# LECTURE NOTES OF FABRIZIO ZILIBOTTI

Note: these notes are produced for didactical purposes only. They are supposed to help the students to identify the topics which receive most stress in the course. They contain no original contribution of the Author, and should not be quoted. They should not be regarded as a substitute of the textbook (Barro R. and X. Sala-i-Martin: "*Economic Growth*", McGraw-Hill, 1995) from which many parts are drawn, nor of the articles.

## 1 Introduction.

1. Sustained differences in growth rates (even though seemingly small) have a very large impact on standards of living. The yearly average growth rate of GDP p.c. in Sweden between 1870-1990 was about 2% (1.96%, to be precise). If Sweden had grown, instead, at the speed of the UK (1.34%) Sweden's GDP p.c. would now be similar to that of Portugal and Greece, whose GDP p.c. in 1990 was about 48% of that of Sweden. Had Sweden grown like India from 1900-87 (0.84%), it would now be a country with standards of living comparable to Thailand, Costa Rica or Brasil (26.5%). Had it grown at a "Japanese" rate (2.95%) Sweden would be more than 3 times as rich as it is now.

2. Large cross-country differences in both GDP p.c. and growth rates.

   **(a)** In 1992, Chad was the poorest countries in the world, according to the Penn data base (http://www.nber.org/pwt56.html). Its GDP p.c. was 45 times lower than the GDP p.c. of the United States.

   **(b)** In the period 1960-92, six countries (Hong Kong, South Korea, Malta, Romania, Singapore and Taiwan) have experienced growth rates in GDP p.c. higher than 5% per year, and have increased their average living standard by a factor between 4.8 and 7.5.

Twenty countries, to the opposite extreme, were poorer in 1992 than in 1960.

3. Although some countries are successful and others are not, the spread of the distribution of world GDP p.c. across countries is fairly stable. Here is a table reporting percentiles of income p.c. distribution across countries. It is very stable, although there is some tendency of rich countries to "converge".

| perc. | 1960 | 1965 | 1970 | 1975 | 1980 | 1985 | 1990 |
|---|---|---|---|---|---|---|---|
| 5 | 0.04 | 0.035 | 0.033 | 0.037 | 0.033 | 0.032 | 0.031 |
| 25 | 0.078 | 0.076 | 0.069 | 0.07 | 0.068 | 0.064 | 0.067 |
| 50 | 0.139 | 0.137 | 0.146 | 0.174 | 0.182 | 0.157 | 0.16 |
| 75 | 0.29 | 0.288 | 0.333 | 0.402 | 0.393 | 0.334 | 0.373 |
| 95 | 0.757 | 0.741 | 0.745 | 0.797 | 0.781 | 0.797 | 0.822 |

4. Persistence and "disasters".

5. Growth is more volatile in poor than in rich countries.

6. A number of factors have been found to correlate with growth performance. Among them, invt.to GDP ratio; human capital; political instability; property right enforcement; financial development.

7. Invt. to GDP ratio tend to increase with develoment (the US being an exception).

   **Questions:**

   **(a)** why large differences in growth rates?

   **(b)** why persistent differences (productivity differences)?

   **(c)** what drives growth in the world overall?

The neoclassical growth theory (NGT) as developed by Solow (1956) and his followers put the main emphasis on the role of savings and physical capital accumulation. In the late 80's and throughout the 90's, this theory came under the fire of criticism – starting with the work of Paul Romer, Gene Grossman and Elhanan Helpman – which originated a new body of theoretical and empirical work that has become known as "endogenous growth theory". The new theory has moved from two main objections to the traditional approach:

1. from an empirical standpoint, NGT is argued to fail to explain in a satisfactory way the enormous disparities of level and growth rates of per capita income across countries;

2. from a theoretical standpoint, NGT is argued to fail to explain the determinants of technological advancement, which is the most important factor to understand the long-run performance of modern economies. This failure cannot be addressed by simple extensions of the traditional model. In particular, it is necessary to abandon the environment in which traditional theory was developed, i.e. perfect competition.

