Optimal taxes on fossil fuel in general equilibrium

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Abstract

We embed a simple linear model of the carbon cycle in a standard neoclassical growth model where one input to the production function, oil, is non-renewable. The use of oil generates carbon emission, the key input in the carbon cycle. Changes in the amount of carbon in the atmosphere drive the greenhouse effect and thereby the climate. Climate change is modeled as a global damage to production and is a pure externality.

We solve the model for both the decentralized equilibrium with taxes on oil and for the optimal allocation. The model is then used to find optimal tax and subsidy polices. A robust model finding is that constant taxes on oil have no effect on the allocation: only time-varying taxes do. A key finding is that optimal ad valorem taxes on oil consumption should fall over time. In the simplified version of the model, optimal taxes per unit of oil should be indexed to GDP. A calibrated, less simplified model also generates declining, and initially rather substantial, taxes on oil.

1 Introduction

In this note we propose a global economy-climate model where taxes, or some other form of government policy, are called for in order to limit the negative impacts of the economy on our climate. The main goal of the note is to show how a reasonable climate externality can be introduced into a growth model yielding a quantatative and transparent characterization of optimal carbon taxes. The background for the work and for our particular approach is that there now is widespread consensus that human activity is an important driver of climate change. First, when fossil fuel is burned, carbon (dioxide) is emitted, and through the carbon cycle this carbon leads to increasing atmospherical carbon concentrations. Second, these higher concentrations influence the global temperature, which in turn is a key determinant of our climate. Third, the direct and indirect damages to humans are largely caused not

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by higher average temperature but by extreme weather outcomes, such as droughts, floods, and storms, but these extreme outcomes are much more frequent at higher average global temperatures. Of course, some of these damages then in turn influence production and thus energy use: there is two-way economy-climate feedback. However, in typical climate projections like those issued by the IPCC, the two-way feedback is not taken into account; there, one takes a "scenario" for energy use as given without asking how it in turn would influence the economy. In the climate-economy model used here, both energy use and climate outcomes are endogenous, and thus any energy projections coming out of the model are consistent with the model simulation of climate damages.

Any emission of carbon adds to a *global* stock of carbon in the atmosphere and it is the *global* concentration that determines *global* temperature. *Local* climates around the world, on the other hand, are a function of geophysical characteristics, i.e., primarily economy-independent factors, and of *global* temperature. This means that when someone burns oil in Uleåborg, to the extent there is an externality, it is global in nature. Thus, a study of the effect of the economy on the climate must involve a study of the global system with a pure externality. The global economy-climate model that we construct in this paper is a natural extension of non-renewable resource models along the lines of **?** to include a climate externality and a carbon cycle. Quite importantly, our model is also an extension in that we study a global competitive equilibrium with an externality, allowing us to discuss explicitly, with standard welfare analysis, how economic policy could and should be used to correct this externality. The prime purpose of the note is indeed to characterize optimal energy taxes in the global decentralized equilibrium economy. We also show that for the case when utility is logaritmic, depreciation is complete and production is Cobb-Douglas, so that consumption is proportional to output net of damages, a very simple closed form solution exists for the optimal tax.

Section 2 describes the model and characterizes the solution to the planning problem. Section 3 then looks at a decentralized world economy and derives the optimal-tax formula. In Section 4 we then use particular functional forms and calibrate the model to obtain our main quantitative conclusions. We discuss some obvious limitations of our work in the concluding Section **??**.

2 The economy and the climate: the planner's perspective

In this section, we describe the central planning problem. This will later be compared to the decentralized solution in order to establish the existence of a policy that replicates the solution to the planning problem as a decentralized equilibrium.

Let us define the planning problem as

$$\max_{\{C_t,K_{t+1},E_t,R_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t) \tag{1}$$
$$\begin{aligned}
C_t + K_{t+1} &= \hat{F}(A_t, K_t, N_t, E_t, A_t^e, S_t) + (1-\delta) K_t \\
R_{t+1} &= R_t - E_t, \quad R_0 \text{ given}, \\
R_t &\geq 0 \forall t, \\
N_t &= 1 \forall t, \\
S_t &= L(\Delta R^t).
\end{aligned}$$

