I. Growth

3. The Samuelson-Diamond-Blanchard OLG model

. Households

Live for two periods and solve

$$\begin{aligned} &\max_{C_{1,t},C_{2,t+1}} U\left(C_{1,t}\right)+\frac{1}{1+\rho}U\left(C_{2,t+1}\right) \\ &s.t. \frac{C_{2,t+1}}{1+r_{t+1}}+C_{1,t}=\text{labor income in } t. \end{aligned} \tag{1}$$

. Firms

Hire labor and capital on competitive market and combine them in a CRS production function to produce the only good. This is sold on a competitive market to the households.

Firms solve

$$\max_{K_t,L_t} F(K_t,A_tL_t)-(w_tA_tL_t+r_tK_t) \tag{2}$$

where A_t is a productivity index that grows geometrically at rate *g*. We can think of *AL* as the number of *effective* units of labor. Also *L* grows at a geometric rate *n*. So

$$\begin{aligned} A_{t+1} &= (1+g)A_t \\ L_{t+1} &= (1+n)L_t \end{aligned} \tag{3}$$

The competitive market imply that factors are paid their marginal products. $r = F_K$ and $w=F_{AL}$.

Since *F* is H(1) we have

$$\frac{1}{A_tL_t}F\left(K_t,A_tL_t\right)=F\left(\frac{K_t}{A_tL_t},1\right)\equiv f\left(k_t\right) \tag{4}$$

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So f is production per effective labor unit and k is capital per effective labor unit. Note that

$$
\begin{aligned}
&F(K_t, A_t L_t) \equiv A_t L_t f\left(\frac{K_t}{A_t L_t}\right) \\
&r_t = F_{K_t} = A_t L_t f'(k_t)\frac{1}{AL} = f'(k_t)
\end{aligned} \tag{5}
$$

Since we CRS and competitive markets firms make zero profits. We can write this as

$$
\begin{aligned}
&w_t A_t L_t = F(K_t, A_t L_t) - r_t K_t \\
&w_t = f(k_t) - r_t k_t
\end{aligned} \tag{6}
$$

Note that we now see that both wages and interest rates are determined by the current capital to effective labor ratio. We thus write

$$
\begin{aligned}
&w_t = w(k_t) \\
&r_t = r(k_t)
\end{aligned} \tag{7}
$$

. Capital market

The labor income of a young generation in t equals $w_t A_t L_t$. Part of this is consumed ($c_{1,t}$) and the rest is saved. Let s_t denote the share of labor income that is saved. This will, in general depend on labor income and the interest rate. The savings in period t is next periods capital stock

$$
\begin{aligned}
&s_t w_t A_t L_t = K_{t+1} \\
&s_t w_t = \frac{K_{t+1}}{A_t L_t} = \frac{K_{t+1}}{A_{t+1} L_{t+1} / (1+n)(1+g)} \\
&k_{t+1} = \frac{s_t w_t}{(1+n)(1+g)}
\end{aligned} \tag{8}
$$

Now assume that s_t only depends on r_{t+1}. (Which class of utility functions produce this result? What are the effects of higher r?) Then we can write the last line of (8) as

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$$k_{t+1} = \frac{s(r_{t+1}(k_{t+1}))w(k_t)}{(1+n)(1+g)} \tag{9}$$

This (implicitly) defines a difference equation for k, i.e., a relation between k_t and k_{t+1} that has to be satisfied in this model. Note that if we made the Solow assumption of a constant savings rate, the difference equation takes an explicit form. Since in this case the RHS contains no k_{t+1}. Fortunately there is a utility function for which the income and substitution effects of higher interest rates cancel so that the household will choose a constant savings rate regardless of the interest rate.

. Functional specification

Let us look at a particularly simple specification. Assume U is the log function and that that production is (Wicksell-) Cobb-Douglas, $f(k) = k^{\alpha}$.

Now we can write the consumption decision of the consumer as follows.

$$\max_{s_t} = \ln(1-s_t)w_t A_t L_t + \frac{1}{1+\rho}\ln s_t(1+r_{t+1})w_t A_t L_t \tag{10}$$

with foc

$$\begin{aligned} \frac{1}{1-s_t} &= \frac{1}{(1+\rho)s_t} \\ \Rightarrow s_t &= \frac{1}{2+\rho} \end{aligned} \tag{11}$$

Now we only need to specify the wage function to have an explicit version of (9). We see immediately that $w = (1-\alpha)k^{\alpha}$. So (9) becomes

$$\begin{aligned} k_{t+1} &= \frac{(1-\alpha)k_t^{\alpha}}{(2+\rho)(1+n)(1+g)} \\ \ln k_{t+1} &= \ln\frac{1-\alpha}{(2+\rho)(1+n)(1+g)} + \alpha \ln k_t \end{aligned} \tag{12}$$

Can you solve this difference equation and is it stable? Here is an example of a plot of (12).

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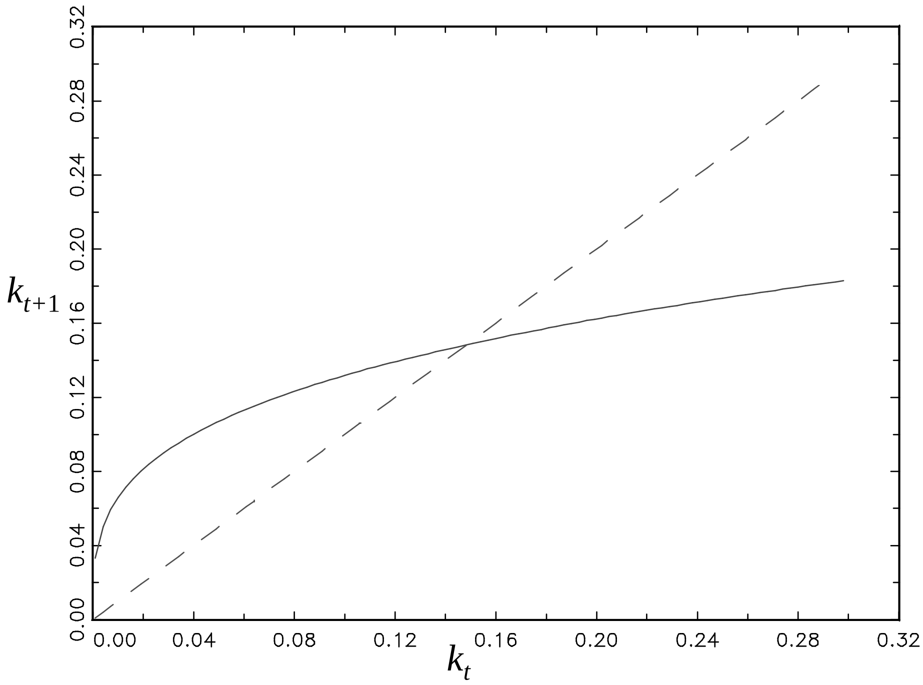

The steady state of (12) is its fixed point

$$k^* = \left(\frac{1-\alpha}{(2+\rho)(1+g)(1+n)} \right)^{\frac{1}{1-\alpha}} \tag{13}$$

The interest rate in steady state is

$$r^* = \alpha (k^*)^{\alpha - 1} = \left(\frac{\alpha(2+\rho)(1+g)(1+n)}{1-\alpha} \right) \tag{14}$$

. Dynamic Inefficiency

In each time period the amount of resources is $K_t + F(K_t, A_t L_t)$. This has to be split into aggregate consumption and next periods capital stock. We can than write

$$K_t + F\left(K_t, A_t L_t\right) = K_{t+1} + C_t \tag{15}$$

where C_t is total consumption of young and old in period *t*. Now divide by $A_t L_t$

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$$k_t + f(k_t) = \frac{K_{t+1}}{A_t L_t} + c_t$$
$$= k_{t+1}(1+g)(1+n) + c_t. \tag{16}$$

c_t is aggregate consumption per unit of current effective unit of labor.

Now consider an economy in a steady state. We then have

$$k^* + f\left(k^*\right) = k^*(1+g)(1+n) + c^*$$
$$\Rightarrow c^* = f\left(k^*\right) - k^*(g+n+gn) \tag{17}$$

Let us find the value of the steady state capital stock that maximizes aggregate consumption per effective unit of labor. The FOC for this problem is

$$f'\left(k^{gr}\right) = n + g + ng \tag{18}$$

This is a variant of the Ramsey Golden Rule. A steady state capital stock above k^{gr} is not Pareto efficient. If we are above k^{gr} we could increase aggregate consumption in all periods now and in the future by reducing forcing the steady state capital stock to be lower.

Now compare (18) and (14)

$$r^* = \left(\frac{\alpha(2+\rho)(1+g)(1+n)}{1-\alpha}\right) \lesseqgtr n + g + ng$$
$$\left(\frac{\alpha(2+\rho)}{1-\alpha}\right) \lesseqgtr 1 - \frac{1}{(1+g)(1+n)} \tag{19}$$

So it is possible that the economy is not on the Pareto frontier and thus dynamically inefficient. What goes wrong here and why does not the first welfare theorem hold? One way of understanding this is to realize that the only way a young generation can reduce the capital stock is by saving less but then consumption would have to be lower for them in the second period. If, however, there was a mechanism that specified that young people saved less but in return got some transfers from the young next period, everybody could be made better off. What does this remind you off?