# 2 The Solow model.

- Some stylized facts about the relationship between capital accumulation and growth (Kaldor, 1963), especially in developed countries:

  1. Physical capital per worker grows over time.
  2. The ratio of physical capital to output is nearly constant (in the long-run).
  3. The shares of labor and physical capital in national income are nearly constant.
  4. The growth rate of output per worker differs substantially across countries

- The Solow model can claim a good success in fitting 1-2-3, and some success (more controversial) in fitting 4.

## 2.1 The environment.

- Robinson Crusoe economy: a household/producer owns inputs and manages the technology transforming inputs into outputs.
- Unique consumption good, which can be turn one-to-one into capital (capital is perfectly reversible)
- Capital depreciates at the constant rate, $\delta > 0$
- Exogenous saving rate.
- Our agent saves a constant fraction of the output flow. The fundamental equation of the Solow model:

$$\dot{K} = I - \delta K = sF(K, L, t) - \delta K$$

- Assume: no technical progress to start with. Thus,

$$F(K, L, t) = F(K, L)$$

- Exogenous exponential population growth: $L_t = L_0 e^{nt} = e^{nt}$ (set $L_0 = 1$)
- The production function has the following properties:

  1. Diminishing returns to each input:

$$\frac{\partial F}{\partial K} > 0, \quad \frac{\partial^2 F}{\partial K^2} < 0,$$
$$\frac{\partial F}{\partial L} > 0, \quad \frac{\partial^2 F}{\partial L^2} < 0.$$

2. Constant Returns to Scale (CRS):

$$F(\lambda K, \lambda L) = \lambda F(K, L).$$

3. Inada Conditions:

$$\lim_{K\to 0}(F_K) = \lim_{L\to 0}(F_L) = \infty,$$
$$\lim_{K\to \infty}(F_K) = \lim_{L\to \infty}(F_L) = 0.$$

- Examples:

1. Cobb-Douglas: $Y = AK^{\alpha}L^{1-\alpha}$
2. CES: $Y = \left(a \cdot K^{\theta} + b \cdot L^{\theta}\right)^{1/\theta}$ ... under which restriction?

- Let $z \equiv Z/L$. Using CRS, we can write:

$$Y/L = F(K/L, 1)$$

$$y = f(k), \quad f'(k) > 0, \quad f''(k) < 0,$$

where, elementary calculus yields:

$F_K = f('k)$

$F_L = [f(k) - k \cdot f('k)]$

- Rewrite the fundamental equation as follows:

$$\dot{K}/L = sF(K, L)/L - \delta K/L = sf(k) - \delta k$$

Then, note that:

$$\dot{k} \equiv \frac{d(K/L)}{dt} = \dot{K}/L - \left(K/L^2\right)\dot{L} = \dot{K}/L - nk$$

Thus, $\dot{K}/L = \dot{k} - nk$, and:

$$\dot{k} = sf(k) - (n + \delta)k$$

FIGURE 1.1 BX

- Steady-state:

$$s \cdot f(k^*) = (n+\delta) k^*$$
$$y^* = f(k^*)$$

No long-run growth in per capita terms ($\gamma_k = \gamma_y = \gamma_c = 0$)!

## 2.2 Golden rule and dynamic inefficiency.

- "Do unto others as you would have others do unto you", i.e., "choose $s$ so as to maximize consumption under the constraint that all future generations can consume at least as much as you do".

- We look for the maximum sustainable steady-state per capita consumption.

$$c^* = (1-s) f(k^*) = f(k^*) - (n+\delta) k^*$$

- Since (given $n, \delta$) there is one-to-one mapping between $s$ and $k^*$, we can, identically, set the problem in terms of finding the steady-state capital, $k^*$, which maximizes $c^*$.This is given by:

$$f'(k_{gold}) = (n+\delta)$$

which implicitly defined $k_{gold}$ (e.g., under Cobb Douglas: $k_{gold} = (\alpha/(n+\delta))^{1/(1-\alpha)}$). Given $k_{gold}$, we can easily derive:

$$s_{gold} = (n+\delta) k_{gold}/f(k_{gold})$$

FIGURE 1.3 BX

- Are some saving rates "more desirable" than others? Here we do not derive savings from an explicit optimization problem. We can say, however, that an economy is **dynamically inefficient** if it is possible to increase consumption of the current generation without reducing consumption of future generations.

