# Incentives and Aggregate Shocks

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This paper presents an incentive-based theory of the dynamics of the distribution of consumption in the presence of aggregate shocks. The paper builds on the models concerning the distribution of income or consumption and incentive problems of Green (1987), Thomas and Worrall (1991), Phelan and Townsend (1991), and Atkeson and Lucas (1992). By incorporating aggregate production shocks, the model allows an examination of the interactions between individual and aggregate consumption series given incomplete insurance. Further, the methodology outlined allows the incorporation of incentive considerations to macroeconomic environments similar to Rogerson (1988) and Hansen (1985).

## 1. INTRODUCTION

This paper presents an incentive-based theory of the dynamics of the distribution of consumption in the presence of aggregate shocks. Its methodological accomplishments are two-fold: First, it displays a method for explicit consideration of aggregate shocks in dynamic general-equilibrium incentive economies such as those of Green (1987), Phelan and Townsend (1991) and Atkeson and Lucas (1992). Second, it displays a method for explicit consideration of incentive conditions in macroeconomic environments, where, for reasons of tractability, incentive conditions are usually ignored. One interpretation of the environment presented here is an incentive-constrained, limited-insurance version of the unemployment models of Hansen (1985) and Rogerson (1988) which, for reasons of tractability, assume full insurance over unemployment risk.

In the model presented, each agent's observable outcomes are a function of his unobservable level of effort, an unobservable idiosyncratic shock, and a publicly observed aggregate shock. While risk-averse agents would wish to pool (and thus insure against) their idiosyncratic risk, such complete insurance is not feasible due to the unobservable effort. Because of the non-incentive compatibility of full-insurance, higher effort levels are achieved by making each agent's consumption a function of his current and past observable outcomes. Agents with favourable current outcomes receive both higher current and future consumption. This contingency of individual consumption on individual outcomes causes the consumption of ex ante identical agents to differ over time. Further, since individual outcomes are also affected by aggregate shocks, the aggregate shocks not only affect the average level of consumption as in a single-agent model, but the efficient distribution of consumption as well.

To this date, the only analytically tractable model available to make predictions regarding the dynamics of the distribution of consumption in the presence of aggregate shocks is the full-information or complete-markets framework. With aggregable preferences, such an economy reduces to an economy with a single representative agent. After solving the representative-agent economy, one can make predictions regarding distributions in economies which reduce to the representative-agent economy relatively straightforwardly. Papers by Mace (1991), Cochrane (1991), and Townsend (1994) examine the

relationship between individual consumption series and aggregate consumption series in light of the predictions of full-information, complete-market models. The general result of these papers is that the full insurance implications of full information models are rejected. This implies the need for models which allow richer interactions between aggregate and idiosyncratic shocks on consumption levels.

Solving dynamic, stochastic models which make predictions regarding consumption distributions is difficult if one strays from assumptions which allow reduction to a representative agent. With more than one type of agent at a given time, market clearing prices generally depend on the entire distribution of agent types. In models where the evolution of the distribution of types is endogenous, this causes intractability. One way of interpreting the aggregation results which allow the reduction of an economy to a single representative agent is that they show one set of special circumstances which cause distributions not to affect prices.

An accomplishment of this paper is the derivation of a different set of assumptions which allow predictions regarding distributions with the full-information assumption relaxed, but still allow the model to be solved. While the predictions of this model differ substantially from the testable (and rejected) full-insurance predictions of the full-information model, they are not without content. Specifically, I show that like the dynamic, incentive-based model of Green (1987) which is void of production and aggregate shocks, individual consumption levels follow a process which is the sum of an i.i.d. term and a term which follows a random walk. Second, I show that while individual consumption changes are not perfectly correlated as with full-insurance, they are positively correlated across individuals, since they depend on a common aggregate shock. Finally, I show that the characteristics of the cross-sectional distribution of consumption itself will depend on the history of aggregate shocks. I show that in contrast to the full-information framework where the current aggregate shock alone determines the distribution of consumption, the frictions assumed here cause the predicted distribution of consumption to follow a Markov process.

This paper builds on the models concerning the distribution of income or consumption and incentive problems of Green (1987), Thomas and Worrall (1991), Phelan (1994), Phelan and Townsend (1991), and Atkeson and Lucas (1992). Unlike each of the papers except Phelan (1994), the long-run distribution of consumption is not degenerate due to the overlapping generations framework adopted here. Banerjee and Newman (1991) also consider the question of consumption distribution and risk-bearing and develop a model with a non-degenerate long-run distribution. They do this by assuming family dynasties with bequest motives where consumption is bounded from below by the fact that agents are limited in the amount of debt they can pass to their children and above by assumptions causing the incentive problem to disappear for agents with high enough wealth levels. Taub (1990) considers linear incentive schemes in an economy with moral hazard and aggregate shocks.

The model in this paper can also be considered an incentive-constrained, limited-insurance model of unemployment similar to the full-insurance unemployment models of Hansen (1985) and Rogerson (1988). Thus while the model presented in this paper is simpler that that of Hansen on dimensions other than the consideration of incentives (specifically, the model in this paper does not have capital) the methods in this paper should be seen as encouraging to macroeconomic theorists who wish to incorporate incentives or dynamic contracting into models with both individual and aggregate risk.

Section 2 presents the technology and preferences assumed in this model. Section 3 presents a fairly lengthy construction of an equilibrium. The key of this section is an

ability to show that market prices can be represented by vectors which remain constant over time, or which are independent of the distribution of types in the population. The aggregate shock affects realized prices only by picking out elements of these constant vectors. (For example, there will be a time-independent vector associating each aggregate shock with a one-period interest rate. The realized interest rate is determined only by the current realized aggregate shock.) Section 4 discusses the characteristics of this equilibrium as compared to the characteristics of the same model but given full information. Section 5 discusses optima, using an extension of the techniques of Atkeson and Lucas (1992). Here the constructed equilibrium of Section 3 is shown to be optimal, and a planning problem is derived to find the plan associated with the constructed equilibrium without searching for market-clearing prices. Section 6 concludes.

## 2. THE MODEL

The economy consists of overlapping generations of identical agents. Each generation consists of a continuum of agents and is of equal size (unity) at birth. At the beginning of each date, each agent takes an unobservable action $a \in A$ where the set $A$ is assumed ordered and finite. This action results in a publicly observed output $q \in Q$ ($Q$ ordered and finite) according to the non-zero probability function $P(q|a, \theta)$ where $\theta \in \Theta$ ($\Theta$ ordered and finite) is a publicly observed aggregate shock drawn independently over time with probability $Z(\theta)$. It is important to note that $\theta$ is observed after the action is taken. After output $q$ occurs, consumption occurs. Let $Y \equiv R$ denote the set of possible consumption amounts. After each period, agents are assumed to die with probability $(1 - \Delta)$, independent of age, and thus the size of the $t$-aged generation equals $\Delta^{t-1}$. Assume at the first calendar date that there exists the steady-state number of agents, $1/(1 - \Delta)$. The calendar dates are numbered $\tau = 1, \ldots, \infty$.

These variables can be interpreted straightforwardly as a model where individuals can "sleep on the job" (take low effort $a$). In this interpretation, $q$ represents the physical, stochastically determined work output of the agent measured in units of the consumption good and $P(q|a, \theta)$ represents the probability of each of these output levels for a given level of effort and aggregate shock. Presumably, low effort levels increase the probability of low outputs. An alternative interpretation is to consider $q$ to be the quality of a match between a worker and a firm (again measured in units of the consumption good and representing the deterministic productivity of the worker in that match) and $a$ to be the unobservable effort taken by an agent to find a productive match. Thus, one can consider an incentive-constrained unemployment model by letting $q$ take on two levels: employment, with a corresponding high productivity $q$, and unemployment, with a corresponding zero or low productivity $q$. The function $P(q|a, \theta)$ is then the probability of a given employment state for an agent given the effort level of the agent at finding a job, $a$, and the aggregate state $\theta$.

At the beginning of his life, each agent enters into a binding lifetime contract with a financial intermediary or firm, in the case of equilibria, or the social planner, in the case of optima. The contract specifies for each date in the agent's life his action level $a$ and his consumption $c$. The contract may make consumption or effort a function of all information available in the economy at the appropriate time. Agents care about their expected discounted stream of point-in-time utilities over consumption and effort where the common point-in-time utility function $U(a, c)$ is the constant absolute risk aversion specification

$U: A \times Y \to \mathbf{R}_-$ such that $U(a, c) = -\exp(-\gamma(c - v(a)))$, where $v(a)$ is any increasing function. Agents discount the future using the constant discount parameter $\beta < 1$. Define the set of possible ex-ante utilities $W \equiv \mathbf{R}_-$.