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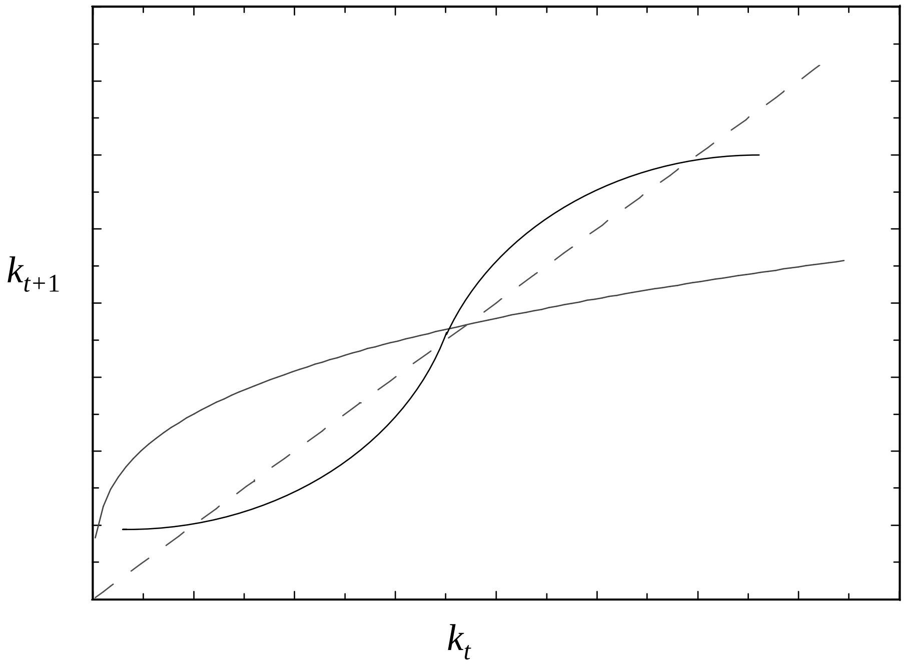
. More general savings functions

In a more general case, the difference equation given by (9) can be less nice. It can have no or multiple steady states. With multiple steady states, some will be non-stable. Multiple steady states could occur if, for example, the savings rate increase as interest rate falls (the income effect dominates).



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I.C. Endogenous Growth

. Prerequisites for Endogenous Growth

Take the simplest neo-classical growth model and assume away the exogenous growth rates. Can we anyway generate sustained long run growth? Let the CRS production function be

$$Y = F(K, AL) \tag{20}$$

and capital accumulation

$$\dot{K} = sF(K, AL) \tag{21}$$

Now

$$\begin{aligned} &\dot{Y} = F_K \dot{K} = F_K sY \\ &\Rightarrow g_y = F_K s \end{aligned} \tag{22}$$

Now , what happens to g_y as time goes to infinity. The RHS of (22) is non-negative. As long as s is strictly positive, capital accumulation continues. So if s is kept positive F_K falls forever since F_{KK}<0. In other words, if F shows CRS in K and AL, then it shows DRS in K alone. If the Inada condition

$$\lim_{K \to \infty} F_K(K, AL) = 0 \tag{23}$$

then the economy thus is bound to approach zero growth due to (23) and the fact that s is finite.

With depreciation or population growth we need that the limit in (23) is large enough to account for depreciation and population growth.

What is the fix? We need to have a production function that shows CRS *in factors of production that are produced* (*i.e., that are accumable*). The most simple and straightforward way is to let the productivity index A depend on how much capital the economy has accumulated. We could think of this as a *Learning by doing* mechanism.

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So in the simplest endogenous growth model we could assume that the level of knowledge, measured by A is given by

$$A_t = aK_t \tag{24}$$

Substituting into the production function we have

$$Y_t = F\left(K_t, aK_t L\right) \tag{25}$$

From the CRS assumption we have

$$F\left(K_t, aK_t L\right) = K_t F(1, aL) \tag{26}$$

The term $F(1,aL)$ is constant so output is linear in K alone, i.e., shows CRS in K alone. In some of the already classic articles by Romer and others the constant was called A. Then we can write $Y=AK$, thus these models go under the name *AK-models.*

The basic ingredient in endogenous growth models is thus to construct a social production function that has CRS in produced factors of production. There are basically to lines of models that achieve this,

1. Knowledge or Human Capital in the production function. Knowledge and/or human capital is surely producable as you hopefully experience during this course. If then the production function has CRS in capital and human capital *together*.

2. Increasing specialization. Since Adam Smith we know that increasing market sizes permits increasing specialization. So as production increases the economy may be more and more efficient. This may be another source of endogenous growth.

. Growth rate in the simplest learning by doing (AK) model

Let the constant $F(1,aL)$ be called $\underline{A}$. Then we have

$$\begin{aligned} Y &= \underline{A}K \\ \dot{Y} &= \underline{A}\dot{K} \\ \Rightarrow g_y &= g_K \end{aligned} \tag{27}$$

So the growth rate of the economy depends on the growth rate of capital which in turn depends on the net savings rate. With a given savings rate s

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$$\dot{K} = sY = s\underline{A}K$$
$$g_K = s\underline{A} = g_y \tag{28}$$

In this simple model it is easy to endogenize the savings decision. With CRRA preferences

$$U = \int_t^{\infty} e^{-\rho(s-t)} u(c_s) ds$$
$$u(c_s) = (1-\sigma)^{-1} c_s^{1-\sigma} \tag{29}$$

the Euler equation, which we can derive using Optimal Control, states that

$$g_c = \frac{u'}{-u''c}(r_t - \rho) = \frac{r_t - \rho}{\sigma} \tag{30}$$

where r_t is the net return on savings. If this equals the full marginal return on capital (why should it not?) – $r_t = A$ so

$$g_c = \frac{\underline{A} - \rho}{\sigma} \tag{31}$$

The only saving rate which is consistent with (31) and the transversality conditions is that the growth rate of consumption equals the growth rate of output. What would happen otherwise? So

$$g_c = g_y$$
$$\Rightarrow \frac{\underline{A} - \rho}{\sigma} = s\underline{A} \tag{32}$$
$$s = \frac{\underline{A} - \rho}{\underline{A}\sigma}$$

Note that this model implies very simple dynamics – the economy is always in its steady state growth path. Shifts in some parameters or a shock to the capital stock implies an immediate jump to the new steady state growth path. Can you explain why?

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## .	Aggregate versus Private Knowledge

In the previous example there was no distinction between aggregate and private knowledge. This meant that the learning by doing effect of accumulating capital was fully internalized in the savings decision by the individuals. This may, reasonably, not be the case. Take the opposite view instead. Assume that there are an infinite number of identical small firms indexed by the rational numbers *i* on a unit interval [0,1]. Now that the production function for the individual firm is given by

$$Y_t^i = F\left(K_t^i, AL_t^i\right) \tag{33}$$

The level of knowledge is proportional to the aggregate stock of capital and is identical to all firms.

$$A_t = a\int_0^1 K_t^i di = aK_t \tag{34}$$

Now the *aggregate* production function can be written

$$\int_0^1 Y_t^i di \equiv Y_t = \int_0^1 F\left(K_t^i, aK_t L_t^i\right)di = \int_0^1 aK_t L_t^i F\left(\frac{K_t^i}{AK_t L_t^i}, 1\right)di = \underline{A}K_t \tag{35}$$

However, when a firm decides to increase its capital slightly, the effect that has on the aggregate capital stock and thus on the stock of knowledge is negligible. This since

$$\frac{\partial A_t}{\partial K_t^i} = \frac{\partial}{\partial K_t^i} a\int_0^1 K_t^i di = 0 \tag{36}$$

So the *private* return on capital is

$$F_K\left(K_t^i, aK_t L_t^i\right) \tag{37}$$

which is smaller than the social which equals

$$F_K\left(K_t^i, aK_t L_t^i\right) + F_L\left(K_t^i, aK_t L_t^i\right)aL = \underline{A} \tag{38}$$

Let, for example, the production function be

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$$
\begin{aligned}
F(K_t, L_t) &= K_t{}^{\alpha}(A_t L)^{1-a} = K_t{}^{\alpha}(aK_t)^{1-\alpha} L^{1-\alpha} \\
&= (aL)^{1-\alpha} K_t = \underline{A} K_t
\end{aligned}
\tag{39}
$$

The private return to capital is in this case

$$
\alpha K_t^{\alpha-1}(A_t L)^{1-a} = \alpha K_t^{\alpha-1}(aK_t L)^{1-a} = \alpha(aL)^{1-a} = \alpha \underline{A} \tag{40}
$$

Since this is the interest rate faced by consumers the Euler equation for consumers with CRRA utility gives

$$
\begin{aligned}
g_c &= \frac{r_t - \rho}{\sigma} = \frac{\alpha \underline{A} - \rho}{\sigma} \\
s_t &= \frac{\alpha \underline{A} - \rho}{\underline{A}\sigma}
\end{aligned}
\tag{41}
$$

So we see that growth in this model is to low compared to the welfare maximum given by (31)

. R&D

In the previous examples knowledge was produced as a by-product of capital accumulation. We can easily change that and introduce a specific sector where knowledge is produced. So let

$$
\begin{aligned}
\dot{A}_t &= b(a_L L)^{\gamma} A_t \\
g_A &= b(a_L L)^{\gamma}
\end{aligned}
\tag{42}
$$

where a_L is the share of labor that is allocated to knowledge accumulation. This sector could be thought of as an *R&D* sector or a schooling sector. The are some decreasing returns to scale in this sector if γ is smaller than unity. The production function for output and the capital accumulation is given by

$$
\begin{aligned}
Y_t &= F(K_t, (1-a_L) A_t L) \\
\dot{K}_t &= sY_t
\end{aligned}
\tag{43}
$$

For now we assume that savings and the share of labor allocated to R&D is fixed and exogenous. Now lets define

