. Equation (22) can therefore be written

$$\begin{aligned} \frac{1}{u'\left(c\left(\theta_u^t\right)\right)} &= p\frac{1}{u'\left(c\left(\theta_u^t \times 1\right)\right)} + (1-p)\frac{1}{u'\left(c\left(\theta_u^t \times 0\right)\right)} \\ \frac{1}{u'(c_t)} &= E_t\frac{1}{u'(c_{t+1})}. \end{aligned}$$

This is the famous "Inverse Euler Equation" (Rogerson, -85 Econometrica)[10]. Note the difference between this and the standard Euler equation.

$$u'(c_t) = E_t u'(c_{t+1}).$$

The inverse Euler equation has an important implication. To see this, first note that Jensen's inequality,

$$E_t\frac{1}{u'(c_{t+1})} > \frac{1}{E_t u'(c_{t+1})} \Rightarrow \frac{1}{E_t\frac{1}{u'(c_{t+1})}} < E_t u'(c_{t+1})$$

since the inverse function is convex. Using this with the Inverse Euler equation gives,

$$u'(c_t) = \frac{1}{E_t\frac{1}{u'(c_{t+1})}} < E_t u'(c_{t+1}).$$

[10] With a difference between subjective and market discount rates (ρ and r, respectively), we would get

$$\frac{1}{u'(c_t)}\frac{1+r}{1+\rho} = E_t\frac{1}{u'(c_{t+1})}.$$

The fact that $u'(c_t) < E_t u'(c_{t+1})$ in the optimal allocation means that the agent would like to save more if he had access to a capital market with interest rate r, i.e., he is savings constrained. The incentive constraint implies that it is optimal to prevent the individual to save as much as he would like to. Suppose, for example, that utility is logarithmic, then we have

$$\frac{1}{c_t} = \frac{1}{E_t c_{t+1}} \Rightarrow c_t = E_t c_{t+1},$$

while the Euler equation, guiding private preferences, implies the privately optimal consumption c_t^* given future consumption is

$$c_t^* = \frac{1}{E_t\left(\frac{1}{c_{t+1}}\right)} < E_t c_{t+1}.$$

The intuition is that with more wealth and higher consumption, it is more costly to implement the incentive constraint. Thus, the benevolent planner want to prevent some wealth accumulation. The standard interpretation of this is that when there are incentive constraints, it may be optimal to tax the returns to savings. However, it may turn out that this tax is nevertheless zero in expectation, thus not creating any revenue for the planner/government (Kocherlakota 2005, Econometrica). How can such a tax discourage savings? Hint: risk premium depends on covariance with marginal utility. Explain!

In the logarithmic example, suppose individuals can save and borrow a gross interest rate r. Consider a marginal tax rate that depends on employment status and last period individual asset holdings, $\tau_{t+1}^e = \tau^e(a_t)$ and $\tau_{t+1}^u = \tau^u(a_t)$. Then, to have the individual

Euler equation satisfied, we need

$$u'(c_t) = \beta E_t u'(c_{t+1})(1+r)(1-\tau(a_t)) \tag{23}$$
$$\frac{1}{c_t} = \left(p\frac{1}{c^e_{t+1}}\left(1-\tau^e_{t+1}\right) + (1-p)\frac{1}{c^u_{t+1}}\left(1-\tau^u_{t+1}\right)\right)$$

The inverse Euler equation requires

$$c_t = pc^e_{t+1} + (1-p)\,c^u_{t+1} \tag{24}$$

Suppose we consider a zero expected tax rate, i.e., $p\tau^e_{t+1} = -(1-p)\,\tau^u_{t+1}$.Then,

$$\tau^e_{t+1} = \frac{-(1-p)}{p}\tau^u_{t+1}. \tag{25}$$

Using (24) to replace c_t in (23) together with (25) yields

$$\tau^u_{t+1} = \frac{p\left(c^e_{t+1} - c^u_{t+1}\right)}{pc^e_{t+1} + c^u_{t+1}(1-p)} = \frac{p\Delta c_{t+1}}{E_t c_{t+1}}$$
$$\tau^e_{t+1} = -\frac{(1-p)\left(c^e_{t+1} - c^u_{t+1}\right)}{pc^e_{t+1} + c^u_{t+1}(1-p)} = -\frac{(1-p)\,\Delta c_{t+1}}{E_t c_{t+1}}$$

These tax rates leads to both the Euler and the inverse Euler equation being satisfied. Note that the tax is *negative* in case the agent becomes employed, while positive if he remains unemployed. That is, it creates a net return that is negatively correlated with marginal utility.

Result: Rendahl (2007)

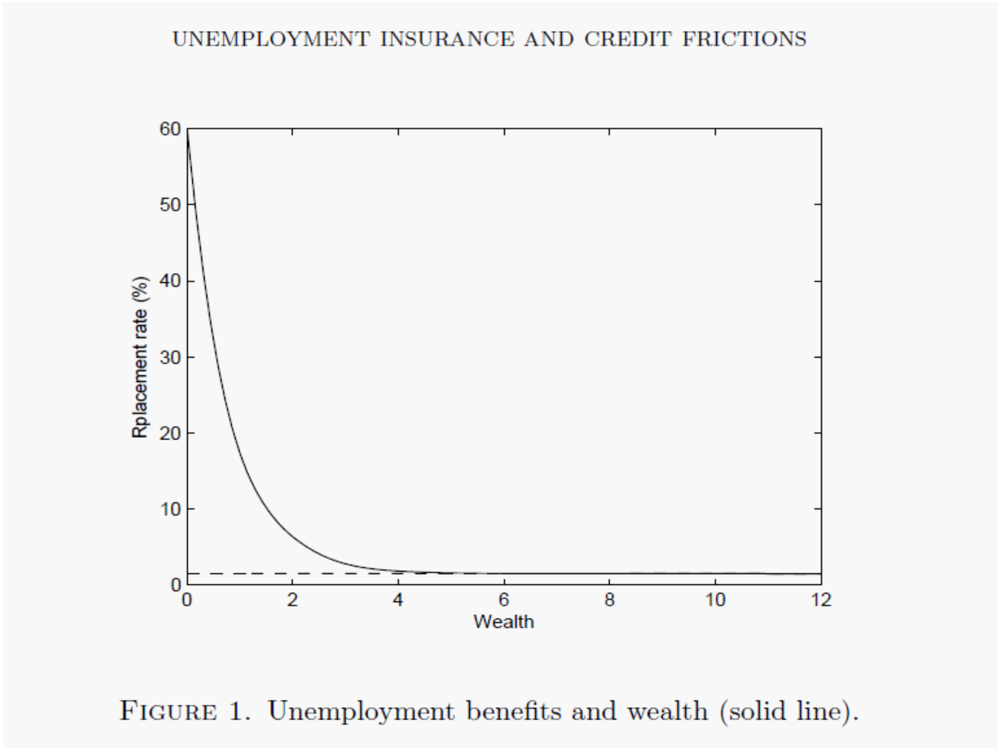


Figure 1: From Pontus Rendahl 2007 (Job market paper)

Consider the repeated H&N economy but where individuals have access to a safe observable bond. It turns out that a tax/transfer that only depends on last period asset holdings and employment status can implement the second-best allocation as the private choices of individuals. Unemployment benefits falls in the asset position of the agent. Over an unemployment spell, unemployment benefits increase but consumption falls.

2.3 The Dynamic approach with unobservable saving

An key assumption in the previous subsection was that the planner can control the consumption level of the individual at all times, the only unobservable is search effort. In reality this assumption seems questionable, given the existence of alternative means of income, capital markets, insurance within an extended family and durable goods.

In this subsection, we assume that the planner cannot control the consumption of the individual – she has access to a perfect market for lending and borrowing at a fixed interest rate and her wealth is unobservable. Of course, this extreme is perhaps equally unrealistic and the truth might be somewhere in between.

An immediate problem is that search decisions in this setting might depend on the unobservable wealth level. Making sure that there is always an incentive to search might then be unfeasible in general. In one special case, the search decision is not dependent on wealth, when individuals have CARA utility. This is the way we go here. Furthermore, we simplify by assuming that search is either one or zero.

Individuals maximize their intertemporal utility, given by

$$E\int_0^\infty e^{-rt}U\left(c_t\right)dt,$$

where

$$U\left(c_t\right)\equiv -e^{-\gamma c_t}.$$

The purpose of the planner is to maximize time zero welfare of an employed agent subject to

1. budget balance expressed as actuarial fairness, i.e., that the expected discounted value of tax payments equals that of benefits (note that this is not the same as a budget balance in a pay-as-you-go system) and to
2. the constraint that agents voluntarily search.

Without loss of generality, we let individuals pay lump-sum taxes, denoted τ, implying that

$$\dot{A}_t = rA_t + y - c_t - \tau, \tag{26}$$

where $y = w$ if the individual is employed, $y = b - s$, if the individual is unemployed and search and $y = b$ if the individual is unemployed without searching. An individual who searches, finds a job with an exogenous instantaneous probability h and a person with a job loses it with probability q. Define the average discounted probabilities (ADP's) of being unemployed (in state 2) as

$$\Pi_2 \equiv r \int_0^\infty e^{-rt} \mu_{2,t} dt$$

It is straightforward to calculate that

$$\Pi_2 \equiv \frac{q}{r+h+q}.$$

where $\mu_{2,t}$ is the probabilities of being unemployed at time t, respectively, conditional on being employed at time zero, provided that unemployed search for a job.