The function U is a standard concave utility function, C is consumption, and $\beta \in (0,1)$ is the discount factor. The second line of (1) is the aggregate resource constraint. The left-hand side is resource use—consumption and next period's capital stock. The first term on the right-hand side is output produced by an aggregate production function $\hat{F}$. The arguments of $\hat{F}$ include the standard inputs K_t and N_t (capital and labor) and A_t: an aggregate measure of technology. In addition, aggregate output depends on the energy input (fossil fuel) E_t, with an associated energy efficiency level A_t^e. We assume throughout that fossil fuel is essential in the sense that the production function satisfies the standard Inada conditions. Finally, we allow a climate variable S_t to affect output. This effect could in principle be both positive and negative, though here the focus is on various sorts of damages that are all captured in the production function. We will specify later how $\hat{F}$ depends on S, but note that we view climate to be sufficiently well represented by one variable, which we take to be the global concentration of carbon in the atmosphere in excess of preindustrial levels. We argue this is reasonable given medium-complexity climate models from natural science; these imply that the climate is quite well described by current carbon concentrations in the atmosphere (e.g., lags due to ocean heating are not so important).

The variable R_t denotes remaining fossil fuel at the beginning of period t and its negative increment is fossil fuel use $E_t = \Delta R_{t+1}$. Finally, we let the climate itself depend on previous use of fossil fuel through the history $\Delta R^t \equiv \{R_1 - R_0 \ldots, R_t - R_{t-1}, R_{t+1} - R_t\}$ via the function L. Later, we will give $L(\Delta R^t)$ a simple structure that we argue reasonably well approximate more complicated models of global carbon circulation. When we consider the decentralized equilibrium, the effect of emissions on climate damages will be assumed to be a pure externality, not taken into account by any private agent. The parameter δ measures capital depreciation and finally we note that we disregard extraction costs for simplicity[1].

2.1 Damages

We assume that the climate damage affects output proportionally:

$$Y_t = S(S_t; \gamma_t) F(A_t, K_t, N_t, E_t, A_t^e) \equiv \hat{F}(.) \,.$$

[1] See our other work where we include extraction costs.

The potentially varying and stochastic parameter γ_t measures the strength of damages given S_t.We will later consider some specific functional forms but here it sufficies to note that the damage function[2] satisfies

$$S(S_t) > 0, S'(S_t) < 0.$$

Thus, we summarize all damages, including direct utility damages or damages to the capital stock, as well as technical change that reduces the damages (adaptation), in the function S.

2.2 Carbon circulation

Carbon emitted into the atmosphere by burning fossil fuel enters the global carbon circulation system, where carbon is exchanged between various reservoirs, like the atmosphere, the terrestial biosphere and different layers of the ocean. Analyzing climate change driven by the greenhouse effect, the concentration of CO_2 in the atmosphere is the key driver and we therefore need to specify how emissions dynamically affect atmospheric CO_2 concentration. A seemingly natural way of doing this would be to set up system of linear difference equations in the amount of carbon in each reservoir. This approach is taken by Nordhaus (1999,2003 and 2007) who specifies three reservoirs; *i.* the atmospehere, *ii.* the biosphere/upper layers of the ocean, and *iii.* the deep oceans. The parameters are calibrated so that the two first reservoirs are quite quickly mixed in a partial equilibrium. Biomass production reacts positively to more atmospheric carbon and the exchange between the surface water of the oceans and the atmosphere also reach a partial equilibrium quickly. The exchange with the third reservoir is, however, much slower. Only a few percent of the excess carbon in the first two reservoirs trickles down to the deep oceans every decade.

An important property of such a linear system is that the steady state shares of carbon in the different reservoairs is independent of the aggregate stock of carbon. The stock of carbon in the deep oceans is very large compared to the amount in the atmosphere and also relative to the total amount of fossil fuel yet to be extracted. Thus, the linear model predicts that also heavy use of fossil fuel will not lead to high rates of atmospheric CO_2 concentration in the long run.

The linear model sketched above abstracts from important mechanisms, in particular regarding the exchange of carbon with the deep oceans. Arguably the most important problem with the linear specification (see, Archer, 2005 and Archer et al., 2009) is due to the so called Revelle buffer factor (Revelle and Suez, 1957). As CO_2 is accumulated in the oceans the water is acidified. This limits its capacity to absorb more CO_2 dramatically making the effective "size" of the oceans as a carbon reservoir decrease by a factor 15, approximately (Archer, 2005). Very slowly, the acidity decreases and the pre-industrial equilibrium can be restored. This process is so slow, however, that we can igore it in economic models. The IPCC 2007 report concludes that "About half of a CO2 pulse to the atmosphere is

[2]The function S is normally called a damage function despite the fact that (proportional) damages are given by $1 - S$.

removed over a timescale of 30 years; a further 30% is removed within a few centuries; and the remaining 20% will typically stay in the atmosphere for many thousands of years" and the conclusion of Archer (2005) is that a good approximation is that 75% of an excess atmospheric carbon concentration has a mean lifetime of 300 year and the remaining 25% stays forever.