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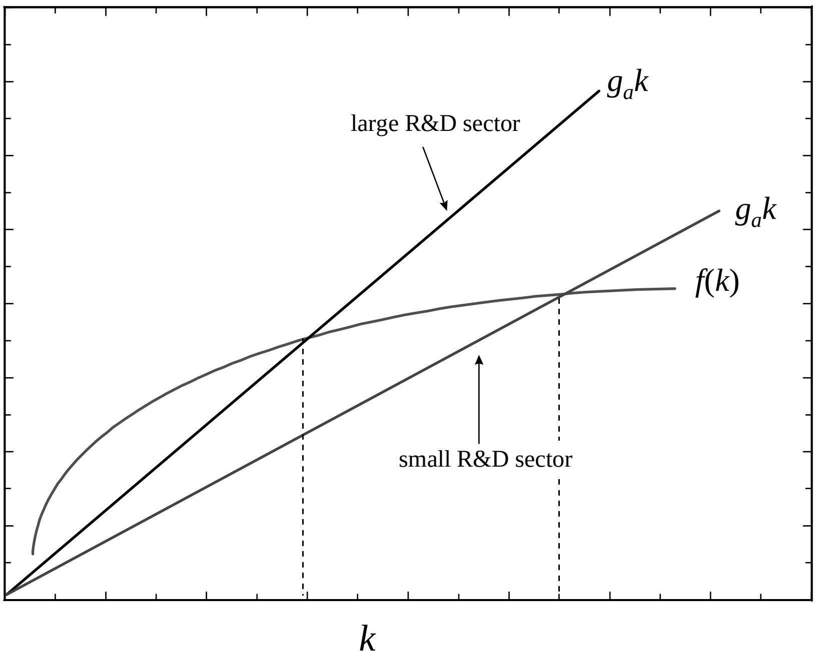
$$k_t \equiv \frac{K_t}{(1-a_L)A_t L}$$
$$f(k_t) \equiv F(k_t,1) \equiv \frac{F\left(K_t,(1-a_L)A_t L\right)}{(1-a_L)A_t L} \tag{44}$$

So, as in the Solow model

$$\dot{k}_t = f(k_t) - g_A k_t \tag{45}$$

Now this reminds us about the Solow model and a steady state occurs when (45) is zero. Let us look at a plot of the components in the RHS of (45).



We see that a large R&D sector and thus a high growth rate of knowledge implies a lower steady state capital/effective labor ratio and vice versa. What would happen if we got a shift from a low to a higher share of labor in the R&D sector?

In this model the savings rate is unimportant for long run growth as in the Solow model. But, now growth is (almost) endogenous. To really explain growth we would like to let a_L be determined within the model. This is the next step in the development of this model. We would like to have some firms who do the R&D and hire labor for this. Obviously we

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need to introduce some property rights for that purpose. Note that in such a model there is no reason to believe that growth is at its socially optimal level.

. Growth from increasing specialization

Now let us turn to the other source of sustained growth – specialization. Let us think of an economy with two sectors. Final output which is produced using labor (with inelastic supply) and a range of intermediate goods. The latter are produced with capital only. The production function for the final good is

$$Y_t = L^\alpha \sum_{i=1}^{M_t} x_{i,t}^{1-\alpha} \tag{46}$$

where each $x_{i,t}$ represents the amount of a particular intermediate input that is used at time t. M_t is the total number of intermediate inputs in use at t. We are going to look at symmetric equilibria where all $x_{i,t}$ are equal and denoted x_t so

$$Y_t = \sum_{i=1}^{M_t} x_t^{1-\alpha} = M_t x_t^{1-\alpha} \tag{47}$$

where we have normalized the labor supply to unity. The specification in (46) implies that there is positive returns to specialization. Consider the case when we double the number of inputs but use each in half the amount. Output is then

$$Y_t = 2M_t \left(\frac{x_t}{2}\right)^{1-\alpha} = 2^\alpha M_t x_t^{1-\alpha} \tag{48}$$

Now look at the intermediate goods production sector. Here capital is transformed into intermediate goods by M_t firms. There is a fixed cost of running a firm (paid in capital) so the capital requirement of firm i which produces $x_{i,t}$ can be written

$$k_{i,t} = h_0 + h_1 x_{i,t} \tag{49}$$

Now we assume that the final goods sector is competitive and that it buys intermediate goods so that its marginal product equals its price

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$$\frac{\partial Y_t}{\partial x_{i,t}} = (1-\alpha) x_{i,t}^{-\alpha} = p_{i,t} \tag{50}$$

This is the inverse demand faced by the intermediate goods producing firms so their profits are

$$\begin{aligned} \pi_{i,t} &= p_{i,t} x_{i,t} - (h_0 + h_1 x_{i,t}) r_t \\ &= (1-\alpha) x_{i,t}^{1-\alpha} - (h_0 + h_1 x_{i,t}) r_t \end{aligned} \tag{51}$$

where r_t is the user cost of capital (interest rate plus depreciation). From (51) follows that the profit maximum for intermediate goods producing firms is

$$\begin{aligned} &(1-\alpha)^2 x_{i,t}^{-\alpha} = h_1 r_t \\ &\Rightarrow r_t = \frac{(1-\alpha)^2 x_{i,t}^{-\alpha}}{h_1} = \frac{(1-\alpha)^2 x_t^{-\alpha}}{h_1} \end{aligned} \tag{52}$$

where the last equality comes from the assumption of a symmetric equilibrium. Now assume that there is no barriers to entry so profits in the intermediate goods sector are zero.

$$(1-\alpha) x_t^{1-\alpha} = (h_0 + h_1 x_t) r_t \tag{53}$$

Lastly, we require that the capital market is in equilibrium so that supply of capital (which is given at any point in time) equals its demand. This implies

$$M_t (h_0 + h_1 x_t) = K_t \tag{54}$$

Substituting from (52) into (53) we get

$$\begin{aligned} &(1-\alpha) x_t^{1-\alpha} = (h_0 + h_1 x_t) \frac{(1-\alpha)^2 x_t^{-\alpha}}{h_1} \\ &\Rightarrow x_t = \frac{1-\alpha}{\alpha} \frac{h_0}{h_1} \end{aligned} \tag{55}$$

Lastly, substituting this into (54) we get

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$$M = \frac{K_t \alpha}{h_0} \tag{56}$$

Substituting these results into the production function for aggregate output we get

$$Y_t = M_t x_t^{1-\alpha} = K_t \frac{\alpha}{h_0}\left(\frac{1-\alpha}{\alpha}\frac{h_0}{h_1}\right)^{1-\alpha} \equiv AK_t \tag{57}$$

so we are back into the simple *AK* model. Now we just have to make some additional assumption about savings, for example that it is chosen optimally by individuals facing some interest rate, for example the one established at the capital market and given by (53) if there are no capital income taxes. That savings rate determines the growth rate of capital which from (57) is identical to the growth rate of output.

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I.D.Fiscal Policy

In the neo-classical model we can think of (at least) four ways the fiscal policy affects the growth path of the economy

1. Government spending directly affecting production and utility functions.

$$\begin{aligned} U &= U(c,l,g) \\ F &= F(K_p,K_G,L) \end{aligned} \tag{58}$$

By changing government consumption utility (production) and marginal utility (products) changes. This would cause relative prices to change. A simplification is achieved if we let government services enter additively

$$U = u(c,l) + v(g) \tag{59}$$

in this case g will not change the marginal utilities of goods and leisure.

2. Change the budget set of agents

3. Non-lump sum taxes which drive a wedge between MRS and MRT

4. Redistribute resources

. The intertemporal budget set – Ricardian Equivalence

Assume the government goods do not interact with other goods in utility or production and disregard exogenous growth in population or technology. We now want to show that the path of financing of government purchases is irrelevant for the consumers. I.e., the time profile of budget deficits and taxes is irrelevant. Only the present value of government spending and the initial debt level is important.

The government budget constraint gives

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$$
\begin{aligned}
&\dot{B}_t = G_t + r_t B_t - \tau_t \\
&e^{-\int_0^t r_s ds}\left(\dot{B} - r_t B_t\right) = \frac{d}{dt} e^{-\int_0^t r_s ds} b_t = e^{-\int_0^t r_s ds}\left(G_t - \tau_t\right) \\
&de^{-R(t)} B_t = e^{-R(t)}\left(G_t - \tau_t\right) dt \\
&e^{-R(t)} B_t - e^{R(0)} B_0 = \int_0^t e^{-R(s)}\left(G_t - \tau_t\right) ds
\end{aligned}
\tag{60}
$$

Now require that the present value of the debt converges

$$
\begin{aligned}
&\lim_{t\to\infty} e^{-R(t)} B_t = 0 \\
&\Rightarrow B_0 + \int_0^\infty e^{-R(s)}\left(G_s - \tau_s\right) ds = 0 \\
&B_0 + \int_0^\infty e^{-R(s)} G_s ds = \int_0^\infty e^{-R(s)} \tau_s ds
\end{aligned}
\tag{61}
$$

So the present value of all future taxes equals the present value of future government spendings plus current debt stock.

Now look at the consumers. Recall that the slope of the consumption path was given from the Euler equation.

$$
\frac{\dot{c}_t}{c_t} = \frac{u'}{-u'' c_t}\left(r_t - \rho\right) \tag{62}
$$

As long as the tax does not affect this the slope of consumption is unaffected by the way of financing. Now look at the budget constraint. The instantaneous budget constraint is

$$
\dot{A}_t = W_t - C_t - \tau_t + r_t A_t \tag{63}
$$

Integrate

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$$de^{-R(t)}A_t = e^{-R(t)}(W_t - C_t - \tau_t)dt$$

$$\lim_{T\to\infty} e^{-R(T)}A_T = 0 = A_0 + \int_0^\infty e^{-R(t)}(W_t - C_t - \tau_t)dt \tag{64}$$

$$\int_0^\infty e^{-R(t)}C_t dt = A_0 + \int_0^\infty e^{-R(t)}W_t dt - \int_0^\infty e^{-R(t)}\tau_t dt$$

Now we can substitute into the last line of (64)

$$\int_0^\infty e^{-R(t)}C_t dt = A_0 + \int_0^\infty e^{-R(t)}W_t dt - B_0 - \int_0^\infty e^{-R(s)}G_s ds \tag{65}$$

so we see that what is important for the households is only the present value of government spending plus initial debt.