## 3. EQUILIBRIA

This section constructs an equilibrium where all potential trading is between infinitely-lived financial intermediaries or "firms" which exist from the first calendar date. When agents are born they are offered lifetime contracts by firms and accept whatever contract gives them the highest expected discounted utility $w$ from the perspective of birth. This is the only market transaction individuals may ever engage in. From then on they simply follow the contract. In effect, agents "sell themselves" to a firm for life. Enough commitment is assumed between firms and individuals for such contracts to be always binding. While for now this is simply an asserted restriction of the interaction between agents in the economy, it will later be shown to non-restrictive in the sense that if the possibility for such commitment exists, it is optimal for society or the contracts themselves to bar agents from trade.

Firms, on the other hand, can trade the consumption good across dates and states at prices they take to be given. For a given promised utility $w_0$, the firm's problem is to maximize arbitrage profits (or, as it will be stated later, to minimize arbitrage losses). Firms do not have preferences. They take as given prices for date and state contingent consumption in terms of a numeraire good and use these prices to transform profits and losses at various dates and states into a single number. Again, only firms operate in the contingent claims and credit markets. Transactions in these markets are assumed observable and thus barring individuals from trade is enforceable.[1]

*Optimal contracts*

I now consider the question of what kind of contracts firms will offer to agents. Instead of assuming time-zero markets allowing firms to trade date- and state-contingent consumption, I assume a sufficient set of spot markets. Let $B_t(\theta_t | \theta_0, \dots, \theta_{t-1})$ be the spot price at the beginning of date $t$ of $\theta_t$-contingent consumption, in terms of date $t$ non-contingent consumption (thus $\sum_{\Theta} B_t(\theta_t | \theta_0, \dots, \theta_{t-1}) = 1$). That is, I assume a spot insurance market at date $t$ that allows firms to buy and sell $\theta_t$-contingent consumption before $\theta_t$ is realized. For intertemporal trade, I assume a credit market. Let $\delta_t(\theta_t | \theta_0, \dots, \theta_{t-1})$ be the spot price at the end of period $t$ (after $\theta_t$ is realized) of date $t + 1$ non-contingent consumption in terms of current ($\theta_t$-contingent) consumption. These spot markets sufficiently replicate the trading opportunities of a complete set of time-zero markets.

The problem of finding efficient contracts can be solved using recursive techniques if these spot prices are independent of history and depend only on the current aggregate shock $\theta$, and can thus be written simply as vectors $B(\theta)$ and $\delta(\theta)$. This is true under any economy which reduces to a representative-agent framework (as does this one if not for unobservable effort), but at this point I have given no reasons why this should be the case for this economy. Nevertheless, it is still a well defined question to ask how contracts would look if these vectors of spot prices were constant. I will later construct an equilibrium where these vectors are, in fact, constant.

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1. Note that unlike most overlapping-generations models the traders here, the firms, have access to all markets.

Optimal contracts are recursive in the sense that the expected discounted utility of an agent from any point time on (say $w$) sufficiently summarizes all information about that agent relevant for constructing a continuation contract between the firm and the agent.[2] When an agent is born, the cost to a firm of contracting with the agent is solely a function of the level of utility $w$ the firm promises the agent. (Here, cost is in terms of present non-contingent consumption.) An agent owed a high expected discounted utility (say $w_0$) will cost more than an agent owed a low $w_0$. After the agent's first day of life, he will have an expected discounted utility (say $w_1$) of following the contract he signed at birth. Since actions and transfers under this contract can depend on the output of the agent in his first period of life, this expected discounted utility $w_1$ will in general differ from the expected discounted utility $w_0$ the agent had when he signed the contract. Nevertheless, from the perspective of the firm, the one-period-old agent looks no different than a newborn agent who was given an initial utility promise of $w_1$. It costs the firm the same to provide a lifetime contract to an agent initially promised $w_1$ as it does to continue a contract that provides the agent with a continuation utility of $w_1$.

Given that the firm considers the expected discounted utility of the agent to be a state variable, its choice variables are as follows. The firm chooses a function $a(w): W \rightarrow A$, that specifies the recommended action as a function of the agent's expected utility from the current date on. Likewise, the firm chooses for each $w \in W$ a consumption transfer function $Y(q, \theta|w): Q \times \Theta \rightarrow R$ which again depends on the agent's expected discounted utility at the beginning of the period, and also the agent's current output $q$ and the current aggregate shock $\theta$. The current period output of the agent and the aggregate shock are known to the firm at the time of the consumption transfer but not at the time actions are recommended. Finally, the firm chooses for each $w \in W$ a function $W'(q, \theta|w): Q \times \Theta \rightarrow W$ which specifies the utility promise to the agent from the perspective of the next period.

The set of functions $[a(w), Y(q, \theta|w), W'(q, \theta|w)]$ completely specifies a lifetime contract for an agent with an initial utility of $w_0$. His recommended action on his first day of life is $a(w_0)$ and his consumption is $Y(q_0, \theta_0|w_0)$, where $q_0$ is the agent's output on his first day of life and $\theta_0$ is the aggregate shock for this date. The promised utility function $W'(q, \theta|w)$ then generates the contract for the next period of life. For instance, $a(W'(q_0, \theta_0|w_0))$ is the agent's recommended action in his second period of life, and $Y(q_1, \theta_1|W'(q_0, \theta_0|w_0))$ is the agent's consumption in his second period of life.

Without loss of generality, I can split the consumption transfer function $Y$ into a transfer which does not depend on $q$ or $\theta$, $z(w)$, and a transfer which can depend on $q$ and $\theta$, $y(q, \theta|w)$, so that $Y(q, \theta|w) = z(w) + y(q, \theta|w)$. Likewise, I can allow the promise generating function $W'$ to be stated as a component $w'(q, \theta|w)$ which may depend on $q$

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2. The proof that the continuation utility of the agent $w$ sufficiently summarizes his history (more formally presented in Spear and Srivastava (1987) and Phelan and Townsend (1991)) can be sketched as follows. One first formulates contracts as specifying the work effort $a$ and consumption $c$ of an agent at each point in time as a function of the agent's entire output history as well as the history of aggregate shocks and consider an efficient contract in this (large) space. After any given history $h$, the agent will have a utility $w(h)$ associated with continuing the contract. One then considers giving the firm a chance to re-optimize or choose a new continuation contract subject to incentive compatibility conditions and a constraint that the agent's continuation utility remain constant. However, one can show that any new continuation contract could have been incorporated into the original contract without upsetting the incentive compatibility for the original contract or the ex ante utility the agent associated with the original contract. This implies that the firm cannot lower its continuation costs by reoptimizing. Any potential gain from re-optimization would have already been incorporated into the original contract. This implies that if after history $h$, the firm were to "forget" the history $h$, but remember the agent's utility $w(h)$ associated with this history, the firm could recover the continuation plan from $h$ by minimizing its continuation costs of providing an incentive compatible continuation contract subject to the agent receiving continuation utility $w(h)$. Thus the agent's utility contains all the information necessary to recover the continuation contract or sufficiently summarizes history $h$.

and $\theta$ scaled by $\exp(-\gamma z(w))$, or $W'(q, \theta|w) = \exp(-\gamma z(w))w'(q, \theta|w)$. For now, simply take this non-contingent payment $z$ and the scaling of utility promises as harmless extra degrees of freedom in the firm's choice problem. The non-contingent payment $z$ can always be chosen to equal to zero and thus $\exp(-\gamma z) = 1$. Further, for $z$ equal to any non-zero constant, the functions $y$ and $w'$ can be chosen to "undo" the effect of the non-zero $z$.

The firm is not free to pick just any contract. For an agent who was promised a specific expected discounted utility of $w$, the firm must pick policies $[z, a, y(q, \theta), w'(q, \theta)]$ which actually deliver this expected utility. This can be expressed in the form of the promise-keeping constraint

$$w = \sum_\Theta \sum_Q \{-\exp(-\gamma(z+y(q, \theta)-v(a))) + \beta\Delta \exp(-\gamma z)w'(q, \theta)\}P(q|a, \theta)Z(\theta). \tag{1}$$

Equation (1) is the lifetime expected discounted utility of the agent from the beginning of the period given that the utility promise $\exp(-\gamma z)w'(q, \theta)$ is actually delivered given that the agent lives to see the next period.

The firm must also respect the unobservability of actions, again for each $w \in W$. This is expressed as incentive constraints that for all $\hat{a} \in A$,

$$\sum_\Theta \sum_Q \{-\exp(-\gamma(z+y(q, \theta)-v(a))) + \beta\Delta \exp(-\gamma z)w'(q, \theta)\}P(q|a, \theta)Z(\theta)$$

$$\geqq \sum_\Theta \sum_Q \{-\exp(-\gamma(z+y(q, \theta)-v(\hat{a}))) + \beta\Delta \exp(-\gamma z)w'(q, \theta)\}P(q|\hat{a}, \theta)Z(\theta). \tag{2}$$

The right-hand side of equation (2) is the lifetime expected discounted utility of following deviation action $\hat{a}$, again given that future utility promises are kept.