A simple, yet reasonable representation of the carbon cycle is that a share φ_L of carbon emitted into the atmosphere stays there forever. Within a decade, a share $1-\varphi_0$ of the remainder has exited the atmosphere into the biosphere and the surface oceans. The remaining part $(1-\varphi_L)\,\varphi_0$ decays at a geometric rate φ. Formally, we can then define a carbon depreciation factor $d\,(s)$ representing the amount of carbon remaining in the atmoshere s periods into the future as

$$d\,(s) = \varphi_L + (1-\varphi_L)\,\varphi_0\,(1-\varphi)^s\,.$$

Baseline calibration of the carbon cycle

Using discrete time interal of a decade, we use the approximation of Archer (2005) to calibrate $\varphi = 1/30$. φ_L is set to 20% as in the IPCC report (Archer's value of 25% will be included in the sensitivity analysis). The remaining parameter φ_0 is set so that $d\,(2) = \frac{1}{2}$. This yields,

$$\begin{aligned}\varphi &= 1/30,\\ \varphi_L &= 0.2,\\ \varphi_0 &= 0.4013.\end{aligned}$$

It should be noted that this paramerization is consistent with a quick mixing between the atmosphere, the biosphere and surface oceans. Within the period, a share $(1-d\,(0)) =$ 47.9% of emitted carbon has left the atmosphere.

Having defined the depreciation structure of atmospheric carbon, the law-of-motion of atmospheric carbon follows

$$S_t = \sum_{s=0}^{t}\left(\varphi_L + (1-\varphi_L)\varphi_0\,(1-\varphi)^{t-s}\right)E_s.$$

2.3 Solving the planning problem

The planner problem is now

$$\begin{aligned}&\max_{\{K_{t+1},R_{t+1},C_t,S_t\}_{t=0}^{\infty}} \mathbb{E}\sum_{t=0}^{\infty}\beta^t U\,(C_t)\\ C_t &= S\,(S_t)\,F\,(A_t,K_t,N_t,R_t-R_{t+1},A_t^e)\\ &\quad+(1-\delta)\,K_t - K_{t+1},\end{aligned}$$

subject to the additional constraints $S_t = L(\Delta R^t)$, and R_t being a non-increasing non-negative sequence.

The first order condition for K_{t+1} yields the standard Euler condition

$$U'(C_t) = \mathbb{E}_t \beta U'(C_{t+1}) \left(\frac{\partial Y_{t+1}}{\partial K_{t+1}} + 1 - \delta \right). \tag{2}$$

Let us also now convenience define

$$\varepsilon_t = \frac{\partial Y_t}{\partial E_t}$$

implying that ε_t is the private value measured in units of the consumption good of a marginal unit of fossil fuel.

After dividing by β^t, the first order condition with respect to R_{t+1} can be now be written

$$\begin{aligned} & U'(C_t)\,\varepsilon_t + \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j U'(C_{t+j}) \frac{\partial Y_{t+j}}{\partial S_{t+j}} \frac{\partial S_{t+j}}{\partial E_t} \\ &= \mathbb{E}_t \beta \left(U'(C_{t+1})\,\varepsilon_{t+1} + \sum_{j=0}^{\infty} \beta^j U'(C_{t+1+j}) \frac{\partial Y_{t+1+j}}{\partial S_{t+1+j}} \frac{\partial S_{t+1+j}}{\partial E_{t+1}} \right) \end{aligned} \tag{3}$$

The first row of the equation is the expected marginal *social* value of a unit of fossil fuel at time t. The first term is the private value, consisting of the marginal product of fossil fue, valued at current marginal utility. The second term is weigthed sum of current and future expected marginal damages caused by marginal unit of carbon emitted in period t with weights given by discounted marginal utilities. The second term is the dynamic marginal externality of fossil fuel emitted in period t. The second row is the expected marginal[3] social value in period $t+1$ discounted with the factor β. The optimality condition thus simply says that the marginal value of using fossil fuel should be the same in period t and $t+1$ when evaluated from period t.

