1 Endogenous growth

- Solow and Ramsey-Cass-Koopmans: no autonomous engine of growth. In the absence of exogenous trend, growth dies off in the long-run.

  1. No theory of determinants of long-run growth;
  2. No theory of determinants of long-run cross-country differences in growth rates;
  3. Policies do not affect long-run growth.

- In the analysis of endogenous growth models, we will set $x = 0$.

1.1 THE AK MODEL

- Production technology:

$$f(k) = Ak$$

- Equilibrium still determined by (*),(**),(***), given k_0

$$\begin{aligned} \dot{c}_t &= \frac{A - \delta - \rho}{\theta} \cdot c_t \\ \dot{k}_t &= Ak_t - c_t - (\delta + n) k_t \\ \lim_{t \to \infty} \left[k_t \cdot e^{-(A - \delta - n)t} \right] &= 0 \end{aligned}$$

- We can obtain an explicit solution:

  (a) Guess a steady-state solution such that c/k is constant (assume $A > \delta + \rho$).

$$\frac{\dot{c}_t}{c_t} = \frac{\dot{k}_t}{k_t} = \frac{A - \delta - \rho}{\theta} = \gamma$$

  (b) Use (**)

$$\frac{\dot{k}_t}{k_t} = \frac{A - \delta - \rho}{\theta} = (A - \delta - n) - c/k.$$

(c) From (a)-(b):

$$\frac{c}{k} = \rho - n - \frac{1-\theta}{\theta} \cdot [A - \delta - \rho]$$

In particular: $c_0 = \left\{\rho - n - \frac{1-\theta}{\theta} \cdot [A - \delta - \rho]\right\} k_0$.

(d) Hence, from (c)-(a):

$$\begin{aligned} c_t &= c_0 \cdot e^{(1/\theta)\cdot(A-\delta-\rho)t} \\ k_t &= k_0 \cdot e^{(1/\theta)\cdot(A-\delta-\rho)t} \end{aligned}$$

(e) TVC (after replacing $k_t$by its solution):

$$\lim_{t\to\infty} \left[k_0 \cdot e^{(1/\theta)\cdot(A-\delta-\rho)t} \cdot e^{-(A-\delta-n)t}\right] = 0$$

provided that the following condition (bounded utility) holds:

$$\rho > n + (1-\theta)(A - \delta - n)$$

- No transitional dynamics.

Prove yourself that all other trajectories satisfying (*) and (**) fail to satisfy either Euler equation or TVC (hint: study the dynamics of the modified system of differential equations in c_t and $\chi_t \equiv (c_t/k_t)$

Policies can now affect long-run growth. Consider the permanent (unexpected) introduction of a proportional tax on returns to capital such that, while firms pay a rental rate equal to $R_t = f'(k_t) = A$, consumers gross return to savings is only $R_t(1-\tau)$ (the proceedings are rebated lump-sum). Thus, $r_t = A(1-\tau) - \delta$. The growth rate becomes:

$$\gamma_\tau = \frac{A(1-\tau) - \delta - \rho}{\theta}$$

Figure: PHASE DIAGRAM AK model

- AK MODEL can be thought as a reduced-form representation. An example with physical and human capital.

  Assume $Y = AK^{\alpha}H^{1-\alpha} = AK\left(H/K\right)^{1-\alpha}$

  Output can be used on a one-for-one basis for consumption, investment in physical capital and investment in human capital. All investments are fully reversible. For simplicity, let depreciation be the same (rate δ) for both types of capital.

  In equilibrium: $R_K = R_H = r + \delta$.

  Firms optimization:

$$R_K = \alpha A\left(H/K\right)^{1-\alpha} = (1-\alpha)\,A\left(H/K\right)^{-\alpha} = R_H$$

  Thus, in equilibrium,

$$\begin{aligned} H/K &= (1-\alpha)/\alpha \\ r &= \alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}A - \delta \\ Y &= \left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} AK \end{aligned}$$

  and all variables in the economy grow at the constant rate

$$\gamma = \frac{\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}A - \delta - \rho}{\rho}.$$

1.2 LEARNING-BY-DOING (Arrow-Sheshinsky-Romer).

- We assume a constant population ($n = 0$).
- Technology:

$$Y_t = F\left(K_{i,t}, \tilde{A}_t L_i\right)$$

  where it is now important to distinguish between firm-level capital, K_i, and aggregate capital, to be denoted by K.

- Labor-augmenting technical progress ($\tilde{A}$) is no longer exogenous, but is a function of aggregate capital accumulation.

$$\tilde{A}_t = \phi K_t$$

  $\tilde{A}$ can be interpreted as public knowledge. Knowledge is assumed to have a non-rival character: when a firm adds to the stock of knowledge, all firms in the economy can benefit from this addition.

  Knowledge is assumed to depend on the cumulated past investments, i.e., on the stock of capital in the economy (see Arrow and Schmookler in the 1960's).

  Knowledge as a pure spillover→free-rider problem. No firm is prepared to pay for the accumulation of this non-rival input. In equilibrium, accumulation of knowledge is endogenous, but unintentional.