DRAW CONSUMPTION PATH
(first jump up, and then decline to a
higher steady-state cons. than the initial level)

## 2.3 Transitional dynamics without technical progress.

- Define $\gamma_z \equiv \frac{d\ln(z_t)}{dt} = \frac{\dot{z}_t}{z_t}$
- The relationship between output growth and capital growth is

$$y_t = f(k_t) \quad \Rightarrow \quad \gamma_y = [k_t \cdot f'(k_t) / f(k_t)]\, \gamma_k$$

- ... particularly simple under Cobb-Douglas:

$$y = Ak^\alpha \Rightarrow \gamma_y = \alpha \cdot \gamma_k$$

From the fundamental equation of the model:

$$\gamma_k \equiv \frac{\dot{k}}{k} = s \cdot f(k)/k - (n + \delta)$$

where $f(k)/k$ is a decreasing function of $k$ since $f(k)$ is concave. Thus, $\frac{\partial}{\partial k}\gamma_k < 0$.

FIGURE 1.4 BX: Transitional dynamics

FIGURE 1.5 BX: effect of an increase of $s$

- 1. **Absolute convergence:** poor countries (low $y$, low $k$) tend to grow faster than rich countries
  2. **Conditional convergence:** for equal values of $s$, $n$ and $\delta$, poor countries (low $y$, low $k$) tend to grow faster than rich countries.

FIGURE 1.9 BX: conditional convergence

- The data: no absolute convergence across countries; absolute convergence across regions in Europe, and across states in the US.

- Before analyzing conditional convergence in detail, we introduce technical progress.

## 2.4 Exogenous technical progress.

- Assume labor-augmenting technical progress (other types of technical progress are not consistent with a steady-state, nor with constant $K/Y$)

$$\dot{K}_t = s \cdot F(K_t, A_t L_t)$$
$$A_t = A_0 e^{xt}$$

- Define $\hat{z}_t \equiv z_t / A_t$. Thus, dividing variables by $A_t L_t$ (rather than by just $L_t$), we rewrite the fundamental equation as:

$$\dot{\hat{k}}_t = sf\left(\hat{k}_t\right) - (n + g + x)\,\hat{k}_t$$

- As before, there is convergence to a steady-state:

$$sf\left(\hat{k}^*\right) = (n + g + x)\,\hat{k}^*$$

  with the following properties:

$$\gamma_{\hat{k}} = \gamma_{\hat{y}} = \gamma_{\hat{c}} = 0$$

  implying, since $\gamma_{\hat{z}} = \gamma_z - x$,:

$$\gamma_k = \gamma_y = \gamma_c = x$$

- The model now predicts long-run growth in per capita terms. The shortcoming is that growth is entirely driven by an exogenous trend, independent of any economic decision (e.g., propensity to savings). In particular, policies affecting the saving rates cannot affect long-run growth.

## 2.5 The "speed of convergence". Mankiw, Romer and Weil (1992).

- Assume Cobb Douglas technology:

$$Y_t = DK_t^{\alpha}\left(A_t L_t\right)^{1-\alpha}$$

Thus, the fundamental equation is:

$$\gamma_{\hat{k}} = sD\hat{k}^{\alpha-1} - (x+n+\delta)$$

which can identically be written as:

$$\frac{d}{dt}\ln\left(\hat{k}_t\right) = sDe^{-(1-\alpha)\ln\left(\hat{k}_t\right)} - (x+n+\delta),$$

or, defining $\xi_t \equiv \ln\left(\hat{k}_t\right)$ :

$$\dot{\xi}_t = sDe^{-(1-\alpha)\xi_t} - (x+n+\delta).$$

- Linearizing around the steady-state yields:

$$\dot{\xi}_t \simeq -(1-\alpha)\, sDe^{-(1-\alpha)\xi^*}\left(\xi_t - \xi^*\right).$$