To see this, note that

$$\mu_{2,t+dt} = qdt\left(1 - \mu_{2,t}\right) + \left(1 - hdt\right)\mu_{2,t}$$

or

$$\mu_{2,t+dt} - \mu_{2,t} = qdt\left(1 - \mu_{2,t}\right) - hdt\mu_{2,t} \tag{27}$$

$$\frac{\mu_{2,t+dt} - \mu_{2,t}}{dt} = q - \left(h + q\right)\mu_{2,t} \tag{28}$$

taking the limit as $dt \to 0$ yields

$$\dot{\mu}_{2,t} = -\left(h+q\right)\mu_{2,t} + q, \tag{29}$$

with root $-\left(h+q\right)$. The steady state is a particular solution, i.e.,

$$\bar{\mu}_2 = \frac{q}{h+q}$$

The solution to the system is then

$$\mu_{2,t} = \left(\mu_{2,0} - \bar{\mu}_2\right) e^{-(h+q)t} + \bar{\mu}_2.$$

Solving for the *ex-ante* case when individuals are born employed ($\mu_{2,0} = 0$) yields $\mu_{2,t} = \bar{\mu}_2 \left(1 - e^{-(h+q)t}\right)$.

Then,

$$\begin{aligned}
\Pi_2 &= r\int_0^\infty e^{-rt}\bar{\mu}_2\left(1 - e^{-(h+q)t}\right)dt \\
&= r\bar{\mu}_2\int_0^\infty e^{-rt}\left(1 - e^{-(h+q)t}\right)dt \\
&= r\frac{q}{h+q}\left(\frac{1}{r} - \frac{1}{r+q+h}\right) = \frac{q}{r+h+q}
\end{aligned}$$

Actuarial fairness the UI system is now a simple linear function of the benefits

$$\tau = \Pi_2 b \tag{30}$$

Under constant absolute risk aversion and stationary income uncertainty, the value functions for the two states $j \in \{1, 2\}$ can be separated

$$V(A_t, j) = W(A_t)\tilde{V}_j(\tau, b), \tag{31}$$

where

$$\begin{aligned} W(A_t) &\equiv \frac{e^{-\gamma A_t}}{r} \\ \tilde{V}_j &\equiv -e^{-\gamma c_j}, \end{aligned} \tag{32}$$

and σ_j are state-dependent consumption constants such that the state dependent consumption functions are

$$c_j(A_t) = rA_t + \sigma_j. \tag{33}$$

The consumption constants σ_j are nonlinear functions of income in all states and thus, depend on the planner choice variables τ, and b. The constants are found as the unique solutions to the Bellman equations for each state:

$$\begin{aligned} \sigma_1 &= w - \tau - \frac{q\left(e^{\gamma\Delta_2} - 1\right)}{\gamma r}, \\ \sigma_2 &= b - s - \tau + \frac{h\left(1 - e^{-\gamma\Delta_2}\right)}{\gamma r}, \end{aligned} \tag{34}$$

where q is the exogenous hiring rate, h is the hiring rate if the agent search actively and

$$\Delta_2 \equiv \sigma_1 - \sigma_2.$$

Let us derive these results;

Conjecturing that the value functions are $-\frac{1}{r}e^{-\gamma(rA_t+\sigma_j)}$, we can write the Bellman equations for the employed as

$$-\frac{1}{r}e^{-\gamma(rA_t+\sigma_1)} = \max_{\sigma} -e^{-\gamma(rA_t+\sigma)}dt - (1-rdt)(1-qdt)\frac{1}{r}e^{-\gamma(rA_{t+dt}+\sigma_1)} \\ - (1-rdt)\,qdt\frac{1}{r}\left[e^{-\gamma(rA_{t+dt}+\sigma_2)}\right].$$

Using the budget constraint, $A_{t+dt} = A_t + r(w-\tau-\sigma)\,dt$, and dividing by $e^{-\gamma r A_t}$, this becomes

$$-\frac{1}{r}e^{-\gamma\sigma_1} = \max_{\sigma} -e^{-\gamma\sigma}dt - (1-rdt)(1-qdt)\frac{1}{r}e^{-\gamma(r(w-\tau-\sigma)dt+\sigma_1)} \\ - (1-rdt)\,qdt\frac{1}{r}\left[e^{-\gamma(r(w-\tau-\sigma)dt+\sigma_2)}\right].$$

Using the first-order linear approximation, $e^{-\gamma(r(w-\tau-\sigma)dt+\sigma_1)} \approx e^{-\gamma\sigma_1} - \gamma r(w-\tau-\sigma)\,dt e^{-\gamma\sigma_1}$, adding $\frac{1}{r}e^{-\gamma\sigma_1}$ to both sides, dividing by dt and letting dt approach zero, yields

$$0 = \max_{\sigma}\left\{-re^{-\gamma(\sigma-\sigma_1)} + r + \gamma r(w-\tau-\sigma)\right\} \\ + q\left(1-e^{-\gamma(\sigma_2-\sigma_1)}\right) \tag{35}$$

Similarly, for the unemployed, we obtain

$$0 = \max_{\sigma}\left\{-re^{-\gamma(\sigma-\sigma_2)} + \gamma r(b_2-s-\tau-\sigma)\right\} \\ + r + h - he^{-\gamma(\sigma_1-\sigma_2)} \tag{36}$$

The right hand sides of (35) and (36) are maximized at $\sigma = \sigma_j$, implying that these values maximize the RHS's of the Bellman equations.

Substituting σ_1 and σ_2 respectively for σ in (35) and (36) solves the maxima. Finally, solving for gives the σ_j^s gives (34), which by construction then solves the Bellman equations.

Clearly, the objective of the planner is now to maximize σ_1, from which also follows time consistency – the welfare of employed at all times is maximized.

The first step is now to derive an expression for σ_1 in terms of Δ_2 where the budget constraint (30) is used to replace the tax rate. For this purpose, we subtract the second line of (34) from the first and solve for b. Then, we use this expression in the budget constraint $\tau = \Pi_2 b$ and substitute for τ in the first line of (34). This yields

$$\sigma_1 = \kappa + \Pi_2 \left(\Delta_2 - \frac{he^{-\gamma \Delta_2}}{\gamma r} \right) - (1 - \Pi_2)\, q \frac{e^{\gamma \Delta_2}}{\gamma r}, \tag{37}$$

where κ is a constant, independent of the choice variables. Straightforward calculus shows that (37) defines σ_1 as a concave function of Δ_2 with a unique maximum at 0. The reason for σ_1 being maximized at $\Delta_2 = 0$ is obvious – when actuarial insurance is available, full insurance maximizes utility. However, $\Delta_2 = 0$ is not incentive compatible. Searching moving will not occur voluntarily. Now, as in Baily approach, we can use the consumption fall upon separation, Δ_2, to evaluate the gain by finding employment.

If the unemployed agent shirks she is unemployed for ever, getting an income $b - \tau$ and a utility

$$-\frac{1}{r} e^{-\gamma r A_t} e^{-\gamma (b - \tau)}.$$

The utility if the individual instead searches is

$$-\frac{1}{r}e^{-\gamma r A_t}e^{-\gamma\sigma_2}.$$

To induce search, we clearly need

$$\sigma_2 \geq b - \tau.$$

Note that the consumption of the unemployed who search is $rA_t + \sigma_2$. Furthermore, her total income net of search costs is $rA_t + b - \tau - s$. Therefore, the search condition implies consumption to be strictly higher than income. Over time, the unemployed depletes her assets and consumption therefore falls, despite the benefits being constant. The celebrated result by Shawell-Weiss and Hopenhayn-Nicolini that consumption should optimally fall over the unemployment spell when the insurer can fully control consumption (no hidden savings) is therefore mimicked in this case, where hidden savings are allowed.

