Problem set 5: The Solow-Swan Model

Due: Tuesday March 2

1. Problem 1: A Solow model

Use the following definitions: s = (exogenous) savings rate, δ = depreciation of capital, n = population growth rate, g = rate of growth of labor augmenting technology, $L = e^{nt}$ = labor supply, $\hat{L} = e^{(g+n)t}$ = effective labor supply. Also, let $k = K/L$, $y = Y/L$, $\hat{k} = K/\hat{L}$, and $\hat{y} = Y/\hat{L}$, where production is of a Cobb-Douglas form: $Y = K^{\alpha}\hat{L}^{1-\alpha}$.

The gross resource constraint for a closed economy is

$$K^{\alpha}\hat{L}^{1-\alpha} = C + \delta K + \dot{K}$$

where C is consumption, $\alpha < 1$, and $\dot{K} \equiv \frac{\partial K}{\partial t}$. We can rewrite the resource constraint as a savings relation:

$$S = sY = Y - C = \delta K + \dot{K}.$$

(a) Rewrite the resource constraint in terms of $\hat{k}$, i.e. write $\dot{\hat{k}}$ as a function of $\hat{k}$ and exogenous parameters/variables.

(b) Define $\gamma_{\hat{k}} = \frac{\dot{\hat{k}}}{\hat{k}}$. The definition of a steady state is that all variables grow at a constant rate, i.e. that γ_x is constant for all variables x.

(i) Show that $\gamma_{\hat{k}} = 0$ in this model.

(ii) What is the steady state growth of k, and K?

(iii) Can you relax a model assumption so that we get $\gamma_{\hat{k}} > 0$ in steady state?

(c) Find the steady state value of $\hat{k}$ (expressed in exogenous parameters/variables).

(d) Is the steady state value of $\hat{k}$ $\left(\hat{k}^*\right)$ dynamically efficient? For which configuration of parameters is $\hat{k}^*$ equal to the "Golden rule capital stock"?

(e) Assume now that $g = 0$. Express the economy's growth rate of GDP per capita $\frac{\dot{y}}{y}$ (γ_y) at a given point in time as a function of the parameters s, n, δ, α, and the current output level y_t. [Hint: start from γ_k and substitute for k by an expression for y. Then rewrite γ_k in terms of γ_y.]

(f) Assume that there are two countries, R (Rich) and P (Poor), which differ only with respect to current income levels. Let GDP per capita of country P be one tenth of that in country R. Assume furthermore that country R is in steady state and that $\alpha = \frac{1}{3}$ and $g = 0$. Provide some reasonable values for s, n, and δ, and calculate the growth rate of country P. Is it reasonable?

2. Problem 2: Endogenous growth with transitional dynamics

Use the following definitions: $s =$ (exogenous) savings rate, $\delta =$ depreciation of capital, $n =$ population growth rate, $L = e^{nt} =$ labor supply, $k = \frac{K}{L}$ and $y = \frac{Y}{L}$, where production is of CES form:

$$Y = A\left\{a\,(bk)^{\psi} + (1-a)\,[(1-b)\,L]^{\psi}\right\}^{\frac{1}{\psi}}$$

where $0 < a < 1$, $0 < b < 1$ and $\psi < 1$.

The gross return constraint for a closed economy is

$$Y = C + \delta K + \dot{K}$$

where C is consumption, and $\dot{K} = \frac{\partial K}{\partial t}$. We can rewrite the resource constraint as a savings relation:

$$S = sY = Y - C = \delta K + \dot{K}$$

(a) Rewrite the resource constraint in terms of k, i.e., write $\dot{k}$ as a function of k and exogenous parameters/variables.

(b) Find $\gamma_k = \frac{\dot{k}}{k}$ as a function of k. What is the steady state growth of k and of K if we assume that $sAba^{\frac{1}{\psi}} > n + \delta$?

(c) Find the economy's growth rate of GDP per capita as a function of s, n, δ, a, b, ψ and the capital level.

(d) Analyze the transitional dynamics with the help of a graph. Is there conditional convergence in this economy?

3. Problem 3: Problem 1.3 in Barro and Sala-I-Martin

Assume that the production function satisfies the neoclassical properties.

(a) Why would the saving rate, s, generally depend on k? Provide some intuition only.

(b) How does the speed of convergence change if $s(k)$ is an increasing function of k? What if $s(k)$ is a decreasing function of k?
Consider now an AK technology.

(c) Why would the saving rate, s, depend on k in this context?

(d) How does the growth rate of k change over time depending on whether $s(k)$ is an increasing or decreasing function of k?

(e) Suppose that the rate of population growth, n, depends on k. For an AK technology, what would the relation between n and k have to be in order for the model to predict convergence? Can you think of reasons why n would be related to k in this manner?

(f) Repeat part **(e)** in terms of the depreciation rate, δ. Why might δ depend on k?

4. Problem 3: Problem 1.5 in Barro and Sala-I-Martin

For a neoclassical production function, show that each factor of production earns its marginal product. Show that if owners of capital save all their income and workers consume all their income, then the economy reaches the golden rule of capital accumulation. Explain the results.