Since $sDe^{-(1-\alpha)\xi^*} = (x+n+\delta)$, then:

$$\dot{\xi}_t \simeq -(1-\alpha)(x+n+\delta)\left(\xi_t - \xi^*\right),$$

Finally, reverting to the original notation:

$$\gamma_{\hat{k}} \simeq -(1-\alpha)(x+n+\delta)\ln\left(\frac{\hat{k}_t}{\hat{k}^*}\right) \equiv -\beta\cdot\ln\left(\frac{\hat{k}_t}{\hat{k}^*}\right)$$

where $\beta \equiv (1-\alpha)(x+n+\delta)$.

- How about convergence in GDP p.c.? Observe that: $\gamma_{\hat{y}} = \alpha\cdot\gamma_{\hat{k}}$ and $\ln\left(\frac{\hat{y}_t}{\hat{y}^*}\right) = \alpha\cdot\ln\left(\frac{\hat{k}_t}{\hat{k}^*}\right)$. Thus:

$$\gamma_{\hat{y}} \simeq -\beta\ln\left(\frac{\hat{y}_t}{\hat{y}^*}\right)$$

- Solve the differential equations:

$$\ln \hat{y}_t = \left(1 - e^{-\beta t}\right) \cdot \ln \hat{y}^* + e^{-\beta t} \ln \hat{y}_0$$

or, identically:

$$\ln \hat{y}_t - \ln \hat{y}_0 = \left(1 - e^{-\beta t}\right) \cdot \left(\ln \hat{y}^* - \ln \hat{y}_0\right)$$

Interpretation: the larger the gap between the initial GDP and steady-state, the larger the growth rate $t$ periods ahead.

- Note that we can detrmine how long it takes to an economy to get half-way to its steady state. Just solve for $t$ such that:

$$1 - e^{-\beta t} = 1/2 \quad \Rightarrow \quad t_{half} = \ln(2)/\beta$$

For example, if $\beta = 0.02$ per year, then $t_{half} = 35$.

- Solving the convergence equation for fundamental parameters and initial conditions only.

  1. Recall:

$$y^* = D\left(k^*\right)^{\alpha} = \left(\frac{sD}{n + \delta + x}\right)^{\frac{\alpha}{1-\alpha}}$$

  2. Replace $y^*$ in the convergence equation, and rearrange terms:

$$\begin{aligned} \ln \hat{y}_{t,i} - \ln \hat{y}_{0,i} = &- \left(1 - e^{-\beta t}\right) \ln \hat{y}_{0,i} + \left(1 - e^{-\beta t}\right) \frac{\alpha}{1-\alpha} \ln s_i \\ &- \left(1 - e^{-\beta t}\right) \frac{\alpha}{1-\alpha} \ln\left(n_i + \delta + x\right) \end{aligned}$$

  3. Then, since $\ln \hat{y}_{t,i} - \ln \hat{y}_{0,i} = \ln y_{t,i} - \ln y_{0,i} - xt$, and $\ln \hat{y}_{0,i} = \ln y_{0,i} - \ln A_{0,i}$ :

$$\begin{aligned} \ln y_{t,i} - \ln y_{0,i} = xt &- \left(1 - e^{-\beta t}\right) \ln y_{0,i} + \left(1 - e^{-\beta t}\right) \frac{\alpha}{1-\alpha} \ln s_i \\ &- \left(1 - e^{-\beta t}\right) \frac{\alpha}{1-\alpha} \ln\left(n_i + \delta + x\right) + \left(1 - e^{-\beta t}\right) \ln A_{0,i} \end{aligned}$$

4. Or, the regression equation:

$$Growth6085_i = a_0 + a_1 \cdot \ln y_{0,i} + a_2 \cdot \ln s_i + a_3 \cdot \ln (n_i + \delta + x) + \varepsilon_i$$

- Predictions for cross-country analysis (**conditional convergence**):