Let $C^*(w)$ denote the cost in terms of current non-contingent consumption of providing an expected discounted utility $w$ to an agent. Optimal contracts can be found by noting that $C^*(w)$ is a fixed point of the operator $T$ defined by the following programming problem. The operator $T$ takes as given a cost function governing utility promises made today for tomorrow on, and generates the cost function governing utility promises from the perspective of today. The objective of the programme is to minimize the value of net consumption transfers in terms of current non-contingent consumption, or,

$$(TC)(w) \equiv \min_{z,a,y(q,\theta),w'(q,\theta)} \sum_\Theta \sum_Q \{z+y(q, \theta)-q$$

$$+ \delta(\theta)\Delta C(\exp(-\gamma z)w'(q, \theta))\}P(q|a, \theta)B(\theta) \tag{3}$$

subject to the promise-keeping constraint (1), and the incentive constraints represented by (2). It is important to note that while $q$ is a random variable for a given agent, the inner summation over $q$ in the objective function (3)—the expected value of the expression inside the brackets given $\theta$—is known with certainty given $\theta$. Firms do not face risk over $q$ realizations since they can contract with a positive mass of agents and thus $P(q|a, \theta)$ represents the certain fraction of agents who realize $q$. Thus the inner sum is the certain cost, for a given $\theta$ realization, of the plan $[z, a, y(q, \theta), w'(q, \theta)]$. Summing over $\theta$ and weighting by the prices $B(\theta)$ gives the total cost of the plan in terms in current non-contingent consumption.

The exponential utility function allows a substantial simplification of this programming problem. The trick is to use the unconditional payment $z$ to handle the utility constraint, and the conditional payment $y(q, \theta)$ and promises $w'(q, \theta)$ to handle the incentive constraints. This is justified as follows.

In the incentive constraint (2), one can pull the expression $\exp(-\gamma z)$ outside of both summations on each side of the inequality and cancel, giving

$$\sum_\Theta \sum_Q \{-\exp(-\gamma(y(q, \theta)-v(a)))+\beta\Delta w'(q, \theta)\}P(q|a, \theta)Z(\theta)$$

$$\geq \sum_\Theta \sum_Q \{-\exp(-\gamma(y(q, \theta)-v(\hat{a})))+\beta\Delta w'(q, \theta)\}P(q|\hat{a}, \theta)Z(\theta). \qquad (4)$$

This shows that if a contract is incentive compatible given one constant payment $z$, it is incentive compatible for all constant payments $z$.

In the promise-keeping constraint (1), one can again pull out from the right-hand side the expression $\exp(-\gamma z)$. If this is set equal to $-w$, or $z(w)=-\log(-w)/\gamma$, this gives (1) as

$$-1=\sum_\Theta \sum_Q \{-\exp(-\gamma(y(q, \theta)-v(a)))+\beta\Delta w'(q, \theta)\}P(q|a, \theta)Z(\theta), \qquad (5)$$

thus removing $z$ and $w$ from the constraint set. The intuition here is that (5) insures that if $z=0$, the functions $y(q, \theta)$ and $w'(q, \theta)$ deliver an expected utility of $-1$. If the policies $[a, y(q, \theta), w'(q, \theta)]$ satisfy (5), this along with the assumption that $z(w)=-\log(-w)/\gamma$ insures that the collection $[z, a, y(q, \theta), w'(q, \theta)]$ satisfies the original promise-keeping constraint (1) for any $w$. This ability to state the constraints independent of the promise $w$ allows the following result.

**Lemma 1.**

(1) *The function $C^*(w)$ takes the form*

$$C(\dot{w})=\frac{1}{1-\bar{\delta}\Delta}\frac{-\log(-w)}{\gamma}+H, \qquad (6)$$

*where $H$ is a constant $(C(-1))$ and $\bar{\delta}\equiv\sum_\Theta \delta(\theta)B(\theta)$ is the price-weighted average (over $\theta$) of the price of one-period ahead consumption, $\delta(\theta)$.*

(2) *There exist efficient policies $(a, y(q, \theta), w'(q, \theta))$ which do not depend on the promised utility $w$.*

*Proof.* First, the true cost function $C^*$ can be shown to be a continuous function which everywhere lies between two particular functions of the form in (6). The first, a lower bound of the true cost function, is the cost function associated with the same problem but without the incentive constraint. The second, an upper bound of the true cost function, is the cost function where the action $a$ is constrained to equal the lowest element of $A$. Each of these problems is static, and thus the true cost functions for these problems can be shown to take the form of (6), with (say), $\underline{H}$ equalling the constant term for lower bound, and $\bar{H}$ equalling the constant term for the upper bound. The space of continuous functions lying everywhere between the bounding functions just described is complete and can be normalized by applying the sup norm. Call this normed vector space $S$. The operator $T$ can be shown to be a contraction on $S$ by examining the objective function (3) and noting that Blackwell's sufficient conditions hold. The right-hand side is monotone in $C$ and $T(c+C)=T(C)+\bar{\delta}\Delta c$ for constants $c$. This insures that there is a unique fixed point of $T$ in $S$. The set of functions actually of the form in (6) constitute a complete subset of $S$, say $\hat{S}$. Showing that $T$ maps $\hat{S}$ to itself completes the proof.

Since $z$ can be set to any constant without harming the minimized value of the objective function, set $z$ equal to the constant $-\log(-w)/\gamma$. Second, consider a function

$\tilde{C} \in \hat{S}$ (and thus of the form in (6)) with constant term $\tilde{H}$. Replacing $z$ with $-\log(-w)/\gamma$ in the objective function (3) and substituting in for $\tilde{C}$ allows us to derive

$$(T\tilde{C})(w) \equiv \frac{1}{1-\bar{\delta}\Delta} \frac{-\log(-w)}{\gamma} + \bar{\delta}\Delta\tilde{H}$$
$$+ \min_{a,y(q,\theta),w'(q,\theta)} \sum_{\Theta} \sum_{Q} \left\{ y(q,\theta) - q + \frac{\delta(\theta)\Delta}{1-\bar{\delta}\Delta} \frac{-\log(-w'(q,\theta))}{\gamma} \right\} P(q|a,\theta)B(\theta), \quad (7)$$

where the minimization is, again, subject to (5) and (4). Since the solution to the minimization problem in (7) subject to (5) and (4) is independent of $w$, the sum of the second and third terms on the right-hand side of (7) is simply a constant. Thus we have shown that the operator $T$ preserves the guessed from of $C^*$ in the statement of the Lemma. The fact that the minimization problem in (7) does not contain $w$ implies that the loss-minimizing policies $a^*$, $y^*(q, \theta)$ and $w'^*(q, \theta)$ will not depend on $w$.  ‖

The true constant (say $H^*$) can be found by using the fact that for the true cost function $C^*$, $T(C^*) = C^*$, and thus the true constant $H^*$ equals $1/(1-\bar{\delta}\Delta)$ times the solution to the minimization problem in (7). Given this $H^*$, equation (6) is a fixed point of $T$.

The fact that the loss-minimizing policies $a^*$, $y^*(q, \theta)$ and $w'^*(q, 0)$ will not depend on $w$ implies that all agents received the same recommended action $a$ and face the same conditional payments $y^*(q, \theta)$. Their unconditional payments are determined by the function $z^*(w) = -\log(-w)/\gamma$. Future utilities are determined by the scaling factor $\exp(-\gamma z^*(w)) = -w$, multiplied by the common updating function $w'^*(q, \theta)$. This separation implies that one part of the cost-minimizing plan for the representative firm is to partially pay the agents in terms of positive and negative constant annuities—fixed payments the agent receives every period for the rest of his life. (This characteristic is in Green (1987) and Atkeson and Lucas (1992) for optimal plans.)

To see this, consider an agent with starting utility $w_0$ and thus an initial non-contingent payment $z^*(w_0) = -\log(-w_0)/\gamma$. This agent's expected utility at the beginning of his second period of life can be written $w_1 = -w_0 w'^*(q_0|a, \theta_0)$. Here $q_0$ is the agent's realization of $q$ in his first period of life, and $\theta_0$ is the economy-wide realization of the aggregate shock for that date. It quickly follows that $z(w_1) = -\log(-w_0)/\gamma - \log(-w'^*(q_0, \theta_0))/\gamma$. An agent who "draws" $q_0$ and $\theta_0$ in his first period of life is delivered the unconditional payment $-\log(-w'^*(q_0, \theta_0))/\gamma$ consumption units in the next period along with his original unconditional payment $-\log(-w_0)/\gamma$. In general, his unconditional consumption payment at any given period $t$ is the sum of payments for all previous periods, including his initial payment $-\log(-w_0)/\gamma$, along with the latest addition $-\log(-w'^*(q_{t-1}, \theta_{t-1}))/\gamma$.

*Construction of constant-price equilibrium*

Everything to this point relies on this assumption that the vectors of spot prices $B(\theta)$ and $\delta(\theta)$ are actually constant over time. I now turn to constructing an equilibrium where this is true. Note that by constant price equilibrium I do not mean that *realized* prices are constant, but only that the vectors that realized prices are drawn from are constant.

