Macro I - Problem Set 6 :
Ramsey Model and Endogenous Growth

Due: Thursday March 11

1. Problem 1: A simple Ramsey model

A hypothetical social planner solves the following Ramsey problem in continuous time under perfect foresight. Maximize the representative household's utility

$$U = \int_0^\infty e^{-(\rho-n)t} \frac{c_t^{1-\gamma}}{1-\gamma} dt,$$

where ρ is the time discount rate, c_t is consumption per capita and n is the population growth rate. Production per capita is

$$f(k_t) = Ak_t^\theta,$$

where $0 < \theta < 1$, and the resource constraint for the economy is

$$\dot{k}_t = f(k_t) - (\delta + n) k_t - c_t,$$

$$k_0 \text{ given.}$$

Here δ is the depreciation rate of physical capital.

(a)

Characterize the solution to the problem with two differential equations, one for k_t and one for c_t.

(b) Solve for the steady state values for the capital stock and consumption. Describe the dynamics around the steady state in a phase diagram.

(c) What happens to the economy if the discount factor increases permanently (i.e. if ρ increases)? Assume that the economy is initially in a steady state. Plot the results in a phase diagram.

2. Problem 2: Taxation in the Ramsey model

Consider a representative individual that solves

$$\max_{\{c_t\}} \int_0^\infty e^{-\rho t} U(c_t)\, dt$$

s.t.

$$\begin{aligned} \dot{K}_t &= f(K_t) - T - c_t, \\ K_0 &= \bar{K} \\ \lim_{T\to\infty} e^{-\rho T} K_T &= 0. \end{aligned}$$

Assume that

$$\begin{aligned} U(c) &= -\frac{e^{-\gamma c}}{\gamma}, \\ f(K) &= K^\theta, \end{aligned}$$

and $0 < \theta < 1$. Here, K is the capital stock, c is consumption, and T is a tax.

(a) Set up the current value Hamiltonian with its necessary conditions for optimality.

(b) Express the optimality conditions and the transition equation for K as a system of two differential equations in K and c. [Hint: take time derivatives of the condition for the maximum of the Hamiltonian.] Present your results in a phase diagram in K and c. Make sure you indicate movements in the phase diagram in all areas and any potential saddle path.

(c) What are the effects of an unexpected permanent increase in the tax rate? Indicate transitions and the long run effect in a phase diagram.

(d) Now consider an unexpected temporary increase in the tax rate, between, say, time t and T. Indicate transitions and the long run effect in a phase diagram. Describe in words what is happening and why.

(e) Lastly, assume that at t it is announced that at T the tax rate will permanently be raised at $T > t$. Again, indicate transitions and the long run effect in a phase diagram.

3. Problem 3: AK model with externality

Consider a continuous-time model of a closed economy without population growth or technical progress. Production per capita, at date t, is given by

$$y_t = Ak_t, \tag{3.1}$$

where k_t denotes capital per capita and A is a productivity parameter. Capital depreciates at the rate δ_t, where δ_t is a convex function of aggregate capital, K_t: $\delta_t = B(K_t)$ with B_K, $B_{KK} > 0$.

Consumers save in the form of capital. The representative consumer chooses an optimal consumption plan to maximize

$$\int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt, \tag{3.2}$$

where c is consumption per capita and ρ is the rate of time preference.

(a) What will the representative firm be paying to rent capital from consumers in equilibrium at a competitive capital market, where each firm takes aggregate capital as given?

(b) Write down the consumer's budget constraint and characterize the solution to her intertemporal maximization problem.

(c) Consider the full competitive market equilibrium in this economy. Normalize the constant labor force to unity such that $k_t = K_t$ at all t. Characterize the equilibrium dynamics of c_t and k_t, both algebraically and in a common diagram. Does this economy converge to a steady state from any initial capital stock k_0?

(d) Is the private market equilibrium socially efficient from the viewpoint of a social planner that seeks to maximize the representative consumer's utility function (3.2)? Explain why or why not, by analyzing the solution for c and k in the social planner's problem.