Some critical assumptions for the Ricardian Equivalence result

1. Perfect capital markets (so the individual can save or borrow to offset any changes in the time path of taxes and deficits. Same interest for government and individuals.
2. Taxes are lump sum.
3. Infinite horizons, not necessarily infinite lives.

. Ricardian Equivalence with Finite lives and altruism

Consider the simplest example of this. Each generation lives for one period, cares about their kids.

$$\begin{aligned} V(A_t) &= \max_{c_t} U(c_t) + \beta V(A_{t+1}) \\ s.t.\ A_{t+1} &= (1+r)(A_t - c_t) \end{aligned} \tag{66}$$

Now we can substitute from future generations problems

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$$
\begin{aligned}
&V(A_t) = \max_{c_t} U(c_t) + \beta\left[\max_{c_{t+1}} U\left(c_{t+1}\right) + \beta V\left(A_{t+2}\right)\right] \\
&s.t. \quad A_{t+1} = (1+r)(w_t + A_t - c_t) \\
&\qquad\quad A_{t+2} = (1+r)(w_{t+1} + A_{t+1} - c_{t+1}) \\
&\Rightarrow \\
&V(A_t) = \max_{\{c_t\}} \sum_{s=0}^{\infty} \beta^s U(c_{t+s}) + \lim_{T\to\infty} \beta^T V\left(A_{t+T}\right) \\
&s.t. \sum_{s=0}^{\infty} (1+r)^{-s}\left(c_{t+s} - w_{t+1}\right) = A_t + \lim_{T\to\infty} (1+r)^{-T} A_{t+T}
\end{aligned}
\tag{67}
$$

As you see, this is mathematically equivalent to dynamic optimization problem with infinite horizons. So the solution to this problem must be unaffected by the financing path of the governments consumption. Note that we require perfect capital markets. In this context it in particular requires that A_t may be negative or that households choose positive values (bequests) voluntarily.

. Borrowing versus Taxing in the standard OLG model

Now let us look at whether Ricardian Equivalence holds in a standard OLG model without altruism. To focus on the path of financing rather than the spending side we consider a government which wants to finance a constant spending level of 0 for convenience. Consider a government that issues debt equal to B_t, gives the proceeds to the currently young, rolls over the debt for ever and the tax the young in all future periods to pay the interest rate minus the growth rate of the economy. This means that b_t will be held constant at a level *b*For simplicity let us use the log utility, Cobb-Douglas production specification.

The saving decision of the currently young given in (11) is unaffected but the transfer has to be added to the wage

$$
\begin{aligned}
C_{1,t} &= (1-s_t)\left(w_t A_t L_t + B_t\right) \\
&= \frac{1+\rho}{2+\rho}\left((1-\alpha)k_t^{\alpha} A_t L_t + B_t\right)
\end{aligned}
\tag{68}
$$

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From this we see that current consumption increases when the policy is started up. Total savings also increase and become

$$S_t = \frac{1}{2+\rho}\left((1-\alpha)k_t^{\alpha} A_t L_t + B_t\right) \tag{69}$$

Note that part of the savings is in form of government bonds. But it is only real savings that translate into next periods capital stock. To put it differently next periods capital stock is equal to private savings plus government savings, the latter being equal to minus B_t

$$\begin{aligned} K_{t+1} &= S_t - B_t \\ &= \frac{1}{2+\rho}\left((1-\alpha)k_t^{\alpha} A_t L_t + B_t\right) - B_t \\ &= \frac{(1-\alpha)}{2+\rho} k_t^{\alpha} A_t L_t - \frac{1+\rho}{2+\rho} B_t \\ \Rightarrow k_{t+1} &= \frac{(1-\alpha)k_t^{\alpha} - (1+\rho)b_t}{(2+\rho)(1+n)(1+g)} \end{aligned} \tag{70}$$

Alternatively we could derive (70) from noting that next periods capital stock equals production plus the capital stock minus consumption.

$$\begin{aligned} K_{t+1} &= K_t + Y_t - C_{1,t} - C_{2,t} \\ &= K_t + k_t^{\alpha} A_t L_t - C_{1,t} - K_t - \alpha k_t^{\alpha} A_t L_t \\ &= (1-\alpha)k_t^{\alpha} A_t L_t - \frac{1+\rho}{2+\rho}\left((1-\alpha)k_t^{\alpha} A_t L_t + B_t\right) \cdot \\ &= \frac{1}{2+\rho}\left((1-\alpha)k_t^{\alpha} A_t L - (1+\rho)B_t\right) \end{aligned} \tag{71}$$

From (70) we see that it is clear that the deficit financed transfer causes next periods capital stock to decrease. This means that the interest rate goes up. This in turn means that the old in the next period can consume more for each dollar saved when young. Since total savings (the sum of real and financial savings) increased, the old in t+1 will consume more. The generation born in time t when the debt was issued will this clearly benefit and we do not have Ricardian Equivalence.

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From next period the government takes away an amount equal to the debt plus interest rate from the young and gives them the rolled over debt. Let H_{t+1} denote total wealth of a generation born at t+1. We then have

$$\begin{aligned} H_{t+1} &= w(k_{t+1})A_{t+1}L_{t+1} - bA_t L_t\left(1+r(k_{t+1})\right) + bA_{t+1}L_{t+1} \\ &= w(k_{t+1})A_{t+1}L_{t+1} - bA_{t+1}L_{t+1}\frac{1+r(k_{t+1})}{(1+n)(1+g)} + bA_{t+1}L_{t+1} \\ &= A_{t+1}L_{t+1}\left(w(k_{t+1}) - b\left(\frac{r(k_{t+1})-(n+g+ng)}{(1+n)(1+g)}\right)\right) \end{aligned} \tag{72}$$

We see that wealth decrease (increase) in b if the economy is dynamically efficient (inefficient). This is for given k_{t+1} Since the introduction of a government debt necessarily implies a lower capital stock this has negative effects on the wealth and utility of the young in t+1. For sufficiently large dynamic inefficiencies it is possible that that the first effect dominates for small debt levels.

Now consider the consumption of the young;

$$C_{1,t+1} = \frac{1+\rho}{2+\rho}A_{t+1}L_{t+1}\left(w(k_{t+1}) - b\left(\frac{r(k_{t+1})-(n+g+ng)}{(1+n)(1+g)}\right)\right) \tag{73}$$

and consumption of the old is

$$\begin{aligned} C_{2,t+1} &= \left(1+r_{t+1}\right)\left(k_{t+1}A_{t+1}L_{t+1} + bA_t L_t\right) \\ &= \left(1+r_{t+1}\right)\left(k_{t+1}A_{t+1}L_{t+1} + b\frac{A_{t+1}L_{t+1}}{(1+n)(1+g)}\right) \end{aligned} \tag{74}$$

Now consider the capital accumulation equation. This equals total resources available minus consumption

$$K_{t+2} = K_{t+1} + Y_{t+1} - C_{2,t+1} - C_{1,t+1} \tag{75}$$

Normalizing and simplifying we get

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$$K_{t+2} = A_{t+1}L_{t+1} - A_{t+1}L_{t+1}b\frac{1+\rho}{2+\rho}$$
$$k_{t+2} = \frac{(1-\alpha)k_t^{\alpha} - b(1+\rho)}{(2+\rho)(1+n)(1+g)} \tag{76}$$

This follows from the fact that next periods capital stock is equal to the share of production that goes to the young minus their consumption. We see that this difference equation implies that next periods k is lower the larger is b. This implies that the new steady state capital stock is necessarily smaller when a debt roll-over is introduced.

. Distortive Taxation

A positive analysis of capital taxation

Now consider the case when taxes are distortive. In this case we cannot be sure that the path of financing of a given path of spendings is irrelevant. First consider a positive analysis of capital taxation in a standard Ramsey model. The households solves

$$\max_{\{c_t\}} \int_0^{\infty} e^{-\rho t} U(c_t)dt$$
$$s.t. \quad \dot{A}_t = w_t + T_t + (1-\tau_t)r_t A_t - c_t \tag{77}$$
$$\lim_{T\to\infty} e^{-rT} A_T = 0.$$

Competitive firms set wages and pre-tax interest rates equal to their respective marginal products. The Hamiltonian for the consumer is

$$\mathcal{H} = U(c_t) + \mu_t\left(w_t - c_t + (1-\tau_t)r_t A_t\right)$$
$$\mathcal{H}_c = U'(c_t) - \mu_t = 0 \tag{78}$$
$$\mathcal{H}_A = \mu_t(1-\tau_t)r_t = -\dot{\mu}_t + \rho A_t$$

Doing the standard trick of taking time derivatives of the optimality condition and substitute we find

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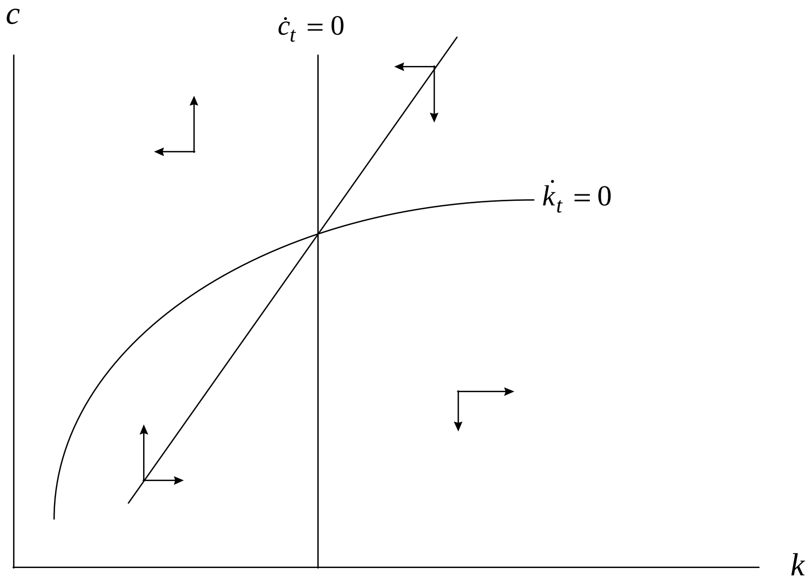
$$
\begin{aligned}
&U''(c_t)\dot{c}_t = \dot{\mu}_t \\
&U'(c_t)(1-\tau_t)r_t = -U''(c_t)\dot{c}_t + \rho U'(c_t) \qquad (79)\\
&\dot{c}_t = \frac{U'(c_t)}{-U''(c_t)}\big((1-\tau_t)r_t - \rho\big)
\end{aligned}
$$

The phase diagram is



Now consider a temporary tax increase that finance a temporary increase in transfers. We assume the tax increase occurs at a time when the economy is in a steady state. In the phase diagram the tax increase implies a leftward shift in the constant consumption curve. At the time of the tax decrease we have to be at the old saddle path again. The only way this can be acheievde is indicated below. Consumption jumps, and this leads to a decrease in the capital stock. consumption then falls and at the time of the shift back it is below the constant capital curve so consumption and capital grows and the old steady state is then approached.