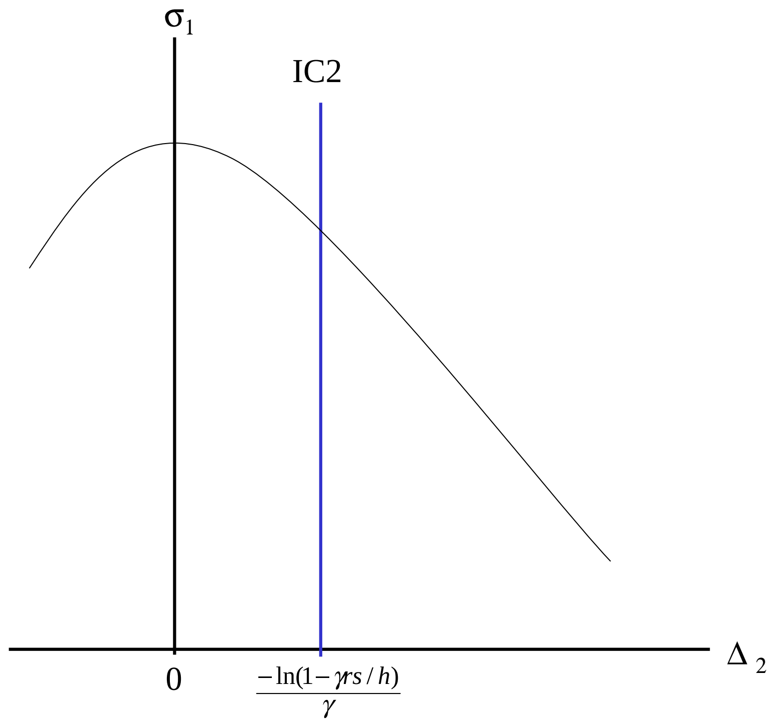
The final part is now to express the search constraint in terms of the consumption difference Δ_2. Using the second line of (34), the search constraint can be written

$$\Delta_2 \geq -\frac{\ln\left(1 - \gamma r \frac{s}{h}\right)}{\gamma}, \tag{38}$$

which we label the *IC2-condition.* We depict this in Figure 2,

A higher r and s and lower h reduce the value of searching, and shifts the constraint to the right.

Figure 1



Finally, we can solve for the value of b that makes the IC2 condition bind exactly. Take the difference between the equations in (34) set it to $-\frac{\ln\left(1-\gamma r\frac{s}{h}\right)}{\gamma}$ and solve for b, which gives

$$-\frac{\ln\left(1-\gamma r\frac{s}{h}\right)}{\gamma} = w-\tau-\frac{q\left(e^{-\ln\left(1-\gamma r\frac{s}{h}\right)}-1\right)}{\gamma r}-\left(b-s-\tau+\frac{h\left(1-e^{\ln\left(1-\gamma r\frac{s}{h}\right)}\right)}{\gamma r}\right)$$

$$\begin{aligned} b &= w+s+\frac{\ln\left(1-\gamma r\frac{s}{h}\right)}{\gamma}-\frac{q\left(-\frac{h}{-h+\gamma rs}-1\right)}{\gamma r}-\frac{h\left(1--\frac{-h+\gamma rs}{h}\right)}{\gamma r} \\ &= w+\frac{\ln\left(1-\gamma r\frac{s}{h}\right)}{\gamma}-\frac{sq}{h-\gamma rs} \end{aligned}$$

In Hassler&Rodriguez (2008), we extend this model and show that it is useful to analyze multiple incentive constraints. It is immediate to show that benefits should optimally be constant over time. This since the incentive constraint does not change over time. We also introduce multiple incentive constraints, showing that if there is also a need to induce some individuals to move to find a job, this is optimally done with an initial period of low benefits.

3 New Public Finance – the Mirrlees approach

3.1 The static Mirrlees model

Consider now a simple two type variant of the model above. Furthermore disregard public good provision. Suppose a share π of the population has high productivity θ_h and the remaining share has productivity $\theta_l \leq \theta_h$. Consider first the first best allocation if the social welfare function is utilitarian

$$\max \pi \left(u\left(c_h\right) + v\left(n_h\right)\right) + \left(1-\pi\right)\left(u\left(c_l\right) + v\left(n_l\right)\right) \tag{39}$$
$$s.t. 0 \leq \pi\theta_h n_h + \left(1-\pi\right)\theta_l n_l - \pi c_h - \left(1-\pi\right) c_l$$

where subscripts denote the type, so c_h, for example, denoted consumption of the high productivity types.

Denoting the shadow value on the resource constraint by λ, we have the first order conditions

$$\pi u'(c_h) - \lambda\pi = 0$$
$$\left(1-\pi\right) u'(c_l) - \left(1-\pi\right)\lambda = 0$$
$$\pi v'(n_h) + \lambda\pi\theta_h = 0$$
$$\left(1-\pi\right) v'(n_l) + \left(1-\pi\right)\lambda\theta_l = 0$$
$$\lambda\left(\pi\theta_h n_h + \left(1-\pi\right)\theta_l n_l - \pi c_h - \left(1-\pi\right) c_l\right) \geq 0$$

Clearly the two first constraints imply that

$$c_h = c_l$$

while the next two implies

$$\frac{v'(n_h)}{v'(n_l)} = \frac{\theta_h}{\theta_l} \geq 1$$

that is the marginal disutility of work is higher for the able individuals, i.e., they work more. Clearly this poses a problem if the planner cannot observe individual productivity and the effort h the individual puts in. *The planner is assumed to only observe income and consumption.*

Furthermore,

$$\theta_h = \frac{-v'(n_h)}{u'(c_h)}$$
$$\theta_l = \frac{-v'(n_l)}{u'(c_l)}$$

with a well-known interpretation.

Consider now the problem of maximizing the utilitarian welfare function subject to the resource constraints and the incentive constraints, i.e., that individuals themselves choose labor supply and savings. A way of finding the second best allocation is to let the planner provide consumption and tell the individual to provide a given amount of income conditional on the ability an individual claims to have. So let's consider a situation where each individual reports her type and the planner then tells her how much income to provide y_i and how much to consume c_i. Let's call the report i_r. The incentive constraint is then that individuals voluntarily report their true ability. According to the *revelation principle,* any incentive compatible allocation can be achieved in this way. Thus we can restrict ourselves to look within the class of allocations that satisfy incentive constraints. Later, we will discuss how to decentralize that, i.e., construct a tax-transfer system such that the optimal incentive-compatible allocation is chosen by the individuals.

The problem is now to solve (39) subject to the truth-telling constraint

$$u(c_i) + v\left(\frac{y_i}{\theta_i}\right) \geq u(c_{i_r}) + v\left(\frac{y_{i_r}}{\theta_i}\right), \forall i_r, i \in \{h, l\}$$

where we have substituted for n by y/θ. Note that we always divide by the true ability. Why?

We will not have both truth-telling constraints binding in the optimal allocation. We conjecture that truth-telling for the more able person binds. Why? Let's call the shadow value on that constraint by λ_I and the resource constraint λ_r. The problem is then

$$\max \pi\left(u(c_h) + v\left(\frac{y_h}{\theta_h}\right)\right) + (1-\pi)\left(u(c_l) + v\left(\frac{y_l}{\theta_l}\right)\right) \tag{40}$$

$$s.t. 0 \leq \pi y_h + (1-\pi) y_l - \pi c_h - (1-\pi) c_l$$

$$0 = u(c_h) + v\left(\frac{y_h}{\theta_h}\right) - u(c_l) - v\left(\frac{y_l}{\theta_h}\right) \tag{41}$$

First order conditions are

$$\pi u'(c_h) - \lambda_r \pi + \lambda_I u'(c_h) = 0$$

$$(1-\pi) u'(c_l) - \lambda_r (1-\pi) - \lambda_I u'(c_l) = 0$$

$$\pi v'\left(\frac{y_h}{\theta_h}\right)\frac{1}{\theta_h} + \pi\lambda_r + \lambda_I v'\left(\frac{y_h}{\theta_h}\right)\frac{1}{\theta_h} = 0$$

$$(1-\pi) v'\left(\frac{y_l}{\theta_l}\right)\frac{1}{\theta_l} + (1-\pi)\lambda_r - \lambda_I v'\left(\frac{y_l}{\theta_h}\right)\frac{1}{\theta_h} = 0$$

These implies

$$\frac{u'(c_h)}{u'(c_l)} = \frac{1-\frac{\lambda_I}{1-\pi}}{1+\frac{\lambda_I}{\pi}}$$

Thus, the higher is the λ_I, the larger is the spread in marginal utilities.

Note also that

$$u'(c_h)\left(1+\frac{\lambda_I}{\pi}\right) = -v'\left(\frac{y_h}{\theta_h}\right)\frac{1}{\theta_h}\left(1+\frac{\lambda_I}{\pi}\right)$$

implying

$$\theta_h = \frac{-v'(n_h)}{u'(c_h)}$$

while

$$-\frac{v'\left(\frac{y_l}{\theta_l}\right)}{u'(c_l)} = \frac{1-\frac{\lambda_I}{1-\pi}}{1-\frac{\lambda_I}{(1-\pi)}\frac{v'\left(\frac{y_l}{\theta_h}\right)}{v'\left(\frac{y_l}{\theta_l}\right)}\frac{\theta_l}{\theta_h}}\theta_l < \theta_l$$

since $1 > \frac{v'\left(\frac{y_l}{\theta_h}\right)}{v'\left(\frac{y_l}{\theta_l}\right)}\frac{\theta_l}{\theta_h}$. Thus the labor leisure choice is distorted for the low ability types but not for the high ability types. The no distortion at the top is a quite general result when the distribution of abilities is bounded.

Take a simple example where $u(c) = \ln c$ and $v(n) = -\frac{n^2}{2}$. Set $\pi = 1/2$ and $\theta_h = 2, \theta_l = 1$.