  1. *Qualitative*: regressing growth rates on initial GDP, controlling for saving rates and population growth rates should give a negative coefficient for initial GDP.
  2. *Quantitative*: taking benchmark values ($x = 0.02$, $n = 0.01$, $\delta = 0.05$, $\alpha = 1/3$), we should expect $\beta \simeq [0.053, 0.056]$. A rate of convergence of 5.6% implies that it takes 12.5 years to cover half the distance to the steady-state (given two economies which only differ in the capital stock, one with GDP equal to a half of the other) the poorer should close half gap in 12.5 years)$\rightarrow$ the model predicts not only convergence, but FAST convergence.

- Empirical evidence. Data 1960-85 (Summers Heston). Results:

  1. If we omit control for $s_i, n_i$, poorer countries do not grow faster than richer countries (lack of absolute convergence);
  2. if we control for $s_i, n_i$, poorer countries do grow faster than richer countries (conditional convergence)
  3. the estimated rate of (conditional) convergence is 2% rather than 5.5%. It takes 35 years (instead of 12.5) to cover half the distance to the steady-state. The Solow's model fails quantitatively.

- Criticism: MRW use OLS. Endogeneity bias. In particular, $s_i$ and $n_i$ might depend on growth rates. More serious problems: assume $A_0$ is systematically larger in rich countries. Since $A_0$ is omitted and correlated with $y_0$, we may have omitted variable bias.

- How could one get the quantitative predictions right? For the model to fit the data, we should have $\alpha = 0.75$. In this case, the model would predict a 2% convergence rate.

- An alternative model with both physical and human capital:

$$Y = DK^{\alpha} H^{\lambda} L^{(1-\alpha-\lambda)}$$

  where we have two fundamental equations:

$$\dot{K} = s_K DK^{\alpha} H^{\lambda} L^{(1-\alpha-\lambda)} - \delta K$$
$$\dot{H} = s_H DK^{\alpha} H^{\lambda} L^{(1-\alpha-\lambda)} - \delta H$$

  MRW argue that a reasonable assumption is that $\lambda = 1/3$.

  In this case (please work it out!), the rate of convergence is

$$\beta \simeq (1 - \alpha - \lambda)(n + \delta + x)$$

  and the model fits the data significantly better (predicted rate of convergence around 2.7%).

  Bottom line: a model where the elasticity of output to accumulable asset is around 1/3 predicts too fast convergence. We need models where the contribution of accumulable asset is around 3/4.

- ENDOGENOUS GROWTH with exogenous saving rates. We go "all the way through" and assume the contribution of accumulable assets (physical, human capital, not raw labor which grows following an autonomous law of motion) to be 100%.

  A simple example with exogenous saving rate: $Y = AK$, where $K$ is a broad notion of capital including human capital

$$\dot{K} = sAK - (n + \delta) K$$

  or:

$$\gamma_K = sA - (n + \delta)$$

FIGURE 1.11 BX: dynamics of AK model

- Features:
    1. No transitional dynamics
    2. Perpetual growth, with a constant growth rate of output, consumption and capital.
    3. Neither absolute, nor conditional convergence

# 3 The Ramsey-Cass-Koopmans model.

- Relax the assumptions that the saving rate is *exogenous* and *constant.*

- Empirical observation: saving rates increase with economic development.

- Richer dynamics of savings imply changes in transitional dynamics and speed of convergence.

- We study the model within a new environment (*market economy*)

    1. Households$\rightarrow$provide labor services (supplied inelastically) in exchange for wages, consume and accumulate assets.
    2. Firms$\rightarrow$have technical know-how to turn inputs into output, rent capital from consumers and hire labor services.