Assume a single price-taking representative firm which in every period offers all new-born agents a contract with utility $w_0$, makes non-contingent payments $z^*(w) = -\log(-w)/\gamma$ to agents whose current expected utilities equal $w$, makes contingent payments according to $y^*(q, \theta)$, and updates utility promises through the function $w'^*(q, \theta)$ as in the previous

section. That is, assume that the representative firm adopts policies which are optimal given constant prices. I now look to see if markets can clear this restriction.

For markets to clear, this representative firm needs at all dates and states to transfer to agents (in aggregate) exactly as much consumption as the agents produce. Put differently, the aggregate deficit of the representative firm (payments $y + z$ minus outputs $q$) must equal zero regardless of the date or state. The strategy I use is to derive conditions which for a given period $\tau$ cause the deficit of the representative firm to be constant across $\theta$ shocks and constant across the consecutive dates $\tau$ and $\tau + 1$. I then show that these conditions imply themselves for one period ahead, and thus mathematical induction implies the conditions guarantee constant deficits for all dates and states. Lastly, by constraining this deficit to be zero, "market clearing" conditions are derived for the consumption good over dates and states.

At the beginning of each period, everything but the distribution of promised utilities in the economy is the same. Let $\Psi^\tau$ be the distribution of promised utilities at date $\tau$. This distribution is assumed to have total measure $1/(1 - \Delta)$, the size of the population. Let $D^*(\Psi^\tau, \theta)$ denote the deficit of the representative firm (or aggregate deficit) at date $\tau$ given distribution $\Psi^\tau$ and realization $\theta$ of the aggregate shock given policies $[z^*, a^*, y^*(q, \theta)]$.

**Lemma 2.** *Relative deficits across $\theta$ shocks are independent of the distribution of utilities $\Psi^\tau$. That is, for all dates $\tau$ and distributions $\Psi^\tau$, the difference $(D^*(\Psi^\tau, \hat{\theta}) - D^*(\Psi^\tau, \tilde{\theta}))$ is a function only of the shocks $\hat{\theta} \in \Theta$ and $\tilde{\theta} \in \Theta$.*

*Proof.* The aggregate deficit can be written

$$D^*(\Psi^\tau, \theta) = \left\{ \int_W \sum_Q \left( \frac{-\log(-w)}{\gamma} + y^*(q, \theta) - q \right) P(q|a^*, \theta) d\Psi^\tau(w) \right\}$$

$$= \left\{ \int_W \frac{-\log(-w)}{\gamma} d\Psi^\tau(w) \right\} + \frac{1}{1 - \Delta} \left\{ \sum_Q \{ y^*(q, \theta) - q \} P(q|a^*, \theta) \right\}. \quad (8)$$

The separation on the right-hand side of (8) occurs because of the separation of optimal payments into a non-contingent payment $-\log(-w)/\gamma$ which depends on $w$ and a contingent payment $y^*(q, \theta)$ which does not depend on $w$. Thus total net payments are the total of non-contingent payments which depends on the distribution of utilities $\Psi^\tau$, plus the total of net contingent payments which depends only on the expected value of the net contingent payment multiplied by the number of agents, $1/(1 - \Delta)$. Differencing (8) across $\theta$ realizations causes the first term on the right hand side to cancel, proving the Lemma. ∥

An immediate corollary is that if a set of polies $[z^*, a^*, y^*(q, \theta)]$ causes the aggregate deficit $D^*(\Psi^\tau, \theta)$ to be constant across $\theta$ realizations for one distribution $\Psi^\tau$, then these policies will induce a constant deficit across $\theta$ shocks for all distributions $\Psi^\tau$.

While the policies $[z^*, a^*, y^*(q, \theta)]$ affect the deficit at date $\tau$, the policy $w'^*(q, \theta)$ (along with $z^*$) affects the date $\tau + 1$ distribution of utilities and thus the date $\tau + 1$ deficit. Let $G^*(\Psi^\tau, \theta)$ denote the date $\tau + 1$ distribution of utilities given the date $\tau$ distribution $\Psi^\tau$ and the date $\tau$ shock $\theta$ under the policies $[z^*, w'^*(q, \theta)]$.

**Lemma 3.** *If for a given distribution $\Psi^\tau$ at date $\tau$, the deficit of the representative firm $D^*(\Psi^\tau, \theta_\tau)$ equals a constant $d$, regardless of the outcome of $\theta_\tau$, and if for any given*

$\bar{\theta}_{\tau+1}$, *the* $\tau+1$ *deficit* $D^*(G^*(\Psi^\tau, \theta_\tau), \bar{\theta}_{\tau+1})$ *equals this same constant d, regardless of* $\theta_\tau$, *then the deficit at all future dates* $(\tau+2, \tau+3, \dots)$ *will also equal d, regardless of the outcomes of shocks* $(\theta_{\tau+1}, \theta_{\tau+2}, \dots)$.

*Proof.* This proof is based on mathematical induction. I show that if one assumes constant deficits across $\theta$ at date $\tau$ and this same equal deficit at date $\tau+1$, then these assumptions imply themselves one period ahead.

The condition that the date $\tau$ deficit equal a constant $d$ regardless of the outcome of $\theta_\tau$ can be stated as, for all $\theta_\tau \in \Theta$

$$D^*(\Psi^\tau, \theta_\tau) = \left\{ \int_W \sum_Q \left( \frac{-\log(-w)}{\gamma} + y^*(q, \theta_\tau) - q \right) P(q|a^*, \theta_\tau) d\Psi^\tau(w) \right\}$$

$$= \left\{ \int_W \frac{-\log(-w)}{\gamma} d\Psi^\tau(w) \right\} + \frac{1}{1-\Delta} \left\{ \sum_Q \{ y^*(q, \theta_\tau) - q \} P(q|a^*, \theta_\tau) \right\} = d. \quad (9)$$

The requirement that the date $\tau+1$ deficit equal $d$ regardless of the outcome of $\theta_\tau$ can be written as, for all $\theta_\tau$ and any given $\bar{\theta}_{\tau+1}$

$$D^*(G^*(\Psi^\tau, \theta_\tau), \bar{\theta}_{\tau+1})) = \left\{ \int_W \frac{-\log(-w)}{\gamma} dG^*(\Psi^\tau, \theta_\tau)(w) \right\}$$

$$+ \frac{1}{1-\Delta} \left\{ \sum_Q \{ y^*(q, \theta_{\tau+1}) - q \} P(q|a^*, \bar{\theta}_{\tau+1}) \right\} = d. \quad (10)$$

Condition (10) need hold only for one possible $\theta_{\tau+1}$ because condition (9) holding for all $\theta$ implies that the expected value of net conditional payments (the second term on the right-hand side of condition (9)) is constant across $\theta$ realizations. Thus condition (9) holding for all $\theta$ implies that if (10) holds for any particular value of $\theta_{\tau+1}$ it holds for all values of $\theta_{\tau+1}$. The next step in the argument is to show that if the conditions in equations (9) and (10) hold for a distribution $\Psi^\tau$, then this implies they also hold for each of the possible successors of $\Psi^\tau$, $G^*(\Psi^\tau, \theta)$.

Equation (10) is simply a restatement of (9) where the distribution $\Psi^\tau$ has been replaced with a successor distribution $G^*(\Psi^\tau, \theta_\tau)$ and the current shock has been set to a particular value. Thus the assumption of conditions (9) and (10) holding for $\Psi^\tau$ immediately implies that condition (9) holds for each possible successor distribution $G^*(\Psi^\tau, \theta_\tau)$.

To show that equation (10) will also hold for all possible $\Psi^{\tau+1} = G^*(\Psi^\tau, \theta)$ requires some more manipulation. Specifically, if I replace in for the successor function $G^*$, the sum of the unconditional $z$ payments can be written as

$$\int_W \frac{-\log(-w)}{\gamma} dG^*(\Psi^\tau, \theta_\tau)(w)$$

$$= \frac{-\log(-w_0)}{\gamma} + \Delta \left\{ \int_W \sum_Q \frac{-\log(ww'^*(q, \theta_\tau))}{\gamma} P(q|a^*, \theta_\tau) d\Psi^\tau(w) \right\}$$

$$= \frac{-\log(-w_0)}{\gamma} + \Delta \left\{ \int_W \frac{-\log(-w)}{\gamma} d\Psi^\tau(w) \right\}$$

$$+ \frac{\Delta}{1-\Delta} \left\{ \sum_Q \frac{-\log(-w'^*(q, \theta_\tau))}{\gamma} P(q|a^*, \theta_\tau) \right\}. \quad (11)$$

Date $\tau + 1$ unconditional payments will equal the payments to newborns, plus $\Delta$ times the unconditional time $\tau$ payments (since only $\Delta$ fraction of the date $\tau$ population survives), plus the new additions to unconditional payments determined by the function $w'^*$. Equation (11) allows the deficit at $\tau + 1$ (Equation (10)) to be written as

$$d = \frac{-\log(-w_0)}{\gamma} + \Delta \left\{ \int_W \frac{-\log(-w)}{\gamma} \, d\Psi^\tau(w) \right\}$$

$$+ \frac{\Delta}{1 - \Delta} \left\{ \sum_Q \frac{-\log(-w'^*(q, \theta_\tau))}{\gamma} P(q|a^*, \theta_\tau) \right\}$$

$$+ \frac{1}{1 - \Delta} \left\{ \sum_Q \{y^*(q, \bar{\theta}_{\tau+1}) - q\} P(q|a^*, \bar{\theta}_{\tau+1}) \right\}. \tag{12}$$