Then, we have

$$\begin{aligned}
\frac{1}{2}c_h^{-1} - \lambda_r \frac{1}{2} + \lambda_I c_h^{-1} &= 0 \\
\frac{1}{2}c_l^{-1} - \lambda_r \frac{1}{2} - \lambda_I c_l^{-1} &= 0 \\
-\frac{1}{2}n_h \frac{1}{2} + \frac{1}{2}\lambda_r - \lambda_I n_h \frac{1}{2} &= 0 \\
-\frac{1}{2}n_l + \frac{1}{2}\lambda_r + \lambda_I n_l \frac{1}{4} &= 0 \\
2n_h + n_l - c_h - c_l &= 0 \\
\ln c_h - \frac{n_h^2}{2} - \left(\ln (c_l) - \frac{\left(\frac{n_l}{2}\right)^2}{2} \right) &= 0
\end{aligned}$$

The solution is: $n_l = 0.73338, \lambda_r = 0.68609, \lambda_I = 0.12896, c_h = 1.8334, c_l = 1.0816, n_h = 1.0908$

Note that $c_h n_h = 2 = \theta_h$, while $c_l n_l < 1 = \theta_l$.

In first best, we instead have

$$\begin{aligned}
\frac{1}{2}c_h^{-1} - \lambda_r \frac{1}{2} &= 0 \\
\frac{1}{2}c_l^{-1} - \lambda_r \frac{1}{2} &= 0 \\
-\frac{1}{2}n_h \frac{1}{2} + \frac{1}{2}\lambda_r &= 0 \\
-\frac{1}{2}n_l + \frac{1}{2}\lambda_r &= 0 \\
2n_h + n_l - c_h - c_l &= 0
\end{aligned}$$

with the solution is: $\{\lambda_r = 0.632\,46, c_h = 1.\,581\,1, c_l = 1.\,581\,1, n_h = 1.\,264\,9, n_l = 0.632\,46\}$, in which case $c_h n_h = \theta_h$ and $c_l n_l = \theta_l$.

3.1.1 Implementation

In the simple case discussed above, we can implement the allocation with a menu of marginal tax rates and transfers. Since the labor-leisure trade-off is distorted (not distorted) for the low (high) ability individuals, we need a tax on labor for only the low ability type. For the low ability type to accept this, we need to give him a larger lump-sum transfer. Thus, individuals are asked to choose either a positive marginal tax and a high transfer or a zero marginal tax and a smaller transfer (typically negative). Think of the intuition for why this is optimal.

Given that the truth telling constraint is satisfied, individuals solve

$$\max\left(u\left(c_i\right)+v\left(n_i\right)\right)$$

$$s.t. c_i=\theta_i n_i\left(1-\tau_i\right)+T_i$$

Implying

$$\theta_i\left(1-\tau_i\right)=\frac{-v'\left(n_i\right)}{u'\left(c_i\right)}$$

In the example, we then have the two private first-order conditions and two budget constraints.

Plugging in the numbers and solving yields

$$[c_h n_h = \theta_h (1 - \tau_h)]_{n_h=1.0908, c_l=1.081\,6, n_l=0.733\,38, c_h=1.8334, \theta_h=2, \theta_l=1}$$

$$[c_l n_l = \theta_l (1 - \tau_l)]_{n_h=1.0908, c_l=1.081\,6, n_l=0.733\,38, c_h=1.8334, \theta_h=2, \theta_l=1}$$

$$[c_h = \theta_h n_h (1 - \tau_h) + T_h]_{n_h=1.0908, c_l=1.081\,6, n_l=0.733\,38, c_h=1.8334, \theta_h=2, \theta_l=1}$$

$$[c_l = \theta_l n_l (1 - \tau_l) + T_l]_{n_h=1.0908, c_l=1.081\,6, n_l=0.733\,38, c_h=1.8334, \theta_h=2, \theta_l=1}$$

The solution is: $\{T_h = -0.348\,06, T_l = 0.499\,87, \tau_l = 0.206\,78, \tau_h = 0\}$

Finally, we need to check whether it is necessary to add some non-linearities in the tax system. Consider the utility if the high transfer, high marginal tax is chosen by the high ability type. The choice then satisfies

$$[c_h n_h = \theta_h (1 - \tau_l)]_{\theta_h=2, \tau_l=0.20678, T_h=-0.34806, T_l=0.49987}$$

$$[c_h = \theta_h n_h (1 - \tau_l) + T_l]_{\theta_h=2, \tau_l=0.20678, T_h=-0.34806, T_l=0.49987}$$

with the solution $\{c_{hdev} = 1.8559, n_{hdev} = 0.85479\}$. Clearly, this gives higher utility and we need to prevent this deviation. This can be done by having another bracket in the tax system. The following tax system could then implement the optimal second-best allocation. The individuals choose from the following menu;

1. A lump sum tax $-T_h = 0.348$. No marginal income tax.
2. A lump sum transfer $T_l = 0.500$. A marginal income tax of $\tau_l = 20.7\%$ up to income $n_l = 0.733$. Above that, a sufficiently high tax rate to deter any benefit claimant to

earn more, e.g., 100%.

3.2 Uniform commodity taxation

An important assumption in the previous subsection was that there is just one good. In reality, there are many goods, both intermediaries and final goods. Then, a key issue becomes; Should different goods be taxed at different rates, i.e., should we use differentiated VAT's? If not, we have seen that it does not matter whether we use a flat consumption tax or a proportional income tax.

One of the most celebrated results in public finance is the Atkinson-Stiglitz *uniform commodity taxation result* (Atkinson & Stiglitz, 1972). This states that under some conditions, most importantly that utility is separable in leisure and an aggregate of market consumption goods, a uniform tax rate should be used. Then, it can, as we have discussed above be replaced by a uniform tax rate on labor income. Loosely speaking, separability means that utility can be written as a function of a consumption aggregate $g(c)$,where c is a vector $[c_1, ..., c_n]$ of consumption goods bought in the market, and labor n (equivalently, leisure) Thus

$$\bar{u}\left(c_1, ...c_n, l\right) = u\left(g\left(c\right), n\right).$$

As above, productivity is unobserved by the planner and he only observes total income, not wages. Due to separability, we can separate the consumers problem in two steps. The last is to maximize $g\left(c\right)$ over the different consumption goods, given disposable income ω and the prices q_i (including taxes).

$$\max_{c} g\left(c_1, ..., c_n\right) \tag{42}$$

$$s.t. \sum_{i} q_i c_i \leq \omega$$

This generates demand functions $d_i(q, \omega)$ and an associated value function $h(q, \omega) \equiv g(d(q, \omega))$. The latter function h, can be thought of has the optimal consumption aggregate, given prices and income.

The first step is then to choose labor supply by solving

$$\max_{y} u\left(h\left(q, \omega(y)\right), \frac{y}{\theta}\right),$$

where $\omega(y)$ i disposable income given gross income y.

Let's follow Boadway and Pestieau (2002) and consider the case were there are two types, $i \in \{h, l\}$ with different planner unobserved productivities (wages), $\theta_h > \theta_l$. We assume that there are two consumption goods, c_1 and c_2 and normalize their relative market price before taxes to unity. Without loss of generality, we assume the policy instrument in terms of consumption taxes is the tax on good 2 and set the other consumption tax to zero. This is w.o.l.g. since a common tax is equivalent to a labor income tax. The price on good 2 faced by consumers is $1 + \tau \equiv q$ implying that the budget constraint of the agent of type i is

$$\omega_i = c_1 + qc_2.$$

The second step problem (42) can now be written

$$h(q,\omega) = \max_{c_2} g(\omega - qc_2, c_2) \tag{43}$$

giving

$$\frac{g_2}{g_1} = q$$

and using the envelope theorem, we have

$$h_\omega = g_1, \tag{44}$$

$$h_q = -g_1 c_2 = -h_w c_2$$

We can now write the planner Lagrangian

$$L = \sum_{i=h,l} \pi_i u\left(h(q,\omega_i), \frac{y_i}{\theta_i}\right) + \lambda_r \sum_{i=h,l} \pi_i \left(y_i + \tau c_2^i - \omega_i\right)$$
$$+ \lambda_I \left(u\left(h(q,\omega_h), \frac{y_h}{\theta_h}\right) - u\left(\left(h(q,\omega_l), \frac{y_l}{\theta_h}\right)\right)\right)$$

The first constraint is the budget constraint of the government and the second is the incentive constraint. We conjecture as above that the high productivity type must be induced not to falsely report that he is a low productivity type.