Let us now consider the externality term. From the equation for the law-of-motion for S_t we find that $\frac{\partial S_{t+1+j}}{\partial E_{t+1}} = \left(\varphi_L + (1-\varphi_L)\varphi_0 \left((1-\varphi)^j \right) \right)$. Using the definition of Y_t and dividing by current marginal utility, we define

$$\Lambda_t \equiv -\mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \left(\varphi_L + (1-\varphi_L)\varphi_0 (1-\varphi)^j \right) \frac{U'(C_{t+j})}{U'(C_t)} \frac{S'(S_{t+j})\, Y_{t+j}}{S(S_{t+j})}. \tag{4}$$

Λ_t^s measures the marginal cost of a unit of carbon emitted into the atmosphere in terms of the consumption good. Thus, it measures the present discounted value of the production damages created by a marginal unit of extra carbon in the atmosphere.

We can now write the optimality condition as

$$\varepsilon_t - \Lambda_t = \beta \mathbb{E}_t \frac{U'(C_{t+1})}{U'(C_t)} (\varepsilon_{t+1} - \Lambda_{t+1}). \tag{5}$$

[3] Note that we here use the extraction cost of the first unit extracted in period $t+1$. This is the relevant unit since we are holding R_{t+2} constant in this exersize.

Together with the transversality condition, this uniquely defines the optimal alloaction. In the case with no uncertainty, (5) simplifies to

$$\frac{\varepsilon_{t+1}-\Lambda_{t+1}}{\varepsilon_t-\Lambda_t}=\frac{U'(C_t)}{\beta U'(C_{t+1})}=\frac{\partial Y_{t+1}}{\partial K_{t+1}}+1-\delta.$$

This expression is a variant of the famous Hotelling rule[4], stating that the return on capital should be set equal to the return to postponing extraction of a marginal unit of fossil fuel to the next period. We can think of this as a portfolio choice problem: how should the wealth we are accumulating for ourselves and for future generations be split into capital, on the one hand, and, on the other, oil resources left in the ground? They should be accumulated in such as way as to equalize returns.[5]

Already at this point, let us point to some important features of Hotelling's formula. First, abstract from the climate externality so that we can think of this formula immediately in terms of market outcomes. Then the formula says that the price of oil, which through proper market pricing must equal ε, should rise over time at a rate equal to the real rate of interest. Second, a special case of some interest is that where we allow a constant extraction costs q and where real interest rate is constant. In such a case, it is easy to show that $\varepsilon_t=\frac{\partial Y_t}{\partial E_t}-q$. Then, the gross price of oil must grow at a declining rate over time (and then converge to a rate of the real rate of interest): postponing extraction now has the benefit of spending the extraction cost later, so the price increase does not have to be so large for the producer to be indifferent.

2.3.1 Backstop technology

Suppose now that we consider the case of a backstop technology such that as in **?**, an alternative non-exhaustable, energy source becomes available at time T. From this point in time, energy is produced with a clean technology. Specifically, we assume that energy is produced with a specific capital good good K_t^e. For simplicity, we assume that the introduction of the clean technolgy is drastic so that fossil fuel is no longer used. Even though $E_{T+s}=0$ for all $s\geq 0$, S_{T+s} remains positive if $S_T>0$ and $\varphi<1$.

The necessary conditions above remain valid for $t<T$, but we now get an end-condition for R_T, namely

$$\left(\varepsilon_{T-1}-\Lambda_{T-1}^s\right)R_T=0.$$

This condition says that either all remaining fossil fuel is used in period $T-1$, i.e., $R_T=0$, or $\varepsilon_{T-1}-\Lambda_{T-1}^s=0$. In the latter case, the marginal social value of the last unit of fossil fuel used should be set to zero, i.e., the private value (the marginal product of fuel minus any marginal extraction cost) should be set equal to the present discounted value of the damage caused by a marginal unit of fossil fuel burning.

[4] The original Hotelling rule, derived in **?**, applied to a monopolistic resource owner. **?** and **?** derive an analogous condition for the case of perfect markets and no externalities, in which case the market implements the optimal extraction path. Finally, **?** shows how to include an externality in the condition, arguing that this naturally leads to slower extraction than in *laissez-faire*.

[5] See Sinn (2008) for a derivation of the Hotelling rule above and for the portfolio-choice interpretation.