## 3.1 Households.

- Dynasties of households.
- Each household maximizes

$$U_0 = \int_0^{\infty} u\left[c_t\right] \cdot L_t \cdot e^{-\rho t}\, dt$$

  Note that the weight received by each "generation" within the dynasty depends on:

  1. how far in time a generation is (discounting)
  2. the size of each generation ($L_t$)

- Assume:

  1. Preferences parameterized by:

$$u\left[c_t\right] = \frac{c_t^{1-\theta} - 1}{1-\theta}$$

     where $\theta > 0$. Constant elasticity of the marginal utility (the inverse of the intertemporal elasticity of substitution, see p.64)

$$\frac{-u''(c) \cdot c}{u'(c)} = \theta$$

     When $\theta \to 1$, then $u[c] \to \ln(c)$.

  2. Population dynamics

$$L_t = L_0 e^{nt} = e^{nt}$$

- Rewrite the objective function as:

$$U_0 = \int_0^{\infty} \frac{c_t^{1-\theta} - 1}{1-\theta} \cdot e^{-(\rho - n)t}\, dt$$

- Intertemporal budget constraint:

$$\dot{a}_t = w_t + r_t a_t - c_t - n a_t$$

where, recall, lower cases denote p.c. variables.

- What is $a$? There are two types of assets in the economy:

  1. Ownership claim on capital;
  2. Loans (negative loans are debt).

  **NB:** Since both assets are riskless, they are perfect substitutes, and in equilibrium they must yield the same rate of return.

- Households behave *competitively*, i.e., take the path of market interest rate ($[r_t]_{t\in[0,\infty]}$) and wage rate ($[w_t]_{t\in[0,\infty]}$) as given.

- No Ponzi game: rule out the possibility that agents borrow to finance present consumption and then use future borrowings to roll over the debt and pay the interest.

$$\lim_{t\to\infty}\left[a_t \cdot e^{-\int_0^t [r_\nu - n]\,d\nu}\right] \geq 0$$

which means that, asymptotically, an agent's debt cannot grow at a rate faster (or equal) to $r - n$. Or, what is the same, the level of debt cannot grow at a rate larger than $r$.

- Hamiltonian (present-value):

$$J = \frac{c_t^{1-\theta} - 1}{1-\theta} e^{-(\rho-n)t} + \mu_t \left[w_t + (r_t - n)\, a_t - c_t\right]$$

where $\mu_t$ (costate variable) is the shadow price of income, i.e., the value of an increment of income received at time $t$ in units of utils at time 0. The FOCs:

$$\begin{aligned} \frac{\partial J}{\partial c} &= 0 \Rightarrow e^{-(\rho-n)t} \cdot c_t^{-\theta} = \mu_t \\ \dot{\mu}_t &= -\frac{\partial J}{\partial a} \Rightarrow \dot{\mu}_t = -\left(r_t - n\right)\mu_t \end{aligned}$$

The latter is known as the Euler equation.

Differentiate the former FOC w.r.t. time:

$$\frac{\dot{\mu}_t}{\mu_t} = -\theta \cdot \frac{\dot{c}_t}{c_t} - (\rho - n)$$

Then substitute into the Euler equation to get:

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\theta}$$

Interpretation:

1. an agent chooses a path of increasing, decreasing or constant consumption depending on whether $r_t$ is larger or smaller than $\rho$.
2. $r_t = \rho + \theta\gamma_c \rightarrow$ if consumption is growing (falling), since agents prefer to smooth consumption, agents demand a positive (negative) premium on the time discount factor in order to forego consumption.

- Transversality condition: the PDV of assets at the "end of life" (infinity) has to be non-positive $\rightarrow$ cfr. finite horizon: it is irrational to have any valuable assets left over after death

$$\lim_{t\to\infty} [a_t \cdot \mu_t] = \lim_{t\to\infty} \left[a_t \cdot e^{-\int_0^t (r_\nu - n)\, d\nu}\right] = 0$$

## 3.2 Firms

- Production technology:

$$Y_t = F\left(K_t, A_t L_t\right)$$

where $A_t = A_0 e^{xt} = e^{xt}$ (exogenous technical progress). Rewrite as:

$$\hat{y}_t = f\left(\hat{k}_t\right)$$

where:

$$\begin{aligned} \frac{\partial Y}{\partial K} &= f'\left(\hat{k}\right) \\ \frac{\partial Y}{\partial L} &= \left[f\left(\hat{k}\right) - \hat{k} \cdot f'\left(\hat{k}\right)\right] e^{xt} \end{aligned}$$

- Profits of the representative competitive firm:

$$\Pi = F\left(K_t, A_t L_t\right) - R_t K_t - w_t L_t$$

where R is the rental rate for capital.