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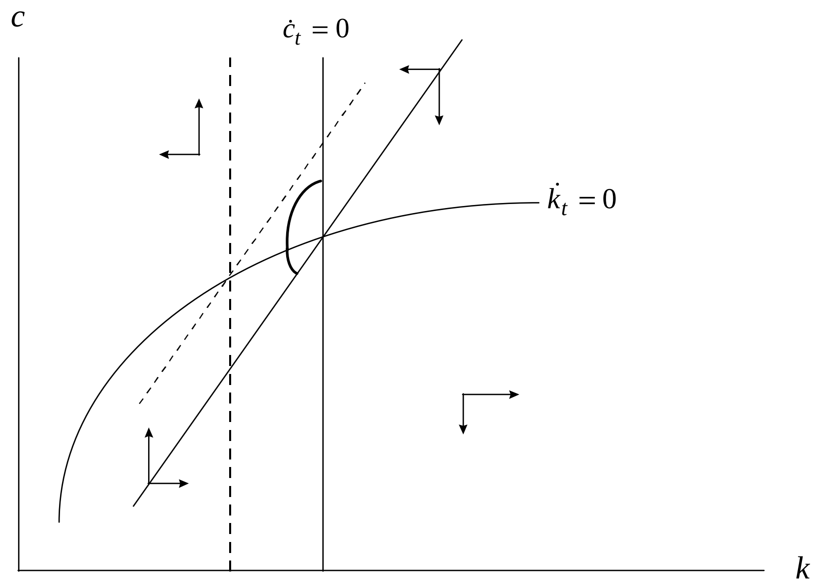

A normative analysis of labor taxation

To study this consider a benevolent government that wants to finance a given but arbitrary stream of spendings $\{g_t\}$ with a tax on an elastic labor supply. First consider the consumer. He solves

$$\max_{\{c_t\}} \int_0^\infty e^{-rt}\big(U(c_t)+V(x_t)\big)dt$$
$$s.t. \quad \dot{A}_t = w(1-x_t)(1-\tau_t)-c_t+rA_t \qquad (80)$$
$$\lim_{T\to\infty} e^{-rT}A_T = 0.$$

where c is consumption, x is leisure and τ is a potentially time varying labor tax. For simplicity we have assumed an additive utility function so that the marginal utilities are independent of other consumption.

The current value Hamiltonian is

$$\mathcal{H} = U(c_t)+V(x_t)+\mu_t\big(w(1-x_t)(1-\tau_t)-c_t+rA_t\big) \qquad (81)$$

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with necessary condition

$$\begin{aligned} \mathcal{H}_c &= U'(c_t) - \mu_t = 0 \\ \mathcal{H}_x &= V'(x_t) - \mu_t w(1-\tau_t) = 0 \\ \mathcal{H}_A &= \mu_t r = -\dot{\mu}_t + \mu_t r \end{aligned} \tag{82}$$

From the last condition we get that μ_t is constant, which implies that consumption is constant and that

$$\begin{aligned} &V'(x_t) = \mu w(1-\tau_t) \\ &\Rightarrow \tau_t = 1 - \frac{V'(x_t)}{\mu w} \end{aligned} \tag{83}$$

So the individual we work a lot when taxes are low and vice versa.

Now consider the benevolent government. It solves

$$\begin{aligned} &\max_{\{c_t\}} \int_0^\infty e^{-rt}\left(U(c_t) + V(x_t)\right)dt \\ &s.t. \qquad B_t = w(1-x_t)\tau_t - g_t + rB_t \\ &\qquad\qquad \lim_{T\to\infty} e^{-rT} B_T = 0. \end{aligned} \tag{84}$$

Since variations in the tax rate determines variations in labor supply, which is what the government is potentially worried about we can use (83) and substitute into the governments problem.

$$\begin{aligned} &\int_0^\infty e^{-rt}\left(U(c_t) + V(x_t)\right)dt \\ &s.t. \qquad B_t = w(1-x_t)\left(1 - \frac{V'(x_t)}{\mu w}\right) - g_t + rB_t \\ &\qquad\qquad \lim_{T\to\infty} e^{-rT} B_T = 0. \end{aligned} \tag{85}$$

The current value Hamiltonian with necessary conditions for optimality becomes

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$$\mathcal{H} = U(c_t) + V(x_t) + \lambda_t\left(w(1-x_t)\left(1-\frac{V'(x_t)}{\mu w}\right) - g_t + rB_t\right)$$

$$\mathcal{H}_x = V'(x_t) - \lambda_t\left(w - \frac{V'(x_t)}{\mu} + (1-x_t)\frac{V''(x_t)}{\mu}\right) = 0 \tag{86}$$

$$\mathcal{H}_B = \lambda_t r = -\dot{\lambda}_t + \lambda_t r$$

From the last we see that λ_t is constant. Using this in first order condition we get

$$V'(x_t)\left(1+\frac{\lambda}{\mu}\right) = \lambda w + \frac{\lambda}{\mu}(1-x_t)V''(x_t) \tag{87}$$

If this equation has a solution it implies a constant level of x_t. A case when no solution may exist is when *V''* is zero. Why is that?

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I.E.Some Evidence

The neo-classical growth model implies convergence. Growth should be inversely related to starting levels of GDP – poor countries should grow faster than rich. Weak or no correlation, however, is generally found between growth and initial GDP. But other parameters may differ between countries. A poor country could be poor because it is far below its steady state growth path (implying high growth) *or* because its steady state growth path is below other countries (implying same growth as richer countries). Differences in savings rates could explain differences in steady states.

Another *failure* of the standard Solow model is that the implied convergence rate appears to be too fast.

Now take an augmented Solow model with human capital.

$$\begin{aligned} Y(t) &= K(t)^{\alpha} H(t)^{\beta} (A(t)L(t))^{1-\alpha-\beta} \\ \dot{k}(t) &= s_k y(t) - (n+g+\delta)k(t) \\ \dot{h}(t) &= s_h y(t) - (n+g+\delta)k(t) \end{aligned} \tag{88}$$

You can then show that

$$\begin{aligned} k^* &= \left(\frac{s_k^{1-\beta} s_h^{\beta}}{n+g+\delta} \right)^{1/(1-\alpha-\beta)} \\ h^* &= \left(\frac{s_k^{\alpha} s_h^{1-a}}{n+g+\delta} \right)^{1/(1-\alpha-\beta)} \end{aligned} \tag{89}$$

and

$$y^* = (k^*)^{\alpha} (h^*)^{\beta} \tag{90}$$

A log approximation of the speed of convergence close to the steady state gives

$$\begin{aligned} \frac{d \ln y(t)}{dt} &= \lambda \left(\ln y^* - \ln y(t) \right) \\ \lambda &\equiv (n+g+\delta)(1-\alpha-\beta) \end{aligned} \tag{91}$$

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Note that the convergence rate decreases in β.

We can solve (91)

$$\ln y(t) = e^{-\lambda t}\left(\ln y(0) - \ln y^*\right) + \ln y^* \tag{92}$$

or

$$\ln y(t) - \ln y(0) = (1 - e^{-\lambda t})\ln y^* - (1 - e^{-\lambda t})\ln y(0) \tag{93}$$

This can be run as a regression. The growth rate should be negatively related to initial GDP. We must, however, control for differences in ln y^*. Now substitute from (89) and (90) into (93). This yields

$$\begin{aligned}\ln y(t) - \ln y(0) = {} & (1 - e^{-\lambda t})\frac{\alpha}{1-\alpha-\beta}\ln s_k + (1 - e^{-\lambda t})\frac{\beta}{1-\alpha-\beta}\ln s_h \\ & -(1 - e^{-\lambda t})\frac{\alpha+\beta}{1-\alpha-\beta}\ln(n+g+\delta) - (1 - e^{-\lambda t})\ln y(0).\end{aligned} \tag{94}$$

So we should include measures of the two savings rates plus n+g+δ in the regression. Mankiw, Romer and Weil (1992) runs that regression on the Summers and Heston dataset (98 countries, 1960-85).

Variable	coeff.	St.dev.
ln(GDP 1960)	-0.289	0.062
ln(average investment rate)	0.524	0.087
ln(Secondary School enrollment 1960-85)	0.233	0.060
ln(n+g+δ)	-0.505	0.288

In a regression with the restriction that the coefficients on the two savings rates and n+g+δ sum to zero α and β and λ are identified. The estimated values are 0.48, 0.23 and 0.0142. This is substantially closer to what we believe is true in reality.

Conclusion.