Consider the case, when $R_T > 0$, implying $\varepsilon_{T-1} = \Lambda_{T-1}$, using this in equation (5) for R_{T-1} we get

$$\varepsilon_{T-2} - \Lambda_{T-2} = 0. \tag{6}$$

This expression has a clear intution; if the last unit of fossil fuel extracted in period $T-1$ optimally has zero social value, this should apply also to a marginal unit in the next to last period. Iterating backward we find that in all periods, the social value should be set to zero.

3 A decentralized economy and implementation of the optimum

A representative individual maximizes

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t U\left(C_t\right)$$
$$s.t.\ C_t + K_{t+1} = \rho_t K_t + \Pi_t^f + \Pi_t^e + T_t,$$

where ρ_t is the rental rate of capital, Π_t^f and Π_t^e are profits from final goods production and resource extraction and T_t are government transfers that we assume are equal to the tax revenues in present value. Here, in equilibrium Π_t^f will be zero, due to perfect competition, but Π_t^e will be positive, essentially delivering the stock value of the oil in the ground.

The first-order condition of interest here, i.e., that for K_{t+1}, delivers the usual

$$U'\left(C_t\right) = \beta \mathbb{E}_t \rho_{t+1} U'\left(C_{t+1}\right). \tag{7}$$

Goods production takes place in perfect competition, implying that the price of the resource—the oil price, p_t^e—is given by its marginal product

$$p_t^e = \frac{\partial Y_t}{\partial E_t}. \tag{8}$$

Competitive goods production also implies that the competitive rental of capital satisfies

$$\rho_t = \frac{\partial Y_t}{\partial K_t} + 1 - \delta. \tag{9}$$

This implies that (7) coincides with the planner solution.

Now consider a representative atomistic resource extraction firm owning a share of fossil fuel resources of all remaining extraction-cost levels. Let us introduce a *per-unit fossil fuel* tax θ_t. The problem of a representative resource extraction firm is to maximize the discounted value of its profits

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t \frac{U'\left(C_t\right)}{U'\left(C_0\right)}\left(\left(p_t^e - \theta_t\right)\left(r_t - r_{t+1}\right)\right)$$
$$\text{s.t. } r_{t+1} \geq 0 \forall t.$$

where r_t is the remaining amount of resources for the representative resource extracting firm. The fact that we assume the oil extracting firms to be atomistic implies that they take all prices and the sequence of capital as exogenous.

Using (8), the first-order condition with respect to r_{t+1} can be written

$$\varepsilon_t - \theta_t = \mathbb{E}_t \beta \frac{U'(C_{t+1})}{U'(C_t)} (\varepsilon_{t+1} - \theta_{t+1}) \tag{10}$$

where we have assumed that there is a unit mass of representative resource extractors implying that $r_t = R_t$. Together with the transversality condition, this defines a profit maximizing extraction path. Clearly, setting $\theta_t = \Lambda_t$, implements the optimal allocation. This is straightforward to understand and is a an old lession due to Pigou − if there is an externality, a tax equal in value to the externality makes agents internalize the externality implying an optimal market allocation.

In the case of a backstop-technology arriving at some known date T, the transversality condition is $(\varepsilon_{T-1} - \theta_{T-1}) r_T = 0$. Using this implies that if r_T optimally is larger than zero, we can write (10) for $t = T - 2$ as

$$\varepsilon_{T-2} - \theta_{T-2} = 0$$

and

$$\varepsilon_t - \theta_t = 0.$$

Thus rents (profits), at t are equal to zero for all periods.

4 An analytical example with a calibration

We know that with log utility, full depreciation and Cobb-Douglas production, there is a closed form solution to the neoclassical growth model. Let us therefore use the same assumptions in the case of a non-renewable resource with externalities, since this model as well has a closed-form solution so long as extraction costs are zero. Key in this analytical derivation is a proportionality result: marginal utility is inversely proportional to output at any time. As we will see, this implies that the model's implications for fossil fuel use, and for optimal fossil fuel taxes, are invariant to the key driver of output growth: improvements in total-factor productivity (TFP) and population growth. Thus, we can shut down TFP growth here since it does not alter any of our results.[6]

More importantly, however, one can argue that these functional-form assumptions are not wildly at odds with what would seem to be quantitatively reasonable assumptions. First, logarithmic curvature for utility is in line with most applied macroeconomic studies. Second, full depreciation is not on short horizons, but with the 10-year periods we will use here, it is not too far from a reasonable rate. Third, though one would have trouble over shorter time horizons with the assumption that energy enters like capital and labor in a Cobb-Douglas production function—since it seems reasonable to assume that installed equipment