Capital depreciates at the rate $\delta > 0$.

Then, the *real* return accruing to the owner of one unit of capital is $R_t - \delta$.

But, since loans and capital are perfect substitute, they must yield the same return:

$$r_t = R_t - \delta$$

Thus, rewrite profit function as:

$$\begin{aligned} \Pi_t &= F\left(K_t, A_t L_t\right) - \left(r_t + \delta\right) K_t - w_t L_t = \\ &= A_t L_t \left[ f\left(\hat{k}_t\right) - \left(r + \delta\right) \hat{k}_t - w_t \cdot e^{-xt} \right] \end{aligned}$$

- Two remarks:

1. (a) Although firms maximize intertemporal profits, this is identical to maximizing profit in each period (since there is no adjustment cost for capital).
2. The scale of production is undetermined, as usual with competitive firms with CRS technology.

- We can express the profit-maximizing choice as the choice of an optimal $\hat{k}$ :

$$f'\left(\hat{k}_t\right) = r + \delta$$

- Determination of $w$. Given the optimality condition $f'\left(\hat{k}_t\right) = r + \delta$, profits depend on $w \cdot e^{-xt}$. In equilibrium:

1. profits could never be positive since this is a competitive industry;
2. if profits were negative, the only possible equilibrium would be with zero production.[1]
3. consider a candidate equilibrium with zero profits and positive production. For $\Pi = 0$, wages must satisfy the following condition:

$$\left[f\left(\hat{k}_t\right) - \hat{k}_t \cdot f'\left(\hat{k}_t\right)\right] e^{xt} = w_t$$

(cfr. Euler theorem).

## 3.3 General equilibrium.

1. Take the Euler equation, and replace $r$ with the equilibrium rate:

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\theta} \quad \Rightarrow \quad \frac{\dot{\hat{c}}_t}{\hat{c}_t} - x = \frac{f'\left(\hat{k}\right) - \delta - \rho}{\theta},$$

hence:

$$\frac{\dot{\hat{c}}_t}{\hat{c}_t} = \frac{f'\left(\hat{k}\right) - \delta - \rho - \theta x}{\theta} \qquad (*)$$

2. Take the intertemporal budget constraint (IBC). In equilibrium, there is no borrowing/lending between households, and all assets held are claims on capital. Thus, $a_t = k_t$. The IBC can then be rewritten as:

$$\dot{k}_t = w_t + (r_t - n)\, k_t - c_t$$

Hence, through a familiar procedure:

$$\dot{\hat{k}}_t = w_t e^{-xt} + (r_t - n - x)\, \hat{k}_t - \hat{c}_t$$

Next, substitute to $w_t$ and $r_t$ their equilibrium value:

$$\dot{\hat{k}}_t = f\left(\hat{k}_t\right) - (\delta + n + x)\, \hat{k}_t - \hat{c}_t \qquad (**)$$

[1]A general equilibrium argument can be used to rule out this equilibrium with zero output. In particular, since households supply labor inelastically, there would be an excess supply of labor, and wage with fall until production is profitable.

3. Finally, the TVC. From $a_t = k_t = \hat{k}_t \cdot e^{-xt}$, it follows:

$$\lim_{t\to\infty} \left[\hat{k}_t \cdot e^{-\int_0^t \left(f'\left(\hat{k}_\nu\right)-\delta-n-x\right)d\nu}\right] = 0 \qquad (***)$$

- (*) and (**) defines a planar autonomous dynamic system in $\left(\hat{c}, \hat{k}\right)$. Note that $\hat{k}_0$ is predetermined whereas $c_0$ is endogenous.