Since we know that (9) holds for both $\Psi^\tau$ and each of its possible successors $G^*(\Psi^\tau, \theta_T)$, the first term on the right-hand side of (9) (the total of non-contingent payments) must be equal between $\Psi^\tau$ and each of its possible successors. This allows the replacement of $\Psi^\tau$ in (12) with any of its possible successors $\Psi^{\tau+1}$. Reversing the derivations from (10) to (12) derives for all $\theta_\tau$

$$D^*(G^*(\Psi^{\tau+1}, \theta_\tau), \bar{\theta}_{\tau+1})) = \left\{ \int_W \frac{-\log(-w)}{\gamma} \, dG(\Psi^{\tau+1}, \theta_\tau)(w) \right\}$$

$$+ \frac{1}{1 - \Delta} \left\{ \sum_Q \{y^*(q, \bar{\theta}_{\tau+1}) - q\} P(q|a^*, \bar{\theta}_{\tau+1}) \right\} = d. \tag{13}$$

Thus the assumption of conditions (9) and (10) for the distribution $\Psi^\tau$ imply themselves for each possible successor distribution $\Psi^{\tau+1}$. Mathematical induction then implies the constant deficit $d$ at all dates $\tau, \tau + 1, \ldots$ regardless of the outcome of shocks $(\theta_\tau, \theta_{\tau+1}, \ldots)$. ∥

Lemma 3 implies that if a set of constant prices $\{\delta(\theta), B(\theta)\}$ induces policies which satisfy (9) and (10) for any given distribution at any given date, this then ensures constant deficits across all states at all future dates. The prices necessary to induce a constant deficit over all dates and states do not depend on the distribution $\Psi$. Thus we can pick any convenient distribution and find the prices which cause constant deficits for that distribution.

In particular, consider equations (9) and (12) under the assumption that the distribution of utilities is for all generations degenerate at a point $w_0$—the date zero distribution of utilities. Equation (9) becomes for all $\theta$

$$\frac{-\log(-w_0)}{\gamma} + \sum_Q (y(q, \theta) - q) P(q|a^*, \theta) = (1 - \Delta)d. \tag{14}$$

Equation (12) becomes for all $\theta$ and any given $\bar{\theta}$

$$\frac{-\log(-w_0)}{\gamma} + \sum_Q (y(q, \bar{\theta}) - q) P(q|a^*, \bar{\theta})$$

$$+ \Delta \sum_Q \frac{-\log(-w'^*(q, \theta))}{\gamma} P(q|a^*, \theta) = (1 - \Delta)d, \tag{15}$$

or, from noticing that equation (15) is simply equation (14) plus an extra term, (15) becomes for all $\theta$

$$\sum_Q \frac{-\log(-w'^*(q, \theta))}{\gamma} P(q|a^*, \theta) = 0. \qquad (16)$$

Note that equation (16) implies that the aggregate of new unconditional $z$ payments equals zero. Further, since equation (16) is also the expected value of additional $z$ payments given $a^*$ and the realization of $\theta$, (and this holds for all $\theta$), an individual's unconditional $z$ payment follows a random work. Further still, note that an agent's realization of his conditional payment $y(q, \theta)$ is independent and identically distributed over time. Thus his consumption is the sum of a random walk component and an i.i.d. component and thus has the same property of every-increasing cross-sectional variance as a pure random walk.[3]

This implies a step-by-step method of constructing an equilibrium. Step 1 is to find price vectors $\{\delta(\theta), B(\theta)\}$, and loss-minimizing policies given these prices $\{a^*, y^*(q, \theta), w'^*(q, \theta)\}$ such that equation (16) holds and the aggregate deficit is independent of the realization of $\theta$ or for any given $\bar{\theta}$ and all $\theta \neq \bar{\theta}$,

$$\sum_Q \{y^*(q, \bar{\theta}) - q\} P(q|a^*, \bar{\theta}) = \sum_Q \{y^*(q, \theta) - q\} P(q|a^*, \theta). \qquad (17)$$

This condition, equation (17), is a direct implication of (14) holding for all $\theta$. If $\theta$ can take on $n$ possible values, equation (17) implies $n-1$ conditions. The condition that new aggregate $z$ payments sum to zero for all $\theta$ (equation (16)) implies $n$ conditions. These are the correct number of conditions necessary to pin down the $2n$ prices $\{\delta(\theta), B(\theta)\}$ given the restriction that $\sum_\theta B(\theta) = 1$.

Step 2 is to find the initial utility promise $w_0$ which causes markets to clear and which results in zero losses for the representative firm. Since the representative firm is the sole supplier and demander of contingent consumption, if prices are such that the representative firm chooses to run a zero deficit at all dates and states (which is implied by equations (17) and (16)) then markets clear at all dates and states. Further, if the representative firm runs a zero deficit at all dates and states, then its losses equal zero. Thus one can find $w_0$ by simply solving equation (14) for $w_0$ given a constant deficit $d = 0$.

The equilibrium is then defined as price vectors $\{\delta(\theta), B(\theta)\}$, the loss-minimizing policies giving these prices $\{z^*(w), a^*, y^*(q, \theta), w'^*(q, \theta)\}$ and the initial utility $w_0$ constructed as above.

## 4. CHARACTERISTICS OF EQUILIBRIUM

*Comparison to full-information framework*

How do consumption profiles differ in this economy relative to its full information counterpart—the same economy but with observable effort? First, with full information, it is possible there is a higher efficient level of effort, $a^*$. I ignore this in this section and focus only on consumption profiles holding $a^*$ constant. Consumption profiles given full information are characterized by full insurance over idiosyncratic shocks. This implies $y(q_1, \theta) = y(q_2, \theta)$ for all $(q_1, q_2, \theta)$ and $w'(q, \theta) = -1$ for all $q$ and $\theta$. That is individual consumption payments $y$ depend only on the aggregate shock $\theta$ and agents start each

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3. By the sequence $\{z_t\}$ being a random walk, it is meant that first differences are i.i.d. over time and $E(z_{t+1} - z_t | z_t) = 0$).

period with the same expected utility. Individual specific variables such as $q$ cannot be used to predict consumption or consumption changes.

This is not true given unobserved effort. In the equilibrium presented, an individual's consumption at date $t$, $c_t$ can be expressed (where $s$ is the agent's date of birth) as

$$c_t = y(q_t, \theta_t) - \frac{\log(-w_0)}{\gamma} - \sum_{\tau=s}^{t-1} \frac{\log(-w'(q_\tau, \theta_\tau))}{\gamma}, \tag{18}$$

and the first difference, $c_{t+1} - c_t$ as

$$c_{t+1} - c_t = y(q_{t+1}, \theta_{t+1}) - y(q_t, \theta_t) - \frac{\log(-w'(q_t, \theta_t))}{\gamma}. \tag{19}$$

Under unobserved effort, consumption changes depend on individual realization of $q_t$ and $q_{t+1}$, as well as $\theta_t$ and $\theta_{t+1}$. This holds not only because contingent payments change depending on $q_t$ and $q_{t+1}$, but also because of the permanent consumption payment $-\log(-w'(q_t, \theta_t))/\gamma$.

With regard to the distribution of consumption, with identical individuals each causing zero profits for the hiring firm, full information implies that each agent consumes the same as every other. Consumption varies across dates only due to the aggregate shock. If one assumes that individuals start life with different $P(q|a, \theta)$ functions, then it is possible to generate cross-sectional inequality in the full information model. Nevertheless, full insurance implies that the distribution of consumption is simply a function of the current shock $\theta_t$. Under unobserved effort, the distribution of consumption depends on the entire history of aggregate shocks $(\theta_1, \ldots, \theta_t)$. In fact, the evolution of consumption distributions is a Markov chain.

*Long-run properties*

Given that individual consumption levels have a random walk component, one would suspect that the long-run distribution of consumption is degenerate since as time goes on, the variance of consumption across a cohort increases without limit. However, the assumption of a constant death rate and overlapping generations allows more reasonable long-run properties. Overlapping generations and a constant death rate allows the distribution of each generation to continually spread without causing a degenerate limiting distribution of the population as a whole.

The long-run properties of this economy can be characterized as follows: First, one can note that set of continuation utilities an agent can have is countable since there are a finite number he can have at any age. The distribution of promised utilities for the economy at a point in time then is element of the $1/(1-\Delta)$ simplex of $R^\infty$ from the fact that the assumed population size is $1/(1-\Delta)$. This is the state of the economy. (The distributions for each generation do not matter since agents of all ages are identical.)