Now, we focus on the FOC for the disposable incomes ω_i and the the consumer price q. To not have to write out the arguments of all functions, we use superscript on functions to denote type and hat's on functions denote for an h type who pretends to be of type l. We then get

$$\omega_l;\pi_l u_h^l h_\omega^l - \lambda_r \pi_l \left(1-\tau\frac{\partial c_2^l}{\partial \omega_l}\right) - \lambda_I \hat{u}_h^h \hat{h}_\omega^h = 0$$
$$\omega_h;\pi_h u_h^h h_\omega^h - \lambda_r \pi_h \left(1-\tau\frac{\partial c_2^h}{\partial \omega_h}\right) + \lambda_I u_h^h h_\omega^h = 0$$
$$q;\sum_{i=h,l}\pi_i u_h^i h_q^i + \lambda_r \sum_{i=h,l}\pi_i \left(c_2^i+\tau\frac{\partial c_2^i}{\partial q}\right) + \lambda_I \left(u_h^h h_q^h - \hat{u}_h^h \hat{h}_q^h\right) = 0$$

Now, multiply the first equation by c_2^l and the second by c_2^h and use (44). Giving

$$\omega_l;-\pi_l u_h^l h_q^l - \lambda_r \pi_l \left(1-\tau\frac{\partial c_2^l}{\partial \omega_l}\right) c_2^l - \lambda_I \hat{u}_h^h \hat{h}_\omega^h c_2^l = 0$$
$$\omega_h;-\pi_h u_h^h h_q^h - \lambda_r \pi_h \left(1-\tau\frac{\partial c_2^h}{\partial \omega_h}\right) c_2^h - \lambda_I u_h^h h_q^h = 0$$

Add these two to the FOC for q; This gives

$$\lambda_r \tau \sum_{i=h,l}\pi_i \left(\frac{\partial c_2^i}{\partial q}+\frac{\partial c_2^i}{\partial \omega_i}c_2^i\right) - \lambda_I \hat{u}_h^h \left(\hat{h}_\omega^h c_2^l + \hat{h}_q^h\right) = 0.$$

Now, consider the parenthesis in the second term, $\hat{h}_\omega^h c_2^l + \hat{h}_q^h$. Spelling out the arguments, we write this

$$h_\omega\left(q,\omega_l\right)c_2^l + h_q\left(\left(q,\omega_l\right)\right).$$

From (44) we know this is zero. Recall that this term comes from the cheating high productivity types, but since he consumes as much of good 2 as the the low productivity types, the same envelope condition holds. This would not be the case if also leisure entered in this expression, since the two types consume different amounts of leisure. We thus end up with

$$\lambda_r \tau \sum_{i=h,l} \pi_i \left(\frac{\partial c_2^i}{\partial q} + \frac{\partial c_2^i}{\partial \omega_i} c_2^i \right) = 0$$

Note that $\frac{\partial c_2^i}{\partial q} + \frac{\partial c_2^i}{\partial \omega_i} c_2^i$ is the derivative of the compensated demand function for c_2, i.e., the effect on demand of a marginal increase in the price dq together with an income transfer of dqc_2. Provided this is not zero, the tax must be zero.

The intuition for the result is that the planner wants to distort only margins that can help him identify the low productivity individuals (equivalently, the cheaters). If the marginal rate of substitution is the same for low and high productivity individuals for some pair of goods, there is no point in distorting it. One can, of course think of cases where this is not the case. For example, a cheating high productivity individual consumes a lot of leisure. Suppose there is one good that is a complement to leisure, like vacation trips. Such a good should then be taxed higher because it reduces the value of cheating for the high productivity individual.

A related result to the A-S is the Diamond-Mirrlees production efficiency result (Diamond & Mirrlees, 1972). This result states that production, in the sense the use of different inputs in production, should not be distorted. This result builds on a similar separability. If consumers care of the final product, not of how it is produced, distorting production cannot help the planner doing anything good.

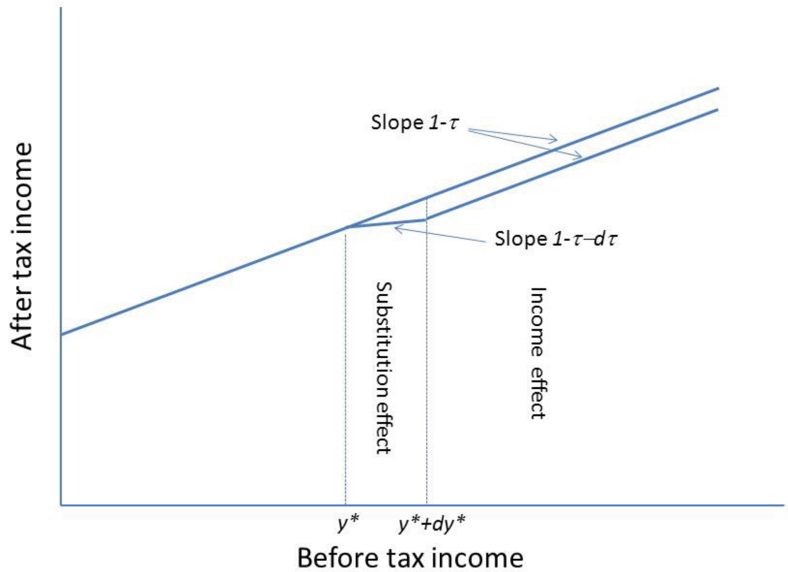
3.3 The direct approach

An alternative to the Mirrleesian approach is to work directly with the tax system and derive optimal properties of that. Saez (2001) show that this can be done using observed characteristics as labor supply elasticities and the actual income distribution. To understand the intuition behind the fairly complicated formulas, consider a tax system $T(y)$ where y is gross income and $T(y)$ is the tax payment. Define $\tau(y) \equiv T'(y)$ and let $H(z)$ be the share of individuals with income at or below z, with a density denoted $h(z)$.

Consider the effects of a small increase in the marginal tax rate $d\tau$ over the small interval y^* to $y^* + dy^*$. This change is illustrated in the figure below. Clearly, individuals with income below y^* are not affected by the change. Individuals in the interval $[y^*, y^* + dy^*]$ face a change in their marginal tax τ, but the average tax is not changed. Thus, there is only a substitution effect and the change in labor supply depends on the *compensated* income elasticity. Thus, an increase in that tax rate reduces labor supply. This is a negative effect seen from the point of view of a benevolent planner and the importance of it depends on the density of individuals $h(y^*)$.

Above $y^* + dy^*$, the marginal income tax rate τ is unchanged but the average income is increased by $dy^* d\tau$. This has a mechanic direct effect on revenues and an endogenous on labor supply that depends on the income elasticity of labor supply. Assuming leisure is a normal good, the higher tax increases labor supply. Provided the value of government revenue is higher than the value of private spending for individuals with income above y^*,both these effects are positive for the planner. The strength of them depends positively (loosely speaking) on total income above $y^* + dy^*$ and therefore on $(1 - H(y^* + dy^*)$.

We have now defined positive and a negative effects of increasing the slope at y^*. If $T(y)$ is optimal, these effects should balance each other exactly. Furthermore, this should be true at all income levels y. Letting $dy^* \to 0$, this then defines a differential equation that must



be satisfied. Together with, e.g., a financing requirement or any other condition that pins down the total tax requirements, this defines the optimal tax. We note that the marginal tax $\tau(y)$, tends to be high;

- if the compensated elasticity at y is low,
- if $h(y)$ is low,
- if total income above y, i.e., $\int_z^\infty yh(y)\,dz$, is high.
- if income elasticity above y is high.
- if the planner's value of money is high relative to the value the planner attach to marginal income of individuals with income above y.

Let us now consider the dynamic Mirrlees approach to optimal taxation. Here, individuals are assumed to be different. These differences can be either in their productivity or in their value of leisure. Such differences imply that there is differences between individuals in their trade-off between leisure and work. It is assumed that the government cannot directly observe this differences, only observe the individuals market choices. For example, governments observe income, but not the effort exerted to get this income.

Consider a simple two-period example.

Individual preferences are:

$$E\left(u\left(c_1\right)+v\left(n_1\right)+\beta\left(u\left(c_2\right)+v\left(n_2\right)\right)\right)$$

where c_t is consumption and n_t is labor supply/work effort. u is increasing and concave and v decreasing and concave. Individuals differ in their ability, denoted θ. It is assumed that there is a finite number $i \in \{1, 2, ..., N\}$ of ability levels and ability might change over time. We will interchangeably use type and ability to denote θ. Output is produced in competitive firms using a linear technology where each individual i produces

$$y_t\left(i\right)=\theta\left(i\right)n_t\left(i\right).$$

There is a continuum of individuals of a unitary total mass. In the first period, individuals are given abilities by nature according to a probability function $\pi_1\left(i\right)$. The ability can then change to the second period. Second period ability is denoted $\theta\left(i, j\right)$ and the transition probability is $\pi_2\left(j|i\right)$.

There is a storage technology with return R. Finally, the government needs to finance some spendings G_1 and G_2. At first, we analyze the case of no aggregate uncertainty.

The aggregate resource constraint is

$$\sum_i \left(y_1(i) - c_1(i) + \sum_j \frac{y_2(i,j) - c_2(i,j)}{R} \pi_2(j|i) \right) \pi_1(i) + K_1 = G_1 + \frac{G_2}{R} \tag{45}$$

where K_1 is an aggregate initial endowment.

