Problem set 4: Asset pricing

Due: Tuesday February 23

Problem 1: The Consumption CAPM Model

Consider a competitive economy in which the representative household allocates its wealth, b_t, across n different types of risky assets that pay an uncertain return. In deciding how to allocate their portfolio intra-temporally and consumption intertemporally, households seek to maximize the utility function

$$\max_{c,s_i} \mathrm{E}_t \sum_{s=0}^{\infty} \beta^s u\left(c_{t+s}\right) \tag{0.1}$$

subject to the budget constraint

$$c_t + \sum_{i=0}^{n} \left(q_{i,t}\right) s_{i,t+1} = w_t + \sum_{i=0}^{n} \left[\left(d_{i,t} + q_{i,t}\right) s_{i,t}\right] \equiv b_t \tag{0.2}$$

where E_t is the expectations operator conditional on information available at time t, c_t is consumption of the (nondurable) good, w_t is exogenous wage income and b_t is total funds (wealth) available at time t. For the ith. asset, $d_{i,t}$ is the dividend paid at the beginning of time t, and $q_{i,t}$ is the price. Thus, an agent that purchases an asset during period t pays q and does not receive any dividends until the beginning of the following period. Finally, $s_{i,t}$ is the agent's holdings of asset i entering period t. The path of dividends is stochastic. Assume that the agent has a CARA (constant absolute risk aversion) utility function,

$$u\left(c\right) = \frac{e^{-\gamma c}}{-\gamma},$$

where $\gamma > 0$ is the degree of absolute risk aversion.

(a) Interpret the budget constraint (0.2).

(b) Formulate the problem as a dynamic programming problem and specify the Bellman's equation.

(c) Derive the first-order condition with respect to $s_{i,t+1}$, and show that it is given by

$$\mathrm{E}_t\left(\mu_t R_{i,t}\right) = 1 \tag{0.3}$$

where

$$R_{i,t} = \frac{d_{i,t+1} + q_{i,t+1}}{q_{i,t}}$$

and

$$\mu_t = \beta e^{-\gamma\cdot(c_{t+1}-c_t)}$$

(d) Suppose that the nth. asset is riskless. Show that the following relationship must hold in equilibrium

$$E_t\left[R_{i,t}\right] > R_n, \quad \text{iff } cov_t\left(R_{i,t}, \mu_t\right) < 0$$

and

$$E_t\left[R_{i,t}\right] < R_n, \quad \text{iff } cov_t\left(R_{i,t}, \mu_t\right) > 0$$

where $cov_t\left(x, y\right)$ is the covariance of x and y conditional on information available at time t. Give an intuitive explanation of the relationship.

Hint: given two random variables x, y, $cov_t\left(x, y\right) = \mathrm{E}[xy] - \left[\mathrm{E}\left(x\right)\right]\left[\mathrm{E}\left(y\right)\right]$

0.1. Problem 2: International risk sharing

Suppose we have the two-country, two-period, S state endowment setup of section 5.2 in OR, and of the $N = 2$ case from section 5.3. Now, however, in both Home and Foreign, agents have the exponential period utility function $u\left(C\right) = \frac{-e^{-\gamma C}}{\gamma}$, with $\gamma > 0$, rather than CRRA period utility.

(a) Calculate equilibrium prices and consumption levels for the case of complete markets (paralleling section 5.2 in OR).

(b) Suppose that instead of having access to complete markets, agents are restricted to trading riskless bonds and shares in Home and Foreign period 2 outputs. Show that the resulting allocation is still efficient (paralleling section 5.3 in OR). Show also that Home and Foreign consumption on both dates are given by $C = \frac{1}{2}Y^W - \mu$, $C^* = \frac{1}{2}Y^W + \mu$, where Y^W is world output and μ is a time-invariant constant.