One can show that if the same aggregate shock is received at all dates, then the distribution of utilities converges to a non-degenerate distribution. For an economy at least $\tau$ dates old, let $\Psi^\tau$ denote the distribution of utilities for agents weakly younger than $\tau$. ($\Psi^\tau$ is an element of the $\sum_{s=1}^\tau \Delta^{s-1}$ simplex of $R^\infty$, again because the size of this segment of the population equals $\sum_{s=1}^\tau \Delta^{s-1}$.) If the aggregate shock is the same at all dates, then distribution $\Psi^\tau$ is independent of the calendar date. The $\lim_{\tau \to \infty} \Psi^\tau$ exists because the total mass of agents older than $\tau$ becomes arbitrarily small.

For general realizations of the sequence of aggregate shocks, the probability of realizing an infinite sequence of aggregate shocks such that the distribution of utilities converges is zero. Young generations constitute a positive percentage of the population and are affected only by recent aggregate shocks, and thus the distribution of utilities is continually buffeted. If an infinite sequence of a particular shock $\theta$ causes a relatively tight limiting distribution of consumption, then a single realization of this shock causes the consumption distribution to tighten. If an infinite sequence of a different shock $\theta$ causes a relatively wide limiting distribution of consumption, then a single realization of this shock causes the consumption distribution to spread.

Even though there is not a limiting distribution of consumption, there is a limiting probability distribution over the distribution of consumption. This is proved by proving the existence of a limiting probability distribution over the distribution of utilities. For an economy at least $\tau$ periods old, let $\mu_{w_i}^{\tau}$ denote the probability measure over the mass of agents weakly younger than $\tau$ periods on utility point $w_i$.[4] That is if $\mu_{w_i}^{\tau}([x, y]) = p$, then there is a probability $p$ that the mass of agents weakly younger than $\tau$ on utility point $w_i$ is in the interval $[x, y]$. This probability does not depend on the date since the distribution of utilities for agents weakly younger than $\tau$ periods depends only on the $\tau$ most recent shocks.

The collection of probability measures $\mu_{w_i}^{\tau}$ over all possible utility points $w_i$ defines a probability measure over the possible utility distribution for agents weakly younger than $\tau$. Again, the $\lim_{\tau \to \infty} \mu_{w_i}^{\tau}$ exists because the total mass of agents older than $\tau$ becomes arbitrarily small. This implies a limiting measure on utility distributions and thus consumption distributions.

## 5. OPTIMA

This section considers the social planner's problem for this economy and whether the equilibrium of the previous section solves this problem. The first result presented is that, like the representative firm's problem, the social planner's problem for this economy takes a recursive form (Lemma 4). This then allows proof that the equilibrium of the previous section does indeed solve the social planner's problem (Lemma 5). Besides being of interest in and of itself, this is desirable from a purely practical viewpoint since analyzing and computing optima tends to be easier than analyzing and computing equilibria. Stating the social planner's problem in a form useful for computing equilibrium outcomes and prices, however, requires somewhat further argument. To this end, a functional form for the cost function of the social planner is derived (Lemma 6). Given this result, the section concludes by presenting a reformulation of the social planner's problem such that the equilibrium prices fall out as functions of the Lagrange multipliers on the constraints faced by the social planner.

The general methodology used in this section (following Atkeson and Lucas (1992)) is to consider the social planner as a sort of "residual claimant" on the resources of the economy. Specifically, consider the social planner as having an ability to contribute a constant amount of the consumption good (positive or negative) to the economy at each date and state. The social planner's problem is then to minimize this constant contribution to the economy (or maximize his constant extraction from the economy) subject to constraints regarding the ex ante utilities of the agents as well as the incentive conditions

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4. Measure $\mu_{w_i}^{\tau}$ maps the Borel subsets of $[0, \sum_{s=1}^{\tau} \Delta^{s-1}]$ to $[0, 1]$.

regarding unobserved effort. If the representative firm's plan from the previous section is not an optimal social plan, then the social planner should be able to extract a positive constant amount from the economy and still give each agent the ex ante utility he expects from the equilibrium. On the other hand, if the representative firm's plan from the previous section is optimal, the maximum amount the social planner can extract (again subject to utility and incentive constraints) should be zero.

It is assumed that the planner must treat all agents of all generations equally, or that each agent receives the same ex ante expected discounted utility $\bar{w}_0$.[5] In the course of proving that the equilibrium plan is optimal subject to this constraint, it is first shown that the social planner's problem takes a recursive form where the state variables of the economy are the distribution of utilities owed by the social planner to currently living agents, and (implicitly) the ex ante utility owed by the planner to agents born in future periods, $\bar{w}_0$. To this end, let $B$ be the set of Borel-measurable distributions with total measure $1/(1-\Delta)$, the size of the population of living agents at any time, and let $\Psi \in B$ represent a distribution of continuation expected discounted utilities for currently living agents.

Define $X^*(\Psi): B \to R$ as the minimum constant contribution by the social planner which allows the provision through an incentive compatible plan of $\Psi$ to currently living agents and the delivery of $\bar{w}_0$ to agents born in future periods. I define an efficient plan (or an optimum) as a specification of contingent consumption and recommended effort levels that achieves a given distribution of utilities $\Psi$ with the minimum constant contribution by the social planner, $X^*(\Psi)$.[6]

The claim that the social planner's problem is recursive is essentially that the cost function $X^*(\Psi)$ can be stated as a function of itself in a Bellman-type equation. Given this, the distribution of promised utilities to currently living agents $\Psi$ completely summarizes history, allowing the planner to choose one-period plans conditional on this distribution and a rule to generate new promised utilities.

In this recursive formulation, for a given distribution $\Psi$, let the choice variables for the social planner be the same as for firms, except now let each of the functions depend explicitly on $w$. That is, the social planner chooses functions $z(w)$, $a(w)$, $y(w, q, \theta)$, and $w'(w, q, \theta)$. As with the representative firm's problem, let $z(w)$ represent an unconditional payment and let promised future utilities equal $\exp(-\gamma z)w'(w, q, \theta)$. (Again, $z$ can always be set to zero.)

The following Lemma proves that $X^*(\Psi)$ indeed solves a Bellman-type equation. Essentially, this Lemma argues that the minimum constant contribution necessary to support a plan $[z(w), a(w), y(w, q, \theta), w'(w, q, \theta)]$ is the *maximum* of the contribution necessary today for each possible $\theta$ realization, and the constant contribution necessary to support each of the possible utility distributions for tomorrow. Again, let $G(\Psi, \theta)$ denote the date $\tau + 1$ distribution of utilities if the date $\tau$ distribution is $\Psi$ and shock $\theta$ occurs, given the policies $z(w)$ and $w'(w, q, \theta)$ for generating the $\tau + 1$ utility distribution.

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5. This assumption throws out all true optima if all of them have the property that generations are treated unequally. That is, with overlapping generations it might be possible through intergenerational transfers to raise the utility of later generations without lowering the utility of earlier generations. Since this model has an initial date $\tau = 0$, discounting, and equal size generations (after the first), I believe I have ruled out such schemes.

6. Atkeson and Lucas (1992) define the cost of a given policy as the *supremum* over dates of the necessary gift, and the true cost of a distribution as the infimum of the cost of all possible policies. Thus they do not assume that any cost can actually be attained, only arbitrarily closely attained. This did not substantially change their results, so here I simply assume for convenience that all maxima are attained.

**Lemma 4.** *The cost function $X^*(\Psi)$ is a fixed point of the following operator $\bar{T}$ defined as follows:*

$$(\bar{T}X)(\Psi) = \min_{z(w),a(w),y(w,q,\theta),w'(w,q,\theta)} \max_{\theta} \max \left\{ X(G(\Psi, \theta)), \right.$$

$$\left. \times \int_W \sum_Q \{z(w) + y(w, q, \theta) - q\} P(q|a(w), \theta) d\Psi(w), X(G(\Psi, \theta)) \right\} \quad (20)$$

*subject to the promise-keeping constraints implied by* (1) *and the incentive constraints implied by* (2) *holding for $w \in W$.*

*Proof.* The proof is nearly identical to the proof in Atkeson and Lucas (1992), Lemma 4.1, and is in the Appendix. ‖

An optimum can be defined as the policies $[z(w|\Psi)$, $a(w|\Psi)$, $y(w, q, \theta|\Psi)$, $w'(w, q, \theta|\Psi)]$ which solve (20) subject to (1) and (2) for all possible distributions $\Psi$, and an initial utility $\bar{w}_0$ such that $X(\bar{\Psi}_0) = 0$, where $\bar{\Psi}_0$ is the distribution where all mass lies at the point $\bar{w}_0$. This result now allows the proof that equilibrium of the previous sections is indeed an optimum.

**Lemma 5.** *The equilibrium policies of the representative firm $[z^*(w) = -\log(-w)/\gamma, a^*, y^*(q, \theta), w'^*(q, \theta)]$, together with the equilibrium initial promised utility $w_0$, are an optimum.*

*Proof.* The proof argues first that the representative firm's policies are a feasible solution to the social planner's problem and imply a cost $X^*(\bar{\Psi}_0) = 0$ for the social planner. If these policies are not an optimum then there must exist another set of feasible policies which allow the social planner to extract a constant positive amount from the economy. This is shown to contradict the assumed cost minimization by the representative firm. A more detailed argument is in the appendix. ‖

At this point, the Bellman equation (20) is not in a particularly useful form for calculating or analyzing equilibria by finding optima. The following Lemma solves for the social planner's cost function $X^*$ up to constant.