[6] To be clear, higher TFP increases the demand for energy, but with Cobb-Douglas production it will simply increase the price of energy one-for-one, and the time path for energy will be unaffected.

and structures have rather fixed energy requirements—but on a longer horizon, since the style of capital can be adjusted in response to energy prices, it is not so unreasonable with a Cobb-Douglas technology. In fact, it is also what Nordhaus uses in his RICE model, which is entirely quantitative in nature. Fourth, zero extraction costs is obviously an exaggeration but the purpose of setting them to zero is only to make consumption proportional to output. With extraction costs, consumption is not exactly proportional to output but since extraction costs are and are likely to remain small relative to aggregate output, we can trust that our results regarding optimal policy are not sensitive to our assumption.

Production is thus assumed to be

$$Y_t = S\left(S_t; \gamma_t\right) F\left(K_t, E_t, A_t\right) = S\left(S_t, \gamma_t\right) A_t K_t^{\alpha} E_t^{\nu}.$$

This together with logarithmic utility implies an Euler equation for physical capital investment that reads

$$1 = \mathbb{E}_t \beta \frac{C_t \alpha Y_{t+1}}{C_{t+1} K_{t+1}} \tag{11}$$

Now it is straightforward to see that $C_t = (1 - \alpha\beta) Y_t$ implying $K_{t+1} = \alpha\beta Y_t$ solves (11). Thus, in every period

$$U'\left(C_{t+j}\right) Y_{t+j} = \frac{1}{1-\alpha\beta}.$$

Using this in the definition of the marginal externality cost (4) yields

$$\Lambda_t = -Y_t \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \left(\varphi_L + (1-\varphi_L)\varphi_0 \left(1-\varphi\right)^j\right) \frac{S'\left(S_{t+j}, \gamma_{t+j}\right)}{S\left(S_{t+j}, \gamma_{t+j}\right)}. \tag{12}$$

The key insight here is that with full depreciation, log utility and Cobb-Douglas production, the shadow value on the damage externality is completly determined by current output and the expectation of a a weighted sum of current ant future marginal proportional damages. Future values of consumption and output are irrelevant, regardless of whether they are stochastic or not. Furthermore, the formula implies a certain form of certainty equivalence. The expected value of future marginal damages determines the the value of Λ_t and thus the optimal tax – the degree of uncertainty is irrelevant.

4.1 An exponential damage function

Since the optimal tax is determined by the expected values $\frac{S'\left(S_{t+j},\gamma_{t+j}\right)}{S\left(S_{t+j},\gamma_{t+j}\right)}$ it is natural to analyze the case of exponential damage functions, since in that case, this ratio is independent of S_{t+j}. Therefore, suppose that

$$S\left(S_t; \gamma_t\right) = e^{-\gamma_t S_t},$$

implying that

$$\frac{S'\left(S_t; \gamma_t\right)}{S\left(S_t; \gamma_t\right)} = -\gamma_t.$$

This paper is not about the estimation of damage functions, but we of course want to compare this specification with current state-of-the art damage function. We take this to be Nordhaus (DICE, 2007) who uses a proportional damage function driven by the global mean temperature T specified as

$$S_N\left(T_t\right) = \frac{1}{1+\theta_2 T_t^2}$$

with $\theta_2 = 0.0028388$. A standard climate sensitivity of 3.0 degrees Celcius per doubling of the atmosheric carbon content gives

$$T\left(S_t\right) = 3\frac{\ln \frac{S_t}{S_0}}{\ln\left(2\right)}$$

where S_0 is the amount of carbon in the atmosphere before industrialization started.

In the Figure 2, we show Nordhaus damage function $S_N\left(T\left(S_t\right)\right)$ (dashed) together with an exponential damage function with parameter $\gamma_t = 5.3\times10^{-5}$. The range of the X-axis is large, 600 gigagtons corresponds to preindustrial levels while 3000 Gigatons of carbon corresponds to the case when most of predicted stocks of fossil fuel are burned over fairly a short period of time. Still, we see that the two curves are close indicating that the exponential case may be worth considering.