The problem is now to maximize the utilitarian welfare function subject to the resource constraints and the incentive constraints, i.e., that individuals themselves choose labor supply and savings. A way of finding the second best allocation is to let the planner provide consumption and work conditional on the ability an individual claims to have (and if relevant, the aggregate state). Here this is in the first period $c_1(i), y_1(i)$ and in the second, $c_2(i,j), y_1(i,j)$. Individuals then report their abilities to the planner. The strategy of an individual is his first period report and then a reporting plan as a function of the realized period 2 ability. Let's call the report i_r and $j_r(j)$, where the latter is the report as a function of the true ability. The incentive constraint is then that individuals voluntarily report their true ability. According to the *revelation principle,* this always yields the best incentive compatible allocation. The *truth-telling* constraint is then that

$$\begin{aligned} &u(c_1(i)) + v\left(\frac{y_1(i)}{\theta_1(i)}\right) + \beta \sum_j \left(u(c_2(i,j)) + v\left(\frac{y_2(i,j)}{\theta_2(i,j)}\right)\right) \pi_2(j|i) \\ &\geq u(c_1(i_r)) + v\left(\frac{y_1(i_r)}{\theta_1(i)}\right) + \beta \sum_j \left(u(c_2(i_r, j_r(j))) + v\left(\frac{y_2(i_r, j_r(j))}{\theta_2(i,j)}\right)\right) \pi_2(j|i) \end{aligned} \tag{46}$$

for any possible reporting strategy $i_r, j_r(j)$. Note that the θ_s are the true ones in both sides of the inequality. Note also that *truth-telling* implies that

$$u\left(c_2\left(i,j\right)\right)+v\left(\frac{y_2\left(i,j\right)}{\theta_2\left(i,j\right)}\right)\geq u\left(c_2\left(i_r,j_r\left(j\right)\right)\right)+v\left(\frac{y_2\left(i_r,j_r\left(j\right)\right)}{\theta_2\left(i,j\right)}\right)\forall j, \tag{47}$$

otherwise utility could be increased by reporting j_r if the second period ability is j. The planning problem is to maximize

$$\sum_i\left(u\left(c_1\left(i\right)\right)+v\left(\frac{y_1\left(i\right)}{\theta_1\left(i\right)}\right)+\beta\sum_j\left(u\left(c_2\left(i,j\right)\right)+v\left(\frac{y_2\left(i,j\right)}{\theta_2\left(i,j\right)}\right)\right)\pi_2\left(j|i\right)\right)\pi\left(i\right)$$

subject to (45) and (46).

Letting stars * denote optimal allocations. We can now define three wedges (distortions) that the informational friction may cause. These are the consumption-leisure (intratemporal) wedges

$$\tau_{y_1}\left(i\right)\equiv 1+\frac{v'\left(\frac{y_1^*(i)}{\theta_1(i)}\right)}{\theta_1\left(i\right)u'\left(c_1^*\left(i\right)\right)},$$
$$\tau_{y_2}\left(i,j\right)\equiv 1+\frac{v'\left(\frac{y_2^*(i,j)}{\theta_2(i,j)}\right)}{\theta_2\left(i,j\right)u'\left(c_2^*\left(i,j\right)\right)},$$

and the intertemporal wedge

$$\tau_k\left(i\right)\equiv 1-\frac{u'\left(c_1^*\left(i\right)\right)}{\sum_j\beta R u'\left(c_2\left(i,j\right)\right)\pi_2\left(j|i\right)}.$$

Clearly, in absence of government interventions, these wedges would be zero by perfect competition and the first-order conditions of private optimization.

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3.4 The inverse Euler equation

We will now show that if individual productivities are not always constant over time, the intertemporal wedge will not be zero. The logic is as follows and similar to what we have done above. In an optimal allocation, the resource cost (expected present value of consumption) of providing the equilibrium utility to each type, must be minimized. Consider the following perturbation around the optimal allocation for a given first period ability type i. Increase utility by a marginal amount Δ for all possible second period types $\{i, j\}$ the agent could become. To compensate, decrease utility by $\beta\Delta$ in the first period.

First, note that expected utility is not changed.

Second, since utility is changed in parallel for all ability levels the individual could have in the second period, their relative ranking cannot change. In other words, if we add Δ to both sides of (47) it must still be satisfied.

Thus, the incentive constraint is unchanged. However, the resource constraint is not necessarily invariant to this peturbation. Let

$$\tilde{c}_1(i;\Delta) = u^{-1}\left(u\left(c_1^*(i)\right) - \beta\Delta\right),$$

$$\tilde{c}_2(i,j;\Delta) = u^{-1}\left(u\left(c_2^*(i,j)\right) + \Delta\right)$$

denote the perturbed consumption levels. The resource expected resource cost of these are

$$\tilde{c}_1(i;\Delta) + \sum_j \frac{1}{R}\tilde{c}_2(i,j;\Delta)\,\pi_2(j|i)$$
$$= u^{-1}\left(u\left(c_1^*(i)\right) - \beta\Delta\right) + \sum_j \frac{1}{R} u^{-1}\left(u\left(c_2^*(i,j)\right) + \Delta\right)\pi_2(j|i).$$

The first-order condition for minimizing the resource cost over Δ must be satisfied at $\Delta = 0$, for the * consumption levels to be optimal. Recall that the inverse function theorem says that if

$$u = u(c) \text{ and}$$
$$c = u^{-1}(u),$$

then

$$\frac{\partial u^{-1}(u)}{\partial u} = \frac{1}{u'(c)} = \frac{1}{u'\left(u^{-1}(u)\right)}.$$

Thus,

$$0 =$$
$$= \frac{-\beta}{u'\left(c_1^*(i)\right)} + \sum_j \frac{1}{R}\frac{1}{u'\left(c_2^*(i,j)\right)}\pi_2(j|i)$$
$$\Rightarrow \frac{1}{u'\left(c_1^*(i)\right)} = E_1 \frac{1}{\beta R u'\left(c_2^*(i,.)\right)},$$

which we note is an example of the *inverse Euler equation.*

From Jensen's inequality, we find that

$$u'\left(c_1^*(i)\right) < E\beta R u'\left(c_2^*(i,.)\right)$$

$$\Rightarrow \tau_k(i) > 0,$$

if and only if there is some uncertainty in c_2^*. Note that this uncertainty would come from second period ability being random and the allocation implying that second period consumption depends on the realization of ability. If second period ability is non-random, i.e., $\pi_2(j|i) = 1$ for some j, then $\tau_k(i) = 0$.

3.5 A simple logarithmic example: insurance against low ability.

Suppose in the first period, ability is unity and in the second $\theta > 1$ or $\frac{1}{\theta}$ with equal probability.Disregard government consumption – set $G_1 = G_2 = 0$, although non-zero spending is quite easily handled. The problem is therefore to provide a good insurance against a low-ability shock when this is not observed.

The first best allocation is the solution to

$$\max_{c_1, y_1, c_h, c_l, y_h, y_l} u(c_1) + v(y_1) + \beta\left(\frac{u(c_h) + v\left(\frac{y_h}{\theta}\right)}{2} + \frac{u(c_l) + v\left(\frac{y_l}{\frac{1}{\theta}}\right)}{2}\right)$$

$$s.t. 0 = y_1 + \frac{y_h + y_l}{2R} - c_1 - \frac{c_h + c_l}{2R}$$

First order conditions are

$$u'(c_1) = \lambda, v'(y_1) = -\lambda$$
$$\beta u'(c_h) = \frac{\lambda}{R}, \beta u'(c_l) = \frac{\lambda}{R}$$
$$\beta v'\left(\frac{y_h}{\theta}\right)\frac{1}{\theta} = -\frac{\lambda}{R}, \beta v'(\theta y_l)\,\theta = -\frac{\lambda}{R}$$

3.5.1 A simple example

Suppose for example that $u(c) = \ln(c)$ and $v(n) = -\frac{n^2}{2}$ and $\beta = R = 1$. Then, we get

$$\frac{1}{c_1} = \lambda, \frac{1}{c_h} = \lambda$$
$$\frac{1}{c_l} = \lambda, y_1 = \lambda$$
$$\frac{y_h}{\theta^2} = \lambda, y_l\theta^2 = \lambda$$
$$c_1 + \frac{c_h + c_l}{2} - y_1 - \frac{y_h + y_l}{2} = 0$$

We see immediately that $c_1 = c_h = c_l$ while $y_h = \theta^2 y_1$ and $y_l = \frac{y_1}{\theta^2}$ and $y_1 = \sqrt{\frac{2}{\left(1+\frac{1}{2}(\theta^2+\theta^{-2})\right)}} = n_1$. Therefore, $n_h = \frac{y_h}{\theta} = \theta n_1$ and $n_l = y_l\theta = \frac{n_1}{\theta}$. Thus, if the individual becomes of high ability in the second period, he should work more but don't get any higher consumption. Is this incentive compatible?