- For simplicity, we restrict attention to Cobb-Douglas technology:

$$F\left(K_i, \tilde{A}L_i\right) = K_i^{\alpha}\left(\tilde{A}L_i\right)^{1-\alpha} = AK_i^{\alpha}\left(KL_i\right)^{1-\alpha}$$

  where $A \equiv \phi^{1-\alpha}$

- When deciding factor rentals, every (infinitesimal) firm takes $\bar{K}$ as parametric. Thus, the equilibrium rates of return are:

$$\begin{aligned} R &= r + \delta = \alpha A\left(\frac{KL_i}{K_i}\right)^{1-\alpha} \\ w &= \frac{(1-\alpha)\, AK_i^{\alpha}\left(KL_i\right)^{1-\alpha}}{L_i} \end{aligned}$$

- We assume a continuum of firms with total measure equal to one. Thus, in a symmetric equilibrium, $K_i = K$ and $L_i = K$. Observe, however, that in order to characterize the competitive equilibrium, one has to substitute in these equilibrium conditions AFTER finding the equilibrium factor prices, rather than before (i.e., rather than directly into the production function). This is crucial to capture the idea that firms act in an uncoordinated fashion.

$$\begin{aligned} r &= \alpha A L^{1-\alpha} - \delta \\ w &= \frac{(1-\alpha) AK}{L^{\alpha}} = (1-\alpha) \cdot k \cdot L^{1-\alpha} \end{aligned}$$

- The dynamic equilibrium conditions, (*), (**), (***) are, then:

$$\begin{aligned} \dot{c}_t &= \frac{\alpha A L^{1-\alpha} - \delta - \rho}{\theta} \cdot c_t \\ \dot{k}_t &= \left(A L^{1-\alpha} - \delta \right) k_t - c_t \end{aligned}$$

$$TVC: \quad \lim_{t \to \infty} \left[k_t \cdot e^{-\left(\alpha A L^{1-\alpha} - \delta\right) t} \right] = 0$$

The dynamics of this model are isomorphic to those of the AK model. But there are two differences:

1. Scale effects.
2. Pareto non-optimality (to be discussed in an exercise).

1.3 Government and growth

- Another example of a model which yields the AK as a reduced form.
- Assume, again, $n = 0$.
- The government taxes agents or firms and uses the proceedings to provide free public services to producers. The government spend its tax income into a "samuelsonian" (non-rival and non-excludable) public good, which can be used by all firms simultaneously, and with no congestion effect.
- Production function:

$$Y_i = A L_i^{1-\alpha} K_i^{\alpha} G^{1-\alpha}$$

Two features:

1. Constant returns to scale to the private inputs (L_i, K_i), and increasing returns overall;
2. Constant returns to *reproducible* inputs (K, G). If G and K grow at a constant rate, the return to capital does not fall over time.

- Note: if the exponent of G were smaller than $1-\alpha$, then we would have diminishing returns to reproducible inputs, and no sustained growth (neoclassical convergence).
- The government runs a balanced deficit:

$$G = \tau Y$$

where τ is tax rate which is assumed to be levied on the value of production of each firm. This implies (replacing Y by its expression, and solving for G) that:

$$G = (\tau A L)^{1/\alpha} \cdot k$$

- The firms' after tax-profit is:

$$\Pi_i = L_i \left[(1-\tau) A k_i^{\alpha} G^{1-\alpha} - w - (r+\delta)\, k_i\right]$$

- Thus, profit maximization implies that:

$$r + \delta = \alpha(1-\tau) A \left(G/k_i\right)^{1-\alpha}$$

- Hence, substituting G by its expression, we obtain:

$$r + \delta = \alpha(1-\tau) A^{1/\alpha} \left(\tau L\right)^{(1-\alpha)/\alpha}$$

- From the standard Euler equation, we have then:

$$\begin{aligned} \frac{\dot{c}_t}{c_t} &= (1/\theta) \cdot \left[\alpha(1-\tau) A^{1/\alpha} \left(\tau L\right)^{(1-\alpha)/\alpha} - \delta - \rho\right] \\ \dot{k}_t &= A^{1/\alpha} \left(\tau L\right)^{(1-\alpha)/\alpha} k_t - \delta k_t - c_t \end{aligned}$$

- The equilibrium has, as usual, constant growth, given by:

$$\gamma = (1/\theta) \cdot \left[\alpha(1-\tau)A^{1/\alpha}\,(\tau L)^{(1-\alpha)/\alpha} - \delta - \rho\right]$$

- Government expenditure ($\tau = G/Y$) has two opposite effects on growth:

  1. $(1-\tau) \rightarrow$ negative effect of taxation on the marginal product of capital (growth- depressing distortionary effect);
  2. $\tau^{(1-\alpha)/\alpha} \rightarrow$ positive effect of public good on the marginal product of capital (growth- enhancing effect of public services)

- Inverse U-shaped relationship between growth and government expenditure. The maximum is achieved in correspondence of the condition $\tau = G/Y = (1-\alpha)$. Interpretation: equating the marginal cost of capital (1) to marginal benefit ($\partial Y/\partial G = (1-\alpha) \cdot Y/G = (1-\alpha) \cdot \tau$).