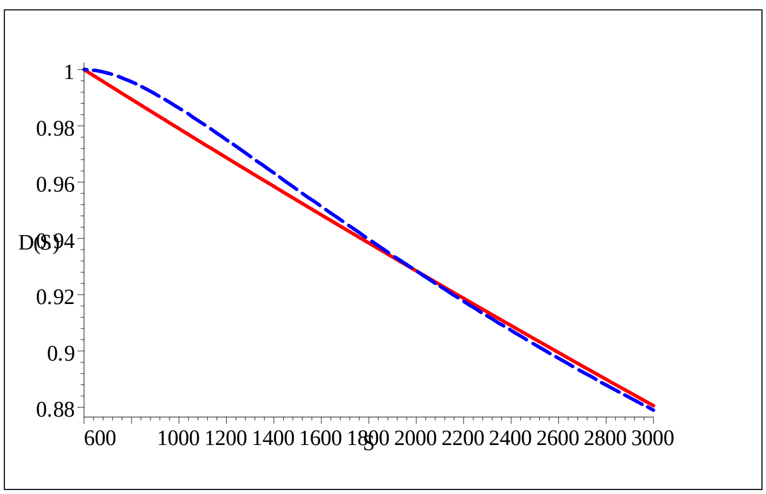


Figure 1. Nordhaus DICE 2007 damage function (dashed) and exponential damages.

Given expectations about the path of γ_t, we can now easily find an expression for the optimal tax. Consider, for example, the case when the the expected value of γ_t is constant at $\bar{\gamma}$. Applying (12) yields,

$$\Lambda_t = Y_t\bar{\gamma}\left(\frac{\varphi_L}{1-\beta}+\frac{\left(1-\varphi_L\right)\varphi_0}{1-\beta\left(1-\varphi\right)}\right) \tag{13}$$

Thus, the optimal tax per unit of fossil fuel should be proportional to output in every period, with a proportionaly factor given by the expected value of the parameter of the damage function, subjective discounting and the parameters determining the depreciation of atmospheric carbon. The formula lends itself very easily to calibration, as we will demonstrate below.

Obviously, (13) is expressed in terms of current output which is endogeneous and itself dependent on the tax. For practical purposes, this is not very important since at least moderate variations in the tax has quite limited effects on output. Specifically, the elasticity of output with respect to fossil fuel use is equal to the fossil fuel income share, which is in the order of a few percent. Thus, we may take current output as exogeneous when calculating the optimal tax as long as the resulting tax change does not influence current output much.

However, we can easily go further and calculate the optimal tax in terms of predetermined variables by deriving expressions for the endogeneous value of E_t. Consider the case when we expect that the optimal use of fossil fuel implies that some fuel will be left unused forever.[7]Under the maintained assumption that extraction costs are small enough to be disregarded, there is then no scarcity rent of fossil fuel in the optimal allocation and the social value of fossil fuel should therefore be zero in every period. Formally,

$$\frac{\partial Y_t}{\partial E_t} = \Lambda_t.$$

Using the assumptions in this section, this yields,

$$\frac{\nu Y_t}{E_t} = Y_t \bar{\gamma} \left(\frac{\varphi_L}{1-\beta} + \frac{(1-\varphi_L)\,\varphi_0}{1-\beta\,(1-\varphi)} \right) \Rightarrow E_t^* = \frac{\nu}{\bar{\gamma}\left(\frac{\varphi_L}{1-\beta} + \frac{(1-\varphi_L)\varphi_0}{1-\beta(1-\varphi)} \right)} \tag{14}$$

Note that this implies that fossil fuel use should be constant and inversely proportional to the expected value of the damage parameter. Using this in the production function, we have

$$Y_t = e^{-\gamma(S_{t-1}+E_t^*)} A_t K_t^{\alpha} \left(E_t^*\right)^{\nu}$$

which together with (13) determines the optimal tax in terms of the predetermined variables S_{t-1}, K_t, the exogeneous A_t and parameters.

4.2 Calibration with uncertainty

Previous work has not treated uncertainty explicitly in the model. As we have seen, however, uncertainty poses no particular problem to our analysis. As an illustrative example, we will assume that there is uncertainty with respect to the strength of the externality. Using the exponential damage function, this means that there is uncertainty regarding future values of γ_t. Specifically, we assume that until some random future date there is uncertainty regarding the long-run value of γ. At that date, uncertainty is resolved and either it turns out that γ will be equal to γ^H or equal to γ^L, with $\gamma^H > \gamma^L$. The *ex−ante* probability of the high value is denoted p. For simplicity, but not necessity, we assume that until the long-run value of γ is learned, the current value $\gamma_t = p\gamma^H + (1-p)\,\gamma^L \equiv \bar{\gamma}$.