FIGURE 2.1 BX

- STEADY-STATE. Characterized by two equations:

$$\begin{aligned} f'\left(\hat{k}^*\right) &= \delta + \rho + \theta x \\ \hat{c}^* &= f\left(\hat{k}^*\right) - (\delta - n - x)\,\hat{k}^* \end{aligned}$$

  Is it an equilibrium? For this to be the case, the TVC has to be satisfied. In particular (since $\lim_{t\to\infty} \hat{k}_t = \hat{k}^*$), we need that:

$$f'\left(\hat{k}^*\right) > \delta + n + x$$

  Since $f'\left(\hat{k}^*\right) = \delta + \rho + \theta x$, then the condition can be expressed as:

$$\rho > n + (1-\theta)\,x$$

- Golden rule and dynamic inefficiency.

  Golden rule: $f'\left(\hat{k}_{gold}\right) = \delta + n + x$

Equilibrium: $f'\left(\hat{k}^*\right) = \delta + \rho + \theta x$

If $\rho > n + (1-\theta)\,x$, then $\hat{k}^* < \hat{k}_{gold}$. No dynamic inefficiency.

- Other trajectories satisfying (*) and (**):

  $\hat{c}_0^{'} \rightarrow$ Euler equation violated in finite time

  $\hat{c}_0^{''} \rightarrow$ TVC violated (observe that $\hat{k}^{**} > \hat{k}_{gold}$, thus $f'\left(\hat{k}^{**}\right) < f'\left(\hat{k}_{gold}\right) = \delta + n + x$).

  Conclusion: the stable manifold converging to $\left(\hat{c}^*, \hat{k}^*\right)$ is the UNIQUE equilibrium.

- THE SHAPE OF THE STABLE MANIFOLD.

  1. High $\theta$: strong preference for smoothing consumption. Assuming $\hat{k}_0 < \hat{k}^*$, lower consumption and lower growth.
  2. Low $\theta$: weak preference for smoothing consumption. Agents are more willing to forego consumption in response to (temporarily) high rates of return. Assuming $\hat{k}_0 < \hat{k}^*$, higher consumption and lower growth.

- BEHAVIOR OF THE SAVING RATE.

  Assume Cobb-Douglas. Straightforward to derive steady-state consumption and saving rate (do it!):

$$s^* = \alpha \cdot \frac{x + n + \delta}{\delta + \rho + \theta x}$$

  Result (we do not prove it):

  1. If $1/\theta = s^*$ : constant saving rate (like in Solow model)
  2. If $1/\theta < s^*$ (low intertemporal elasticity): increasing saving rate.

3. If $1/\theta > s^*$ (high intertemporal elasticity): increasing saving rate

   FIGURE 2.1 BX: behavior of the saving rates

- Calibration:

$$\rho = 0.02; \delta = 0.05; n = 0.01; x = 0.02; \alpha = 0.3$$

  Then, for any $\theta < 17$, the saving rate falls – counterfactually – as the economy develops (common wisdom: $\theta \in [1, 5]$).

- A way out is, again, to set $\alpha$ higher. If $\alpha = 0.75$. If we commit to $\theta = 2.5$, for instance, we get $s^* = 0.5 > 0.4 = 1/\theta$, and we obtain the prediction, coherent with the data, of increasing saving rates. Also, the implied rates of convergence are reasonable.

  FIGURE 2.1 BX: speed of convergence in Ramsey model

- ALTERNATIVE ENVIRONMENTS:

  1. Household/producers. Maximize:

  $$U_0 = \int_0^\infty \frac{\hat{c}_t^{1-\theta} - 1}{1-\theta} \cdot e^{-(\rho - n - x(1-\theta))t}\, dt$$

  subject to:

  $$\dot{\hat{k}}_t = f\left(\hat{k}_t\right) - \hat{c}_t - (\delta + n + x)\,\hat{k}_t$$

  The solution is identical to the market economy just seen.

  2. Social planner→ maximize representative agent's utility subject to feasibility (technology) constraint. In this model, the SP solves the same problem as the household/producers. Key features: no externalities (complete markets).