- Substantial evidence for conditional convergence.
- The Solow model does a good job in explaning growth differences when differences in savings rates are taken into account.

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- The implied rate of convergence is reasonable when the production function includes human capital.

Barro (91) runs regressions of growth between 1960 and 1985 against initial GDP level and other variables that may affect growth or the level of the steady state growth path. The sample is 98 countries from Summers and Heston (1988).

Variable	t-stat
GDP in 1960	-6.3
Secondary School enrollment 1960	3.8
Primary School enrollment 1960	4.5
Average Government spending ratio	-4.3
Frequency of revolutions and coups	-3.1
Rate of assassinations	-2.1
Price distortions for investment goods	-2.7

We see that there is quite strong evidence in favor of *conditional convergence*. Also the other variables come in significantly with expected signs. Unclear if this is due to endogenous growth effects of different steady state growth paths with equal growth. The augmented Solow model cannot be rejected. (See Mankiw, Romer & Weil, 1992).

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II. Fluctuations

II.A. Consumption and Investment under Uncertainty

Three steps in the development of consumption modeling

1. **Keynesian**

$$C_i = a + bY_i + \varepsilon_i \tag{2.1}$$

worked well in cross-section studies with $a>0$, $0<b<1$. Not consistent with time series evidence. Led to

2. **Permanent income (Friedman, -57) – Life Cycle (Modigliani) hypothesis.**

$$c_t = \frac{A_t + E_t \sum_{s=t}^{T} y_s}{T - t + 1} \tag{2.2}$$

Distinguishes life time wealth and transitory income changes. Relation between income and consumption depends on relation between income changes and lifetime wealth. Compare

$$\begin{aligned} y_t &= y_{t-1} + \varepsilon_t \\ y_t &= kt + \varepsilon_t \end{aligned} \tag{2.3}$$

No account taken for second moments, effects of uncertainty.

3. **Consumption and investment under uncertainty.**

   a. Precautionary savings. Leland (-68), Kimball (-90)

   b. Irreversible investments. Option value of Waiting. Pindyck (-91), McDonald & Siegel (-86).

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1. Permanent Income, Precautionary Savings and Liquidity Constraints

Consider the problem

$$V_0(A_0) \equiv \max_{\{c_t,\omega_t\}} E\left[\sum_{t=0}^{T}(1+\rho)^{-t}U(c_t)\middle|\Omega_0\right]$$
$$s.t. \quad A_{t+1} = (A_t + \tilde{y}_t - c_t)\left(R_{t+1}\omega_t + \tilde{Z}_{t+1}(1-\omega_t)\right), \tag{2.4}$$
$$A_{T+1} \geq 0,$$
$$A_0 \text{ given.}$$

Use the Bellman equation

$$V_t(A_t) = \max_{c_t,\omega_t} E_t\left[U(c_t) + (1+\rho)^{-1}V_{t+1}\left((A_t + \tilde{y}_t - c_t)\left(R_{t+1}\omega_t + \tilde{Z}_{t+1}(1-\omega_t)\right)\right)\right] \tag{2.5}$$

FOC:

$$FOC\, c_t;$$
$$U'(c_t) = (1+\rho)^{-1}E_t\left[V'_{t+1}(A_{t+1})\left(R_{t+1}\omega_t + \tilde{Z}_{t+1}(1-\omega_t)\right)\right] \tag{2.6}$$

Use the Envelope theorem

$$V'_t(A_t) = (1+\rho)^{-1}E_t\left[V'_{t+1}(A_{t+1}))\left(\left(R_{t+1}\omega_t + \tilde{Z}_{t+1}(1-\omega_t)\right)\right)\right] \tag{2.7}$$

So

$$U'(c_t) = E_t\left[(1+\rho)^{-1}U'(c_{t+1})\left(\left(R_{t+1}\omega_t + \tilde{Z}_{t+1}(1-\omega_t)\right)\right)\right] \tag{2.8}$$

This is the Euler equation for consumption.

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$$
\begin{aligned}
&FOC\,\omega_t;\\
&E_t\left[(1+\rho)^{-1}V'_{t+1}(A_{t+1})\right)\left(A_t+y_t-c_t\right)\left(R_{t+1}-\tilde{Z}_{t+1}\right)\Big]=0\\
&\Rightarrow E_t\left[U'(c_{t+1})\right)\left(R_{t+1}-\tilde{Z}_{t+1}\right)\Big]=0
\end{aligned}
\tag{2.9}
$$

From (2.9) we can also derive the result that all agents should invest in the risky asset if it has an expected return that is higher than R_{t+1}. This also if they have very high risk aversion. To see this write (2.9) as

$$
\begin{aligned}
R_{t+1}E_t\left[U'\left(c_{t+1}\right)\right] &= E_t\left[U'\left(c_{t+1}\right)\tilde{Z}_{t+1}\right]\\
&= E_t\left[U'\left(c_{t+1}\right)\right]E_t\left[\tilde{Z}_{t+1}\right]+Cov\left[U'\left(c_{t+1}\right),\tilde{Z}_{t+1}\right]
\end{aligned}
\tag{2.10}
$$

The Covariance term is zero when ω is unity. So (2.10) cannot hold then.

Some particular Euler Equations

Non-stochastic interest rate.

$$
\begin{aligned}
&U'\left(c_{t+1}\right)=\frac{1+\rho}{1+r}U'\left(c_t\right)+\varepsilon_{t+1}\\
&E\left[\varepsilon_{t+1}\middle|\Omega_t\right]\equiv E_t\left[\varepsilon_{t+1}\right]=0.
\end{aligned}
\tag{2.11}
$$

Quadratic utility

$$
U\left(c_t\right)=ac_t-\frac{bc_t^2}{2}
\tag{2.12}
$$

$$
\begin{aligned}
&E_t\left(a-bc_{t+1}\right)=\frac{1+\rho}{1+r}\left(a-bc_t\right)\\
&c_{t+1}=\frac{a}{b}\frac{r-\rho}{1+r}+\frac{1+\rho}{1+r}c_t+\varepsilon_{t+1}
\end{aligned}
\tag{2.13}
$$

This is the Hall equation. No variable known in t is correlated with ε_{t+1}. Can be tested using OLS.

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The Hall Equation as a First Order Linear Approximation

$$
\begin{aligned}
E_t U'(c_{t+1}) &= \frac{1+\rho}{1+r} U'(c_t) \\
E_t U'(c_{t+1}) &\approx E_t\left[U'(c_t) + U''(c_t)(c_{t+1} - c_t)\right] \\
&= U'(c_t) + U''(c_t) E_t(c_{t+1} - c_t) \\
&= \frac{1+\rho}{1+r} U'(c_t) \\
\Rightarrow \Delta c_{t+1} &= \frac{r-\rho}{1+r} \frac{U'(c_t)}{-U''(c_t)} + \varepsilon_{t+1}
\end{aligned} \tag{2.14}
$$

Note that low relative risk aversion and $r>\rho$ gives high consumption growth. Why? Note also that by taking a first order approximation we disregard third moments in utility and second moments in consumption – certainty equivalence.

Examples of analytical solutions.

A. Quadratic utility, constant interest rate, only income risk, finite horizon.

Simplify $\rho=r$.

$$
\begin{aligned}
&c_{t+1} = c_t + \varepsilon_{t+1} \\
&\Rightarrow E_t c_{t+s} = c_t \forall s \geq 0.
\end{aligned} \tag{2.15}
$$

From intertemporal (collapsed) budget we know that

$$
\begin{aligned}
&\sum_{s=t}^{T} \frac{c_s}{(1+r)^{s-t}} = A_t + \sum_{s=t}^{T} \frac{y_s}{(1+r)^{s-t}} \\
&E_t \sum_{s=t}^{T} \frac{c_s}{(1+r)^{s-t}} = A_t + E_t \sum_{s=t}^{T} \frac{y_s}{(1+r)^{s-t}} \equiv A_t + H_t
\end{aligned} \tag{2.16}
$$

Then using (2.15) we get

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$$c_t = \frac{r}{(1+r)\left(1-(1+r)^{-T+t-1}\right)}\left(\overbrace{\{A_t + H_t\}}^{W_t}\right) \tag{2.17}$$

Certainty equivalence.

Look at ρ=r=0. Then perfect smoothing

$$c_t = \frac{W_t}{T-t+1} \tag{2.18}$$

This is the *Modigliani Life Cycle Hypothesis.*

Let T=∞ and ρ=r>0. Then

$$c_t = \frac{rW_t}{1+r} = \bar{y}_t^P \tag{2.19}$$

the *Friedman Permanent Income Hypothesis.*

B. No labor income, interest rate risk (*multiplicative*), CRRA (e.g. log), infinite horizon, time autonomous problem (z i.i.d.).

$W_{t+1} = (1+z_{t+1})(W_t - c_t)$. This case we solved in MATFUII and showed that for log utility

$$c_t = \frac{\rho}{1+\rho} W_t \tag{2.20}$$

where 1/(1+ρ) is discount factor. Note that log W_t follows a random walk. A kind of certainty equivalence since for log utility income and substitution effects cancel.

C. Only labor income risk (*additive*) and normal i.i.d. innovations, finite horizon, CARA (exponential) utility. Simplify and set ρ=r=0. The consumer solves

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$$\max_{\{c_t\}} E_0\left[\sum_{t=0}^{T}\frac{e^{-\gamma c_t}}{-\gamma}\right]$$
$$s.t.\quad A_{t+1} = A_t + y_t - c_t \tag{2.21}$$
$$A_0 = A_T = 0.$$

Assume a process for y_t , for example

$$y_{t+1} = y_t + \varepsilon_{t+1}. \tag{2.22}$$

with ε_{t+1} N(0,σ^2). Guess that

$$c_{t+1} = c_t + \frac{\gamma\sigma^2}{2} + \varepsilon_{t+1}. \tag{2.23}$$

By using that if $c \overset{d}{=} N\left(\bar{c},\sigma^2\right)$ then $E\left[e^{-\gamma c}\right] = e^{-\gamma\bar{c}+\frac{\gamma^2\sigma^2}{2}}$ we can check that this satisfies the Euler equation. The budget constraint implies

$$\sum_{s=t}^{T} c_s = A_t + \sum_{s=t}^{T} y_t \tag{2.24}$$

With the expressions for expected consumption and income given by (2.23) and (2.22) (2.24), after taking expected values as of *t*, simplifies to

$$c_t = \frac{1}{T-t+1}A_t + y_t - \frac{\gamma(T-t)\sigma^2}{4} \tag{2.25}$$

Note the problem with long and infinite horizons, consumption may be negative.