**Lemma 6.** *The function $X^*$ takes the form*

$$X(\Psi) = \int_W \frac{-\log(-w)}{\gamma} d\Psi(w) + K, \quad (21)$$

*where*

$$K = \min_{a,y(q,\theta),w'(q,\theta)} \{\sum_Q \{y(q, \bar{\theta}) - q\} P(q|a, \bar{\theta})\}, \quad (22)$$

*for any particular $\bar{\theta} \in \Theta$ and where the minimization is subject to the incentive condition* (4), *the promise keeping condition* (5), *the constant payments conditions for all $\theta \in \Theta$,* (16), *and the constant current deficit condition, equation* (17) *for all $\theta \neq \bar{\theta}$.*

*Proof.* See Appendix. ‖

The formation allows us, with one qualification, to find the market clearing prices $\{\delta(\theta), B(\theta)\}$ using the Lagrange multipliers associated with the constant deficit constraints (17) (to get $B(\theta)$) and the constant payments constraint (16) (to get the $\delta(\theta)$). The qualification is that the constraint set for the minimization problem in (22) must be convex and thus we can invoke the Kuhn–Tucker theorem. One way to avoid convexity problems is to formulate the choice problem in terms of lotteries as in Phelan and Townsend (1991). This is avoided here solely for ease of exposition.

The Kuhn–Tucker theorem allows us to state that there exist multipliers $\lambda_\theta$ for $\theta \neq \bar{\theta}$ and $\mu_\theta$ for each $\theta \in \Theta$ such that the optimal policies $[a^*, y^*(q, \theta), w'^*(q, \theta)]$ solve

$$\min_{a, y(q,\theta), w'(q,\theta)} \left\{ \sum_Q \{y(q, \bar{\theta}) - q\} P(q|a, \bar{\theta})\} + \sum_{\theta \neq \bar{\theta}} \lambda_\theta \{(\sum_Q \{y(q, \bar{\theta}) - q\} P(q|a, \bar{\theta})) \right.$$

$$\left. - (\sum_Q \{y(q, \theta) - q\} P(q|a, \theta))\} + \sum_Q \mu_\theta \left\{ \sum_Q \frac{-\log(-w'(q, \theta))}{\gamma} P(q|a, \theta) \right\}, \quad (23)$$

subject to the incentive condition (4) and the promise keeping condition (5), and, in addition, that each of the market clearing constraints (equations (17) and (16)) holds for all $\theta$.

This Lagrangian is a particularly useful form of the social planner's problem. In this form, the planner's problem has the same constraint set as the firm's problem. Further, the objective functions in the two problems differ only in the coefficients on the moments

$$\sum_Q \{y(q, \theta) - q\} P(q|a, \theta) \quad \text{and} \quad \sum_Q \frac{-\log(-w'(q, \theta))}{\gamma} P(q|a, \theta). \quad (24)$$

Thus to give the firm the same objective function as the planner, one simply needs to choose prices to equate the coefficients on these moments or (from equations (7) and (23))

$$\text{for } \bar{\theta}, B(\bar{\theta}) = 1 - \sum_{\theta \neq \bar{\theta}} \lambda_\theta, \quad \text{for } \theta \neq \bar{\theta}, B(\theta) = \lambda_\theta, \quad \text{and for all } \theta, \frac{\delta(\theta)\Delta}{1 - \bar{\delta}\Delta} = \mu_\theta. \quad (25)$$

Since $\bar{\delta} = \sum_\theta B(\theta)\delta(\theta)$, the $\delta(\theta)$ are implied by the system of equations defined by the last expression in (25) for all $\theta \in \Theta$.[7]

This derivation allows us to easily find market clearing prices by solving this planner's problem. Further, the convexity assumption can be checked ex post. That is, if one derives a solution to the optimum problem allowing for probabilistic choice, but the optimal policies are nevertheless deterministic in the form above, then any non-convexity in the constraint set given deterministic choice is harmless and one can then use the above rules to derive prices $\{B(\theta), \delta(\theta)\}$.

## 6. CONCLUSION

The purpose of this paper was to create a tractable framework to make predictions regarding the dynamics of consumption and distributions of consumption. Of course, it is possible to think of other ways of generating a non-trivial distribution theory. One would be to simply close markets over state-contingent consumption (or insurance markets), but allow trading over dates (or credit markets). This would not necessarily be easier to solve since one would still face the problem that without special assumptions, interest rates will be a

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7. If the $\bar{\delta}$ is brought to the right-hand side by multiplying each side of the expression in (25) by $(1 - \bar{\delta}\Delta)$ then the expression is linear in the $\delta(\theta)$ and thus the system has a unique solution.

function of the distribution of wealth. Further, normative analysis would be hindered by the fact that one has left unexplained why it is that insurance is hindered.

In this model, limited insurance comes directly from the informational assumptions of the model. Given these, the equilibrium is efficient. The equilibrium here most likely has different consumption series implications than those implied by simply closing insurance markets. The best insurance possible given information constraint—the equilibrium presented here—is not only more insurance than would exist given closed insurance markets, but should also be expected to look different in regard to how consumption moves with aggregate shocks. The creation of models with specific differing predictions allows these differences to be evaluated in relation to data.

## APPENDIX

*Proof of Lemma* 4.   Again, let $X^*(\Psi)$ be the true minimum constant gift which allows the attainment of the distribution of utilities $\Psi$ and for the delivery of utility $\bar{w}_0$ to agents born in future periods. The Lemma states that $\bar{T}(X^*) = X^*$. First I show that $\bar{T}(X^*) \geq X^*$.

Suppose that for some $\Psi$, $(\bar{T}X^*)(\Psi) < X^*(\Psi)$. This implies that there exist functions $[z^0(w)$, $a^0(w), y^0(w, q, \theta), w'^0(w, q, \theta)]$ such that

$$X^*(\Psi) > \max_\theta \max \left\{ \int_W \sum_Q \{z^0(w) + y^0(w, q, \theta) - q\} P(q|a^0(w), \theta) d\Psi(w), X^*(G^0(\Psi, \theta)) \right\}, \qquad (26)$$

where $G^0(\Psi, \theta)$ is the distribution of utilities generated by $[z^0(w), w'^0(w)]$ given shock $\theta$.

Let the possibly time-dependent plan $[z_t^1(w), a_t^1(w), y_t^1(w, q, \theta), w'_t^1(w, q, \theta)]_{t=0,\infty}$ denote the policy that actually achieves $X^*(\Psi)$. Define a new time-dependent plan $[z_t^0(w), a_t^0(w), y_t^0(w, q, \theta), w'_t^0(w, q, \theta)]_{t=0,\infty}$ by following plan $[z^0, a^0, y^0, w'^0]$ at the first date and plan $[z_t^1, a_t^1, y_t^1, w'_t^1]$ for each date thereafter. This is a feasible plan that actually delivers the required utilities, and does so at a cost $(TX^*)(\Psi) < X^*(\Psi)$ which is a contradiction. Thus, $(TX^*)(\Psi) \geq X^*(\Psi)$.

Now suppose for some $\Psi$ that $X^*(\Psi) < (\bar{T}X^*)(\Psi)$. Let $[z_t^0(w), a_t^0(w), y_t^0(w, q, \theta), w'_t^0(w, q, \theta)]_{t=0,\infty}$ denote the policy that actually achieves $X^*(\Psi)$. The first date of this plan, $[z_0^0, a_0^0, y_0^0, w'_0^0]$, satisfies the incentive constraints and promise keeping constraints for the recursive formulation. Further, its date-zero deficit must be weakly less than $X^*(\Psi)$ given distribution $\Psi$ and any $\theta$. Further, since $[z_t^0(w), a_t^0(w), y_t^0(w, q, \theta), w'_t^0(w, q, \theta)]_{t=0,\infty}$ has a deficit weakly less than $X^*(\Psi)$ in every period, the cost of each of its possible successors under $[z_0^0, w'_0^0]$, $G^0(\Psi, \theta)$, must be weakly less than $X^*(\Psi)$. This implies that

$$\max_\theta \max \left\{ \int_W \sum_Q \{z_0^0(w) + y_0^0(w, q, \theta) - q\} P(q|a_0^0(w), \theta) d\Psi(w), X^*(G^0(\Psi, \theta)) \right\} \leq X^*(\Psi) < (\bar{T}X^*)(\Psi). \quad (27)$$

Since $[z_0^0, a_0^0, y_0^0, w'_0^0]$ is a feasible recursive plan, this contradicts the minimization in the definition of $\bar{T}$. Thus, $(TX^*)(\Psi) \leq X^*(\Psi)$.   ‖

*Proof of Lemma* 5.   Consider the solution to the firm's problem. There $z^*(w) = -\log(-w)/\gamma$, and the policies $[a^*, y^*(q, \theta), w'^*(q, \theta)]$ solved the minimization problem in (7) subject to (5) and (4). This implies that the combination $[z^*(w), a^*, y^*(q, \theta), w'^*(q, \theta)]$ satisfies (2) and (1), or that the solution to the equilibrium problem is a feasible solution to the optimum problem. The plans offered by the firm are incentive compatible and give each agent his required ex-ante utility.

