Macro I - Problem set 1: Consumption under uncertainty

Due: Tuesday 2nd February

Problem 1

Assume that the consumer has a CRRA (constant relative risk aversion) utility function,

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma},$$

where $\gamma > 0$ is the degree of relative risk aversion and c_t is consumption. The agent maximizes

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$a_{t+1} = y_t - c_t + (1 + r_t) a_t,$$

where β is the agent's time discount rate, a_{t+1} is his financial wealth at the end of period t, y_t is income, and r_t is the interest rate.

(a) Specify the Bellman equation for this problem.

(b) Show that the Euler equation is

$$\mathrm{E}_t \left[\beta (1 + r_{t+1}) \left(\frac{c_t}{c_{t+1}} \right)^{\gamma} \right] = 1. \tag{0.1}$$

(c) Assume that the interest rate is constant, $r_t = r$ for all t. Define $f(c_{t+1}) = \left(\frac{c_t}{c_{t+1}} \right)^{\gamma}$ and make a first order Taylor approximation of $f(c_{t+1})$ around c_t. Show that this approximation, together with (0.1), implies that

$$c_{t+1} - c_t = Ac_t + \varepsilon_{t+1},$$

where

$$\begin{aligned} A &= \frac{1}{\gamma} \left[1 - \frac{1}{\beta (1 + r)} \right], \\ \mathrm{E}_t (\varepsilon_{t+1}) &= 0. \end{aligned}$$

In particular, note that if $\beta = (1 + r)^{-1}$ this implies a random walk for consumption.

(d) (Voluntary) Repeat (c), but use a second order Taylor approximation. Show that if $\beta = (1+r)^{-1}$, then expected consumption is growing over time, i.e. it is not a random walk.

Problem 2

In general, we cannot find closed form solutions for dynamic consumption models under uncertainty. Instead, model implications such as (0.1) can be tested, or the model can be solved numerically. A special case where explicit solutions can be found is the following, where the consumer has quadratic utility:

Assume that the agent maximizes

$$\mathrm{E}_0\left[\sum_{t=0}^{T}\beta^t\left(\alpha c_t - \frac{1}{2}c_t^2\right)\right],$$

s.t.

$$\begin{aligned} a_{t+1} &= y_t - c_t + (1+r)\,a_t, \\ a_{T+1} &\geq 0, \text{ and } a_0 \text{ given.} \end{aligned}$$

Notation is the same as in the previous problem, and $\alpha > 0$.

(a) Specify the Bellman equation for this problem.

(b) Derive the Euler equation.

(c) Iterate the transition equation for a_t forward, and derive the intertemporal budget constraint in expectational terms.

(d) Assume that $\beta = (1+r)^{-1}$, and let $T \to \infty$. Derive the consumption function, $c_t = c\,(a_t, y_t)$.

Problem 3

Let us look at a simple two-period model which illustrates the most important mechanism behind precautionary savings. The agent lives for two periods and solves

$$\max_{c_1, c_2} u\,(c_1) + \mathrm{E}_1 u\,(c_2)\,,$$

subject to

$$\begin{aligned} s &= y_1 - c_1, \\ c_2 &= \tilde{y}_2 + s, \\ y_1 &= \bar{y}, \\ \tilde{y}_2 &= \bar{y} + \varepsilon. \end{aligned}$$

That is, income in period one is fixed at $\bar{y}$ but income in period two is stochastic, $\varepsilon \sim N(0, \sigma^2)$. To simplify, we have assumed that $r = 0$ and $\beta = 1$.

(a) Assume that $u(c) = \alpha c - \frac{1}{2}c^2$, where $\alpha > 0$. Solve for c_1.

(b) Assume that $u(c) = c^{1-\gamma}/(1-\gamma)$, and $\gamma > 0$. Write down the first order condition characterizing the solution. This is the Euler equation for the problem. Show that $\mathrm{E}_1(c_2) > c_1$ if $\sigma > 0$.

Hint: Use Jensen's inequality: If $f(x)$ is strictly concave then $\mathrm{E}[f(x)] < f[\mathrm{E}(x)]$, and if $f(x)$ is strictly convex then $\mathrm{E}[f(x)] > f[\mathrm{E}(x)]$.

(c) Now assume that the agent has a CARA (constant absolute risk aversion) utility function,

$$u(c) = \frac{e^{-\gamma c}}{-\gamma},$$

where $\gamma > 0$ is the degree of absolute risk aversion.

Solve for c_1. Define precautionary savings, $s(\sigma)$, as the difference between consumption with no risk ($\sigma = 0$) and with risk. Is precautionary savings increasing in risk aversion (γ), in risk (σ), in average income ($\bar{y}$)?

Hint: If $x \sim N(\bar{x}, \sigma^2)$, then $\mathrm{E}e^x = e^{\bar{x} + \frac{1}{2}\sigma^2}$.

Problem 4

Look at the CRRA utility function $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, $\gamma > 0$.

(a) What is the degree of absolute risk aversion? How does it depend on income?

(b) What is the coefficient of relative prudence?