Quantifying Precautionary Savings

Take the Euler equation

$$E_t U'\left(c_{t+s}\right) = \left(\frac{1+\rho}{1+r}\right)^s U'\left(c_t\right) \qquad \forall s \geq 0. \tag{2.26}$$

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Note that if *U'* is convex the LHS is increasing in a mean preserving risk increase. Increases in risk thus has to be matched by decreasing consumption today and increasing expected consumption tomorrow. Both helps restore (2.26).

Do Taylor approximation of (2.26) letting ρ=*r*

$$\begin{aligned} &U'(c_t) \\ &= E_t\left[U'(c_{t+1})\right] \approx U'(c_t) + U''(c_t)E_t\left[c_{t+1}-c_t\right] + \frac{1}{2}U'''(c_t)E_t\left[(c_{t+1}-c_t)^2\right] \qquad (2.27) \\ &\Rightarrow E_t\left[c_{t+1}-c_t\right] \approx \underbrace{\frac{-U'''(c_t)}{U''(c_t)}}_{p_a}\frac{1}{2}E_t\left[(c_{t+1}-c_t)^2\right] \end{aligned}$$

or

$$E_t\left[\frac{c_{t+1}-c_t}{c_t}\right] \approx \underbrace{\frac{-U'''(c_t)c_t}{U''(c_t)}}_{p_r}\frac{1}{2}E_t\left[\left(\frac{c_{t+1}-c_t}{c_t}\right)^2\right] \qquad (2.28)$$

p_a and p_r are the absolute and relative coefficients of prudence.

Liquidity Constraints

All derivations above have assumed that the individual can save or borrow to smooth consumption so that the Euler equation is satisfied. But borrowing may, for several reasons, not be available in the required amount to make the Euler equation satisfied. In discrete time it is easy to show that the Euler equation for a period when the individual is liquidity constrained is given by

$$U'(c_t) = \lambda_t + E_t U'(c_{t+1}) \qquad (2.29)$$

where we, just for simplicity assumed that the market interest rate coincides with the subjective discount rate. We let λ_t denote the shadow value (Lagrange multiplier) on the liquidity constraint that financial assets have to be non-negative ($A_t \geq 0\ \forall\ t$). Certainly all extra income the individual receives in *t* will be consumed as long as $\lambda_t > 0$, i.e., as long as marginal utility today is higher than expected marginal utility tomorrow.

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Now consider previous periods. Let us assume that the liquidity constraint is not binding then. For these periods the Lagrange multiplier is zero so the standard Euler equation holds implying

$$U'(c_{t-1}) = E_{t-1}U'(c_t) = E_{t-1}(\lambda_t + U'(c_{t+1})) \tag{2.30}$$

This means that the liquidity constraint affects consumption also in previous periods. Let us think of this as a three period problem. We then understand that some extra income in period *t*-1 will be smoothed out for consumption in period *t*-1 and *t* – over two, rather than three periods as in the non-constrained case. We thus see a higher propensity to consume out of income (but lower than 1) than in the risk-neutral non-constrained case. This is going to look exactly as if the individual reduced precautionary savings in response to a positive income shock. In empirical tests it is thus difficult or impossible to distinguish precautionary savings from potentially binding future liquidity constraints.

2. The Lucas Critique and Some Empirical Consumption Tests and Puzzles

The Lucas Critique

Sample moments between observed macro variables – like consumption, disposable income and output – change when policy change. This since optimum decision rules change with policy. Econometric models can thus only be used in short-term forecasting and can “provide *no* useful information as to the actual consequences of alternative economic policies”.

MPC example

Assume that disposable income follows

$$y_t = \rho y_{t-1} + \varepsilon_t \tag{2.31}$$

Then

$$H_t = \frac{1+r}{1+r-\rho} y_t \tag{2.32}$$

use (2.19) then we may calculate MPC

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$$\frac{\partial c_t}{\partial y_t} = \frac{\partial c}{\partial H_t}\frac{\partial H_t}{\partial y_t} = \frac{r}{1+r}\frac{1+r}{1+r-\rho} = \frac{r}{1+r-\rho} \tag{2.33}$$

If ρ is close to unity, MPC is close to unity. Now let there be a temporary lump sum transfer τ to the house hold. A naive Keynesian would say that this increase consumption almost one for one. But,

$$c_t = \frac{r}{1+r}\left(A_t + H_t + \tau\right) \tag{2.34}$$

Calculate MPC out of the transfer

$$\frac{\partial c_t}{\partial \tau} = \frac{\partial c}{\partial H}\frac{\partial H_t}{\partial \tau_t} = \frac{r}{1+r} \tag{2.35}$$

So if MPC is estimated on income and then used to predict effects of fiscal policy we get wrong results (if $\rho \neq 0$). Must have an *economic* model to understand the effects of policy changes.

A generalization of (2.32).

If

$$A(L) y_t = \varepsilon_t \tag{2.36}$$

then

$$\begin{aligned} H_t &\equiv E_t \sum_{s=t}^{\infty} (1+r)^{-s+t} y_s \\ &= \frac{y_t}{A\left(\frac{1}{1+r}\right)} \end{aligned} \tag{2.37}$$

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3. Tests and Puzzles

Hall -78

$$c_{t+1} = \lambda c_t + \varepsilon_{t+1}$$
$$\varepsilon_{t+1} \perp \Omega_t \tag{2.38}$$

Can be tested by adding variables known in *t* to an OLS regression. Finds no influence from c_{t-1-s} and y_{t-s}. *S&P* stock market index has a significant influence. Suggested explanation – slow adjustments.

Carrol and Summers -89

Strong correlation long run growth in aggregate income and consumption in cross country study. Also at individual level. Appears that consumption grows one for one also with expected growth in income.

Potential explanations;

Liquidity constraints – must in such case be almost everybody. Most people have only very low financial savings.

Flavin -81, JPE

"Excess Sensitivity" to predicted changes in income.

Consumption change to predicted changes in income, e.g., when new pensions are paid out not when they are decided upon.

Campbell and Deaton -89

"Excess Smoothness".

Recall (2.33) and (2.37). There we see that

$$\sigma(\Delta c_t) = \frac{r}{1+r-\rho}\sigma(\varepsilon_t) \tag{2.39}$$

C&D estimates a second order process for income

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$$\begin{aligned}\Delta y_t &= \alpha + \rho \Delta y_{t-1} + \varepsilon_t, \\ A(L) y_t &= \alpha + \varepsilon_t, \\ A(L) &\equiv 1 - (1+\rho) L + \rho L^2.\end{aligned} \tag{2.40}$$

with ρ=0.442. An increase in growth signals future high growth. Then a shift in income today has a very large effect on permanent income so consumption should change very much. In this case

$$\begin{aligned}\sigma_{\Delta c_t} &= \frac{r}{1+r} \frac{1}{\left(1 - \frac{1+\rho}{1+r} + \frac{\rho}{(1+r)^2}\right)} \sigma_{\varepsilon_t} \\ &= \frac{1+r}{1+r-\rho} \sigma_{\varepsilon_t} \approx 1.8 \sigma_{\varepsilon_t}\end{aligned} \tag{2.41}$$

Instead they find that the coefficient is around 1 and only 1/2 for non-durables. So consumption is *excessively smooth*. This relies on non-stationary income.

Excess sensitivity and excess smoothness is two sides of the same coin. When expected future income increases consumption rise less then permanent income but responds when expectations are realized.

Caballero QJE -90 discusses precautionary savings as an explanation for excess sensitivity and excess smoothness. Assume that expected volatility of future earnings increase in the level (e.g., if *y* is a log random walk). Then a positive shock to expected future earnings increase precautionary savings so consumption does not respond one for one. When realized risk disappears so consumption increase.

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II.B.Asset Pricing

In the previous sections we used the Euler equation to derive optimal consumption and investment decisions. Now note that the Euler equation defines a relation between consumption (or other real variables) and prices.

$$\text{Path of prices} \leftrightarrow \text{Path of real variables} \tag{2.42}$$

Previously we took the prices as given and derived the optimal path of consumption. We may, however, use the Euler relation in the other direction. Take the path of real variables, e.g., consumption as given and derive what the prices have to be. A straightforward way to do this is to assume that output is exogenous, like manna from heaven, and cannot be stored. In that environment we may introduce markets for capital and production facilities. This is the setup in the seminal Lucas (*Econometrica*, 1978) paper.

We will later relax the assumption about exogenous and non-storable output. This will give us the basic stochastic growth (or RBC) model.