We conjecture that the binding incentive constraint is for the high ability type. High has to be given sufficient consumption to make him voluntarily choose not to report being low ability. If he misreports, he gets c_l and is asked to produce y_l. The constraint is therefore

$$u(c_1)+v(y_1)+\beta\left(\frac{u(c_h)+v\left(\frac{y_h}{\theta}\right)}{2}+\frac{u(c_l)+v(\theta y_l)}{2}\right)$$
$$\geq u(c_1)+v(y_1)+\beta\left(\frac{u(c_l)+v\left(\frac{y_l}{\theta}\right)}{2}+\frac{u(c_l)+v(\theta y_l)}{2}\right)$$

$$u(c_h)+v\left(\frac{y_h}{\theta}\right)\geq u(c_l)+v\left(\frac{y_l}{\theta}\right)$$
$$\ln c_h-\ln c_l\geq\frac{y_h^2-y_l^2}{2\theta^2}$$

We conjecture this is binding. The problem is then

$$\max_{c_1,y_1,c_h,c_l,y_h,y_l}\ln(c_1)-\frac{y_1^2}{2}+\left(\frac{\ln c_h-\frac{\left(\frac{y_h}{\theta}\right)^2}{2}}{2}+\frac{\ln c_l-\frac{(\theta y_l)^2}{2}}{2}\right)$$
$$s.t. 0=y_1+\frac{y_h+y_l}{2}-c_1-\frac{c_h+c_l}{2}$$
$$0=\ln c_h-\ln c_l-\frac{y_h^2-y_l^2}{2\theta^2}.$$

Denoting the shadow values by λ_r and λ_I the FOCs for the consumption levels are

$$c_1=\frac{1}{\lambda_r}$$
$$c_h=\frac{1+2\lambda_I}{\lambda_r}$$
$$c_l=\frac{1-2\lambda_I}{\lambda_r}$$

from which we see

$$\frac{c_h^*}{c_1^*} = 1 + 2\lambda_I, \frac{c_l^*}{c_1^*} = 1 - 2\lambda_I$$

implying a positive intertemporal wedge if the IC constraint binds.

The intratemporal wedges are found by analyzing the FOC's for the labor supplies

$$\begin{aligned} y_1^* &= \lambda_r \\ y_h^* &= \frac{\lambda_r}{1 + 2\lambda_I}\theta^2 \\ y_l^* &= \frac{\lambda_r}{\theta^4 - 2\lambda_I}\theta^2 \end{aligned}$$

$$\begin{aligned} \tau_{y_1} &= 1 + \frac{v'\left(y_1^*\right)}{u'\left(c_1^*\right)} = 1 - \frac{y_1^*}{\frac{1}{c_1^*}} = 1 - \frac{\lambda_r}{\frac{1}{\frac{1}{\lambda_r}}} = 0, \\ \tau_{y_2}\left(h\right) &= 1 + \frac{v'\left(\frac{y_h^*}{\theta}\right)}{\theta u'\left(c_h^*\right)} = 1 + \frac{-\frac{y_h^*}{\theta}}{\theta \frac{1}{c_h^*}} \\ &= 1 + \frac{-\frac{\frac{\lambda_r}{1+2\lambda_I}\theta^2}{\theta}}{\theta \frac{1}{\frac{1+2\lambda_I}{\lambda_r}}} = 0 \end{aligned}$$

and

$$
\begin{aligned}
\tau_{y_2}(l) &= 1 + \frac{v'\left(\theta y_l^*\right)}{\frac{1}{\theta} u'\left(c_l^*\right)} = 1 + \frac{-\theta y_l^*}{\frac{1}{\theta}\frac{1}{c_h^*}} \\
&= 1 + \frac{-\theta \frac{\lambda_r}{\theta^4 - 2\lambda_I}\theta^2}{\frac{1}{\theta}\frac{1}{\frac{1-2\lambda_I}{\lambda_r}}} = 2\lambda_I \frac{\theta^4 - 1}{\theta^4 - 2\lambda_I} > 0
\end{aligned}
$$

As we see, the wedge for the high ability types is zero, but positive for the low ability type.[11] For later use, we note that

$$
\begin{aligned}
y_1^* c_1^* &= 1 \qquad (48) \\
y_h^* c_h^* &= \frac{\lambda_r}{1 + 2\lambda_I}\theta^2 \frac{1 + 2\lambda_I}{\lambda_r} = \theta^2 \\
y_l^* c_l^* &= \frac{\lambda_r}{\theta^4 - 2\lambda_I}\theta^2 \frac{1 - 2\lambda_I}{\lambda_r} = \frac{1 - 2\lambda_I}{\theta^2\left(1 - 2\lambda_I \theta^{-4}\right)}
\end{aligned}
$$

Before going to the implementation, note that if we eliminate the shadow value on the resource constraint, we have 7 equations and seven unknowns; Geting rid of the shadow

[11] The wedge, asymptotes to infinity as λ_I approach $\frac{\theta^4}{2}$. Can you explain?

value on resources, we have 7 conditions and 7 unknowns

$$
\begin{aligned}
c_1 &= \frac{1}{y_1}, c_h = \frac{1+2\lambda_I}{y_1} \\
c_l &= \frac{1-2\lambda_I}{y_1}, y_h = \frac{y_1}{1+2\lambda_I}\theta^2 \\
y_l &= \frac{y_1}{\theta^4 - 2\lambda_I}\theta^2, \\
0 &= y_1 + \frac{y_h + y_l}{2} - c_1 - \frac{c_h + c_l}{2} \\
0 &= \ln c_h - \ln c_l - \frac{y_h^2 - y_l^2}{2\theta^2}
\end{aligned}
$$

This does not have a nice closed form solution. However, setting $\theta = 1.1$,I numerically found the solution as $c_1 = 0.998\,71, y_1 = 1.001\,3, y_h = 1.108\,9, y_l = 0.88337, c_h = 1.091\,2, c_l = 0.906\,26, \lambda_I = 4.628\,6 \times 10^{-2}$.

As we see, high ability types consume more than low ability types. However, the former consumes less than their income and the latter more, i.e., there is redistribution.

3.6 Implementation

It is tempting to interpret the wedges as taxes and subsidies. However, this is not entirely correct since the wedges in general are functions of all taxes. Furthermore, while there is typically a unique set of wedges this is generically not true for the taxes. As we have discussed above, many different tax systems might implement the optimal allocation. One example is the draconian, use 100% taxation for every choice except the optimal ones.

Only by putting additional restrictions is the implementing tax system found. Let us consider a combination if linear labor taxes and savings taxes that together with type specific transfers implement the allocation in the example. To do this, consider the individual problem,

$$\max_{c_1,y_1,s,y_h,y_l,c_h,c_l} \ln(c_1) - \frac{y_1^2}{2} + \left(\frac{\ln c_h - \frac{\left(\frac{y_h}{\theta}\right)^2}{2}}{2} + \frac{\ln c_l - \frac{(\theta y_l)^2}{2}}{2} \right)$$

$$s.t. 0 = y_1(1-\tau_1) - c_1 - s + T$$

$$0 = y_h(1-\tau_h) + s(1-\tau_{s,h}) - c_h + T_h$$

$$0 = y_l(1-\tau_h) + s(1-\tau_{s,l}) - c_l + T_l$$

with Lagrange multipliers λ_1, λ_h and λ_r.

First order conditions for the individuals are;

$$\begin{aligned} \frac{1}{c_1} &= \lambda_1, y_1 = \lambda_1(1-\tau_1) \\ \lambda_1 &= \lambda_h(1-\tau_{s,h}) + \lambda_l(1-\tau_{s,l}) \\ \frac{y_h}{2\theta^2} &= \lambda_h(1-\tau_h), \frac{\theta^2 y_l}{2} = \lambda_l(1-\tau_l) \\ \frac{1}{2c_h} &= \lambda_h, \frac{1}{2c_l} = \lambda_l \end{aligned} \tag{49}$$

Using this, we see that

$$\frac{1}{c_1} = \frac{1}{2c_h}(1-\tau_{s,h}) + \frac{1}{2c_l}(1-\tau_{s,l})$$

Setting,

$$\tau_{s,h} = -2\lambda_I$$

$$\tau_{s,l} = 2\lambda_I.$$

this gives

$$\frac{1}{c_1} = \frac{1}{2c_h}(1 + 2\lambda_I) + \frac{1}{2c_l}(1 - 2\lambda_I)$$

which is satisfied if we plug in the optimal allocation $c_h^* = c_1^*(1 + 2\lambda_I)$ and $c_l^* = c_1^*(1 - 2\lambda_I)$

$$\frac{1}{c_1^*} = \frac{1 + 2\lambda_I}{2c_1^*(1 + 2\lambda_I)} + \frac{1 - 2\lambda_I}{2c_1^* 1 - 2\lambda_I}$$

Note that the expected capital income tax rate is zero, but it will make savings lower than without any taxes. Why?

Similarly, by noting from (48) that in the optimal second best allocation, we want

$$y_1 c_1 = y_1^* c_1^* = 1,$$

which is implemented by $\tau_1 = 0$. For the high ability type, the second best allocation in (48) is that $y_h^* c_h^* = \theta^2$, which is implemented by $\tau_h = 0$ since (51) implies that $y_h c_h = \theta^2(1 - \tau_h)$.