[7]The calibrations below will suggest that this is the most realistic case. IPCC also strongly argues that burning all fossil fuel is suboptimal.

We will use the work of Nordhaus (2000) to calibrate damage parameters. In line with standard assumptions (reference), we assume the there is a log-linear relation between the atmospheric CO_2 concentration and the global mean temperature in excess of the pre-industrial level, T, such that

$$T_t = T(S_t) = \lambda \ln\left(1 + \frac{S_t}{\bar{S}}\right) / \ln 2, \tag{15}$$

where $\bar{S} = 581$ GtC is the pre-industrial atmospheric CO_2 concentration.

When calibrating the damage function, Nordhaus (2000), uses a bottom-up approach by collecting a large number of studies on various effects of global warming. By adding them up he arrives at an estimate that a 2.5 degree Celsius heating yields an global (output-weighted) loss of .48% of GDP. Furthermore, he argues based on survey evidence that with a probability 6.8% the damages at a 6 degree Celsius heating are catastrophically large at 30% of GDP. Nordhaus calculates the willingness to pay for such a risk and adds it to the damage function. Instead, we directly use his numbers to calibrate γ^H and γ^L. Specifically, we take the 0.48% loss at 3 degrees heating to calibrate γ^L (moderate damages) and the the 30% loss at 6 degrees to calibrate γ^H (catastrophic damages). Using (15) we find that a 2.5 and a 6 degree heating occurs if S_t equals 1035 and 2324, respectively. We thus calibrate γ^L to solve

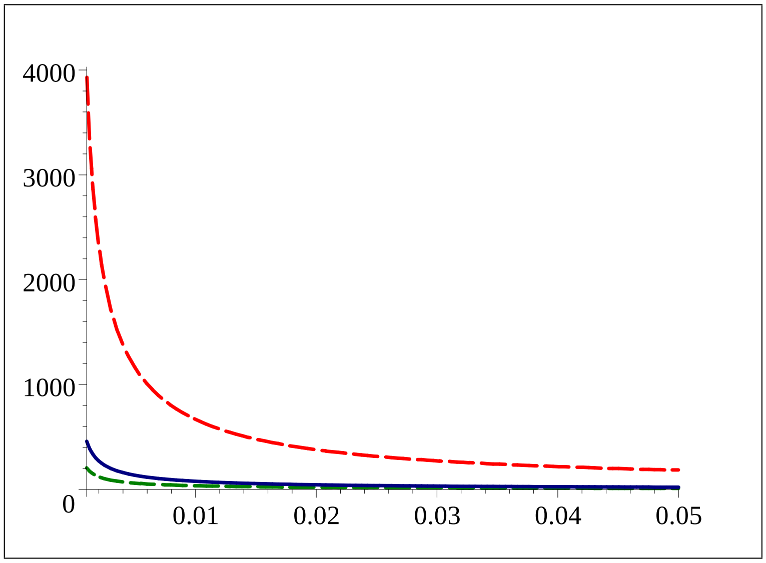
$$e^{-\gamma^L(1035-581)} = 0.9952$$

and

$$e^{-\gamma^H(2324-581)} = 0.70$$

yielding $\gamma^L = 1.060 \times 10^{-5}$ and $\gamma^H = 2.046 \times 10^{-4}$. Using $p = 0.068$,we calculate $\bar{\gamma} = 2.379 \times 10^{-5}$

We can now calculate the optimal taxes before and after we have learnt the long run value γ. We use (13) and express the tax per ton of carbon at a yearly output of 70 trillion dollars. In figure 3, we plot the three tax rates against the yearly subjective discount rate.



Optimal tax per ton C, expected damage (solid), high damage (upper dashed) and low damage (lower dashed).

Two important policy proposals have been made so far, Nordhaus (2000) and in the Stern report (reference). They propose a tax of \$30 and \$250 dollar per ton coal. A key difference between the two proposals is that they use very different subjective discount rates. Nordhaus uses a rate of 1.5% per year and Stern 0.1% per year. For these two values of the discount rate, the optimal taxes using our analysis are \$55.7/ton and \$459/ton respectively. Thus, our calculations suggest a substantially larger optimal tax. The conseqences of learning are dramatic. With a discount rate of 1.5%, the optimal tax rates if damages turns out to be moderate is \$24.8/ton but \$479/ton if they are catastrophic. For the low discount rate, the corresponding values are \$205/ton and \$3950/ton.