Further, recall that the solution to the equilibrium problem delivers zero aggregate deficits at all dates and states. Let $\Psi_0$ be the distribution of utility promises the representative firm faces on the first calendar date (all mass on $w_0$). Since the firm's plan is a feasible plan for the social planner, the planner can at least achieve a maximum deficit over all dates and states of zero, or $X^*(\Psi_0) \leq 0$. This implies that the representative firm's plan is not optimal only if there is another plan which allows a negative constant payment, or that $X(\Psi_0) < 0$.

Now suppose exactly that. This implies there exists a solution to (20) (say $[\hat{z}(w|\Psi), \hat{a}(w|\Psi), \hat{y}(w, q, \theta|\Psi), \hat{w}'(w, q, \theta|\Psi)]$ which delivers strictly negative deficits for all dates and states. Could this plan have been implemented by the representative firm? Since I assumed (and confirmed) that the prices $\{\delta(\theta), B(\theta)\}$ did not depend on the distribution $\Psi$, the representative firm did not condition policies on $\Psi$. Nevertheless, I did not constrain the representative firm to choose policies independent of $\Psi$. It was simply not in its interest to do

so. Further, although the representative firm optimally chooses policies which are independent of $w$, again, this was not arbitrarily imposed, but derived from the choice problem. Thus the representative firm could have chosen to implement $[\hat{z}(w|\Psi), \hat{a}(w|\Psi), \hat{y}(w, q, \theta|\Psi), \hat{w}'(w, q, \theta|\Psi)]$.

Further, if these policies produce negative deficits at every date and every state, then if at least one element of $\{\delta(\theta), B(\theta)\}$ is positive and all others are non-negative (a condition which can be checked for any candidate equilibrium) then this policy will produce positive profits for the representative firm. This contradicts the optimality of $[z^*(w), a^*, y^*(q, \theta), w'^*(q, \theta)]$ from the perspective of the representative firm. This contradiction implies that the equilibrium constructed in the previous section is optimal given that it has non-negative prices and at least one positive price.[8]  ‖

*Proof of Lemma* 6.   As in the proof of Lemma 1, I exploit a bounding function of $X^*$ which takes the form of (21). Let $\underline{X}(\Psi)$ with constant $\underline{K}$ denote the cost function associated with the removal of the incentive conditions.

Imposing restrictions on the minimization problem defining $\bar{T}$ (equation 20) that $z(w) = -\log(-w)/\gamma$ and that policies do not depend on $w$ or $\Psi$ defines a new operator $\tilde{T}(X)$. Applying $\tilde{T}$ to a function of the form in (21) implies

$$(\tilde{T}X)(\Psi) = \int_W \frac{-\log(-w)}{\gamma} d\Psi(w) + \min_{(a, y(q, \theta), w'(q, \theta))} \max_\theta \max \left\{ \frac{1}{1-\Delta} \sum_Q (y(q, \theta) - q) P(q|a, \theta), \right.$$

$$\left. \times \frac{1}{1-\Delta} \sum_Q \frac{-\log(-w'(q, \theta))}{\gamma} P(q|a, \theta) + K \right\}, \tag{28}$$

subject to the promise keeping constraints (5) and incentive constraints (4) from the firm's problem. If the true cost function $X^*$ is of the form in (21) it must be a fixed point of $\tilde{T}$ as well as $\bar{T}$ from Lemma 5. Equation (28) also shows that the operator $\tilde{T}$ preserves the guessed form (21), since the solution to the minimization problem does not depend on the distribution $\Psi$.

Continuing in this line, one can impose condition (16), that new-contingent payments $w'(q, \theta)$ integrate to zero for all $\theta$. Again, this implies a new operator (say $\hat{T}$) for which $X^*$ must again be a fixed point if it is of the form in (21). This implies that (28) becomes

$$(\hat{T}X)(\Psi) = \int_W \frac{-\log(-w)}{\gamma} d\Psi(w) + \min_{(a, y(q, \theta), w'(q, \theta))} \max_\theta \max \left\{ \sum_Q (y(q, \theta) - q) P(q|a, \theta), K \right\}, \tag{29}$$

subject to the promise keeping constraints (5), the incentive constraints (4), and condition (16). This makes transparent that for functions of the form in (21), operator $\hat{T}$ is non-decreasing in the constant term $K$.

Now consider $X_1 = \hat{T}(\underline{X})$, or applying operator $\hat{T}$ to the cost function given no incentive constaints (which is of the guessed form (21) with constant term $\underline{K}$). Since $\hat{T}(\underline{X})$ must be weakly greater than $\underline{X}$ from the incentive constraints in the minimization problem defining $\hat{T}$, one can discard the second part of the inner maximization and write

$$(\hat{T}\underline{X})(\Psi) = \int_W \frac{-\log(-w)}{\gamma} d\Psi(w) + \min_{(a, y(q, \theta))} \max_\theta \sum_Q (y(q, \theta) - q) P(q|a, \theta), \tag{30}$$

where, again, the minimization is subject to conditions (5), (4), and (16). Let $K_1$ be the constant term associated with $X_1$, equal to the minimization in (30). Substituting this definition for $K_1$ into (29) implies that $X_1$ is a fixed point of $\hat{T}$.

At this point, it is clear that the true cost function $X^*$ is a fixed point of $\hat{T}$ if it is of the guessed form (21). The proof continues by showing that all fixed points of $\hat{T}$ other than $X_1$ cannot be the true cost function and thus either $X_1 = X^*$, or $X^*$ is not of the guess form. Further examination of (29) shows that every function of the form in (21) with $K > K_1$ but less than or equal to the upper bound is also a fixed point of $\hat{T}$. Nevertheless, since the policies associated with $X_1$ actually do achieve any distribution of utilities $\Psi$ with a constant gift less than or equal to $X_1(\Psi)$, only $X_1$ (the minimum of this set of fixed-point cost functions) can be a candidate for

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8. Some readers have commented that this result is surprisingly easy to prove given the usual difficulty of proving the First Welfare Theorem in infinite-good settings, especially with overlapping generations. What makes this proof easy is that 1) the firms live in every date, and 2) that an optimum here is defined very differently than usual. Here for one allocation to constitute an improvement over another allocation, it must use fewer resources at every date and state. The special cases that cause problems in the usual First Welfare Theorem proofs will probably cause the duality approach to break down here.

the true cost function $X^*$ among the class of functions of the form of (21). The function $X_1$ is an upper bound of the true cost function.

There are possibly other fixed points of $\bar{T}$ not of the form of (21). Let $\bar{X}$ be such a fixed point. If for some $\Psi$, $\bar{X}(\Psi) > X_1(\Psi)$, then $\bar{X}(\Psi)$ cannot be the true cost function. Again, since $X_1(\Psi)$ is an upper bound on the true cost, any function that is anywhere greater than $X_1$ cannot be the true cost function. Lastly, since $\bar{T}$ is monotonic (if $X_2(\Psi) \leq X_3(\Psi)$ for all $\Psi$ this implies $(\bar{T}X_2)(\Psi) \leq (\bar{T}X_3)(\Psi)$ for all $\Psi$), it is not possible that for all $\Psi$, $\bar{X}(\Psi) \leq X_1(\Psi)$. The statements $(\bar{T}(\bar{X}) = \bar{X})$, $(\bar{T}(X_1) = X_1)$, and (for all $\Psi$, $X_1(\Psi) \geq \bar{X}(\Psi)$) together violate monotonicity. Thus if there are any fixed points of $\bar{T}$ not of the form in (21), they are not the true cost function. Thus $X^* = X_1$.

Given that we have derived $X^*$ as $\hat{T}(\underline{X})$, the final step is to reconcile the expression for the constant term in equation (30) with the constant term in the statement of the Lemma. Since $X^*$ is the true cost function, imposing a constraint that the current period deficit be constant across $\theta$ is harmless because this a characteristic of the constructed market equilibrium. This allows us to specify the social planner's problem as one of minimizing the deficit given a specific $\theta$ realization, say $\bar{\theta}$, subject to the constraint that the deficit given other $\theta$ values equals this amount. Using equation (30) this implies

$$X^*(\Psi) = \int_W \frac{-\log(-w)}{\gamma} d\Psi(w) + \min_{a, y(q, \theta), w'(q, \theta)} \{\textstyle\sum_Q \{y(q, \bar{\theta}) - q\} P(q|a, \bar{\theta})\}, \tag{31}$$

where the minimization is subject to the incentive condition (4), the promise keeping condition (5), the constant payments conditions for all $\theta \in \Theta$, (16), and the constant current deficit condition, equation (17) for all $\theta \neq \bar{\theta}$.  ‖

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