For the low ability type, we want $y_l^* c_l^* = \frac{1-2\lambda_I}{\theta^2(1-2\lambda_I\theta^{-4})}$. From (51), we know $y_l c_l = \frac{1-\tau_l}{\theta^2}$,so we solve

$$\frac{1-\tau_l}{\theta^2} = \frac{1-2\lambda_I}{\theta^2\left(1-2\lambda_I\theta^{-4}\right)}$$
$$\Rightarrow \tau_l = 2\lambda_I \frac{\theta^4-1}{\theta^4-2\lambda_I}.$$

Note that if $\lambda_I = \frac{1}{2}$, $\tau_l = 1$. I.e., the tax rate is 100%. There is no point going higher than that, so λ_I cannot be higher than $\frac{1}{2}$.

Finally, to find the complete allocation, we use the budget constraints. We do not need to use any transfers in the first period. Thus

$$T_h = c_h - y_h - (y_1 - c_1)(1-\tau_{s,h})$$
$$T_l = c_l - y_l - (y_1 - c_1)(1-\tau_{s,l})$$

We should note that $T_l > T_h$ is consistent with incentive compatibility. Why? Because if you claim to be a low ability type you will have to may a high labor income tax which is bad if you are high ability and earn a high income. Thus, by taxing high income lower, we can have a transfer system that transfers more to the low ability types.

To find expressions for the transfers I need to use numerical methods. Using the results for $\theta = 1.1$, we have

$$T_h = 1.0912 - 1.1089 - (1.0013 - 0.99871)\left(1 + 2 * 4.6286 \times 10^{-2}\right)$$
$$= -0.0205$$
$$T_l = 0.90626 - 0.88337 - (1.0013 - 0.99871)\left(1 - 2 * 4.6286 \times 10^{-2}\right)$$
$$= 0.0205$$

3.6.1 Third best – laissez faire.

The allocation in without any government involvements is easily found by setting all taxes to zero.

$$\begin{aligned} \frac{1}{c_1} &= \lambda_1, y_1 = \lambda_1 \\ \lambda_1 &= \lambda_h + \lambda_l \\ \frac{y_h}{2\theta^2} &= \lambda_h, \frac{\theta^2 y_l}{2} = \lambda_l \\ \frac{1}{2c_h} &= \lambda_h, \frac{1}{2c_l} = \lambda_l \end{aligned} \tag{50}$$

Using these and the budget constraints, we get

$$\begin{aligned} y_1 &= \frac{1}{c_1} \\ \frac{1}{c_1} &= \frac{1}{2c_h} + \frac{1}{2c_l} \\ \frac{y_h}{2\theta^2} &= \frac{1}{2c_h} \\ \frac{\theta^2 y_l}{2} &= \frac{1}{2c_l} \end{aligned}$$

$$y_1 = c_1 + s$$

$$y_h + s = c_h$$

$$y_l + s = c_l$$

which implies

$$\begin{aligned} c_1 + s &= \frac{1}{c_1} \\ \frac{1}{c_1} &= \frac{1}{2c_h} + \frac{1}{2c_l} \\ c_h &= \frac{1}{2}s + \frac{1}{2}\sqrt{s^2 + 4\theta^2} \\ c_l &= \frac{\frac{1}{2}s\theta + \frac{1}{2}\sqrt{s^2\theta^2 + 4}}{\theta} \end{aligned}$$

I did not find an analytical solution to this, but setting $\theta = 1.1$ I found the solution $c_1 = 0.997\,75, c_h = 1.102\,3, s = 4.\,504\,5 \times 10^{-3}, c_l = 0.911\,35,\ y_1 = 1.0023, y_h = 1.106\,8, y_l = 0.915\,85$.

As we see, consumption is lower in the first period and labor supply is higher than in second best. Consumption of high ability types is higher and labor supply lower than in second best. For low ability types, consumption is actually higher in *laissez faire* but also labor supply. The second period welfare of low ability types is higher in second best (-0.285 vs. $-0.300\,15$).

3.6.2 Means tested system

Suppose now we want to implement the optimal allocation without a savings-tax but using an asset tested disability transfer instead. That is we set

$$T_l = \begin{cases} T_l \text{ if } s \leq \bar{s} \\ -\bar{T} \text{ else.} \end{cases}$$

where $\bar{T}$ is sufficiently large to deter savings above $\bar{s}$. We set $\bar{s}$ equal to the first best $y_1^* - c_1^*$. Without a savings tax, the cap on savings will clearly bind due to the inverse Euler equation. The problem of the individual is therefore

$$\max_{c_1, y_1, s, y_h, y_l, c_h, c_l} \ln(c_1) - \frac{y_1^2}{2} + \left(\frac{\ln c_h - \frac{\left(\frac{y_h}{\theta}\right)^2}{2}}{2} + \frac{\ln c_l - \frac{(\theta y_l)^2}{2}}{2} \right)$$

$$s.t. 0 = y_1 (1 - \tau_1) - c_1 - \bar{s} + T$$

$$0 = y_h (1 - \tau_h) + \bar{s} - c_h + T_h$$

$$0 = y_l (1 - \tau_l) + \bar{s} - c_l + T_l$$

First order conditions for the individuals are;

$$\begin{aligned}
c_1; \frac{1}{c_1} &= \lambda_1 \\
y_1; y_1 &= \lambda_1 (1 - \tau_1) \\
y_h; \frac{y_h}{2\theta^2} &= \lambda_h (1 - \tau_h) \\
y_l; \frac{\theta^2 y_l}{2} &= \lambda_l (1 - \tau_l) \\
c_h; \frac{1}{2c_h} &= \lambda_h \\
c_l; \frac{1}{2c_l} &= \lambda_l
\end{aligned} \tag{51}$$

giving

$$1 - \tau_1 = c_1 y_1 \tag{52}$$

$$\begin{aligned}
\theta^2 (1 - \tau_h) &= c_h y_h \\
\frac{(1 - \tau_l)}{\theta^2} &= c_l y_l
\end{aligned} \tag{53}$$

We want

$$1 = c_1 y_1 \Rightarrow \tau_1 = 0.$$

We also want

$$\begin{aligned}
c_h y_h &= \theta^2, \\
c_l y_l &= \frac{1 - 2\lambda_I}{\theta^2 (1 - 2\lambda_I \theta^{-4})}
\end{aligned} \tag{54}$$

requiring

$$\tau_h = 0,$$
$$\tau_l = 2\lambda_I \frac{\theta^4 - 1}{\theta^4 - 2\lambda_I},$$

mimicing the results above.

Golosow and Tsyvinski (2006), extend this model and calibrate it to the US. They assume people live until 75 years and start working at 25. The calibrate the probability of becoming permanently disabled for each age group. The problem is substantially simplified by the assumption that disability is permanent. They find the second best allocation in the same way as we have done here working backwards from the last period. As here, they show that the optimal allocation is implementable with transfers with asset limits and taxes on working people. The able should have zero marginal income taxes as in our example. In contrast to our example, the low ability types here have zero labor income and thus face no labor income tax.

An important finding is that asset limits are age dependent and increasing over (most of) the working life.

3.7 Time consistency

Under the Mirrlees approach, the government announces a menu of taxes or of consumption baskets. People then make choices that in equilibrium reveal their true types (abilities) to the government. Suppose the government could then re-optimize. Would it like to do this?

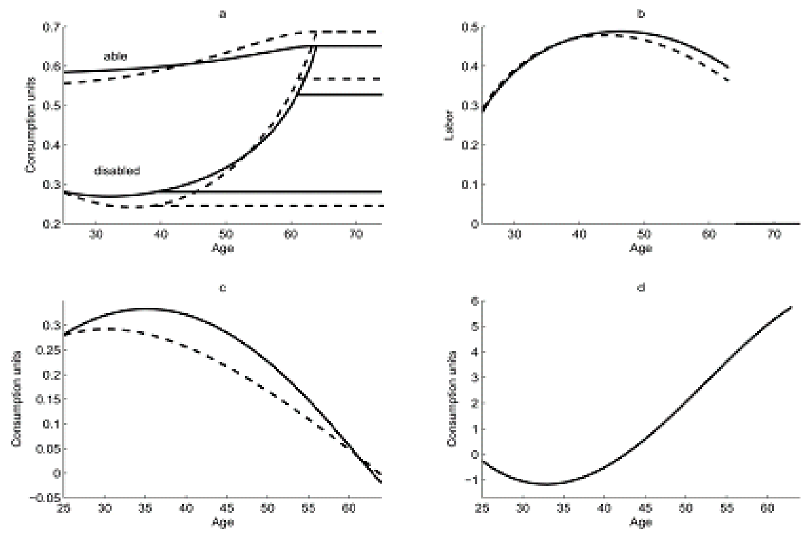

Fig. 2—Optimal disability programs with asset testing (solid lines) and without asset testing (dashed lines): *a*, consumption; *b*, labor; *c*, disability transfers; *d*, asset limits.

Figure 2: Figure from Golosov & Tsyvinski (2006)

The problem is more severe in a dynamic setting provided abilities are persistent. Why?

In a finite horizon economy, there might only be very bad equilibria (Roberts, 84). But better equilibria might arise in infinite horizon.