1. Asset Pricing in the Lucas Tree Model and other CAPM

. Asset Pricing in the Lucas Tree Model

1) Large number of identical agents.
2) Equal number of trees with stochastic crop d_t The distribution of d_t is Markov. Distribution is $F\left(d_t|d_{t-1}\right)=F\left(d_t|d_{t-1},d_{t-2},\ldots\right)$. The process known by all agents.
3) Purpose: find p_t – the price of a tree as a function of the state of the economy (d_t).
4) The gross return on a tree is $\frac{p_{t+1}+d_{t+1}}{p_t}$ (per capita)
5) No safe asset.
6) Perfect market in ownership of trees. All equal so no trade in equilibrium.
7) No storage or foreign trade so consumption $c_t = d_t$

Substitute for $\tilde{z}_{t+1}$ from the Euler equation noting that ω =0 gives

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$$U'(c_t) = E_t\left[(1+\rho)^{-1}U'(c_{t+1})\frac{(p_{t+1}+d_{t+1})}{p_t}\right]$$

or (2.43)

$$p_t = E_t\left[\beta\frac{U'(c_{t+1})}{U'(c_t)}(p_{t+1}+d_{t+1})\right]$$

By 7)

$$p_t = E_t\left[\beta\frac{U'(d_{t+1})}{U'(d_t)}(p_{t+1}+d_{t+1})\right] \tag{2.44}$$

For any type of expectations in (2.44) based on d_t we can compute a price today as a function of d_t Let the individuals subjective expectations of d_{t+1} be described by $F^s(d_{t+1}|d_t)$ and the expectations about the relation between p_{t+1} and d_{t+1} be given by the function $p^s(d_{t+1})$. We then have

$$\begin{aligned} p_t &= \int_{-\infty}^{\infty}\beta\frac{U'(d_{t+1})}{U'(d_t)}\left(p^s(d_{t+1})+d_{t+1}\right)dF^s(d_{t+1}|d_t) \\ &\equiv p(d_t). \end{aligned} \tag{2.45}$$

Now Lucas defined the very powerful concept of *rational expectations*. Let's require that $p(\cdot)\equiv p^s(\cdot)$ and $F(d_{t+1}|d_t)\equiv F^s(d_{t+1}|d_t)$. Lucas proves that this together with (2.45) defines a *unique* and constant pricing function $p(\cdot)$.

Use recursions on (2.44)

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$$p_t = E_t\left[\beta\frac{U'(d_{t+1})}{U'(d_t)}\left(E_{t+1}\left[\beta\frac{U'(d_{t+2})}{U'(d_{t+1})}\left(E_{t+2}\left[\beta\frac{U'(d_{t+3})}{U'(d_{t+2})}(\ldots+d_{t+3})\right]+d_{t+2}\right)\right]+d_{t+1}\right)\right]$$

$$= E_t\left[\beta\frac{U'(d_{t+1})}{U'(d_t)}\left(\left[\beta\frac{U'(d_{t+2})}{U'(d_{t+1})}\left(\left[\beta\frac{U'(d_{t+3})}{U'(d_{t+1})}(\ldots+d_{t+3})\right]+d_{t+2}\right)\right]+d_{t+1}\right)\right] \qquad (2.46)$$

$$= E_t\sum_{j=1}^{\infty}\beta^j\frac{U'\left(d_{t+j}\right)d_{t+j}}{U'(d_t)} + \lim_{j\to\infty}\beta^j\frac{U'\left(d_{t+j}\right)p_{t+j}}{U'(d_t)}$$

A discounted sum of dividends. Stochastic discount rates unless marginal utility is constant.

A simple example with log utility

$$p_t = E_t\sum_{j=1}^{\infty}\beta^j\frac{U'\left(d_{t+j}\right)d_{t+j}}{U'(d_t)} = E_t\sum_{j=1}^{\infty}\beta^j\frac{d_t d_{t+j}}{d_{t+j}} = d_t\sum_{j=1}^{\infty}\beta^j = d_t\frac{\beta}{1-\beta} \qquad (2.47)$$

With d_t i.i.d. so $E_tU'\left(d_{t+j}\right)d_{t+j} = E_tU'(d_{t+s})d_{t+s}\forall j,s>0$

$$p_t = E_t\sum_{j=1}^{\infty}\beta^j\frac{U'\left(d_{t+j}\right)d_{t+j}}{U'(d_t)} = \frac{\beta}{(1-\beta)U'(d_t)}\overbrace{EU'(d_{t+1})d_{t+1}}^{\text{constant}} \qquad (2.48)$$

Note that the price increases in d_t (if U is concave).

Assume an autocorrelation in d_t then $E_t\sum_{j=1}^{\infty}\beta^jU'\left(d_{t+j}\right)d_{t+j}$ depends on d_t Both income and substitution effect, with log utility they cancel.

The Consumption and market CAPM

Previously we found that

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$$U'(c_t)=E_t\left[(1+\rho)^{-1}U'(c_{t+1})\left(\left(R_{t+1}\omega_t+\tilde{Z}_{t+1}(1-\omega_t)\right)\right)\right]$$
$$E_t\left[U'(c_{t+1}))\left(R_{t+1}-\tilde{Z}_{t+1}\right)\right]=0. \tag{2.49}$$

The second equation in (2.49) implies

$$E_t\left[U'(c_{t+1})\tilde{Z}_{t+1}\right]=E_t\left[U'(c_{t+1}))R_{t+1}\right]. \tag{2.50}$$

Substituting into the first equation in (2.49) yields

$$U'(c_t)=E_t\left[\frac{R_{t+1}}{1+\rho}U'(c_{t+1})\right]$$
$$U'(c_t)=E_t\left[\frac{\tilde{Z}_{t+1}}{1+\rho}U'(c_{t+1})\right] \tag{2.51}$$

Consider a similar problem but with *n* risky assets each with a stochastic net return of $\tilde{z}_i$ ($\equiv\tilde{Z}_i$-1). FOC for each risk asset *i* yields after the same substitution as above

$$r_{t+1}E_t\left[U'(c_{t+1})\right]=E_t\left[U'(c_{t+1})\tilde{z}_{i,t+1}\right]$$
$$=E_t\left[U'(c_{t+1})\right]E_t\left[\tilde{z}_{i,t+1}\right]+\text{cov}\left[U'(c_{t+1}),\tilde{z}_{i,t+1}\right]. \tag{2.52}$$

So

$$E_t\left[\tilde{z}_{i,t+1}\right]=r_{t+1}-\frac{\text{cov}\left[U'(c_{t+1}),\tilde{z}_{i,t+1}\right]}{E_t\left[U'(c_{t+1})\right]} \tag{2.53}$$

This is the Consumption CAPM.

So a risky asset can have an expected rate of return that is larger or smaller than the safe return. Note that a positive covariance between consumption and the risky return implies a negative covariance between marginal utility and $\tilde{z}_{i,t+1}$ so that asset will have a risk premium.

Note that if we have CRRA ($U'=c^{-\alpha}$) then we have

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$$
\begin{aligned}
E_t\left[\tilde{z}_{i,t+1}\right] &= r_{t+1} - \frac{\operatorname{cov}\left[c_{t+1}^{-\alpha}, \tilde{z}_{i,t+1}\right]}{E_t\left[c_{t+1}^{-\alpha}\right]} \\
&= r_{t+1} - \frac{\operatorname{cov}\left[c_{t+1}^{-\alpha} / c_t^{-\alpha}, \tilde{z}_{i,t+1}\right]}{E_t\left[c_{t+1}^{-\alpha} / c_t^{-\alpha}\right]} \\
&\approx r_{t+1} + \alpha \frac{\operatorname{cov}\left[c_{t+1} / c_t, \tilde{z}_{i,t+1}\right]}{E_t\left[c_{t+1}^{-\alpha} / c_t^{-\alpha}\right]}
\end{aligned} \tag{2.54}
$$

where I have used that $\operatorname{cov}(f(x), y) \approx f'(\bar{x}) \operatorname{cov}(x, y)$.

Now assume there exists an asset m which return is perfectly negatively correlated with marginal utility. So $U'(c_{t+1}) = -\gamma \tilde{z}_{m,t+1}$ and thus $\operatorname{cov}\left[U'(c_{t+1}), \tilde{z}_{m,t+1}\right]$ $= -\gamma \operatorname{cov}\left[\tilde{z}_{m,t+1}, z_{i,t+1}\right]$.

Then from (2.53)

$$
E_t\left[\tilde{z}_{m,t+1}\right] = r_{t+1} - \frac{\operatorname{cov}_t\left[U'(c_{t+1}), \tilde{z}_{m,t+1}\right]}{E_t\left[U(c_{t+1})\right]} \tag{2.55}
$$

$$
E_t\left[U'(c_{t+1})\right] = \frac{\gamma \operatorname{var}\left[\tilde{z}_{m,t+1}\right]}{E_t\left[\tilde{z}_{m,t+1}\right] - r_{t+1}} \tag{2.56}
$$

Substitute (2.56) into (2.53)

$$
\begin{aligned}
E_t\left[\tilde{z}_{i,t+1}\right] &= r_{t+1} + \frac{\gamma \operatorname{cov}\left[\tilde{z}_{m,t+1}, z_{i,t+1}\right]}{\gamma \operatorname{var}\left[\tilde{z}_{m,t+1}\right]}\left(E_t\left[\tilde{z}_{m,t+1}\right] - r_{t+1}\right) \\
&= r_{t+1} + \beta_{i,t+1}\left(E_t\left[\tilde{z}_{m,t+1}\right] - r_{t+1}\right).
\end{aligned} \tag{2.57}
$$

This is the *market or traditional CAPM*. Note that β_i is the (true) regression coefficient in a regression of asset *i* on *m*. The term $\left(E_t\left[\tilde{z}_{m,t+1}\right] - r_{t+1}\right)$ can be interpreted as the price of aggregate or systematic risk.

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2. Empirics

The Mehra – Prescott Puzzle

Consider a representative household that "maximizes"

$$
\begin{aligned}
&E_1 \sum_{t=1}^{T} \beta^t \frac{c^{1-\alpha}}{1-\alpha} \\
&s.t. \quad c_t = y_t, \\
&\qquad\;\; y_t = \lambda_t y_{t-1}
\end{aligned}
\tag{2.58}
$$

λ is a stochastic growth rate that can take n different values $\{\lambda_1,\ldots,\lambda_n\}$ all >0. The probability of a specific growth rate depends only on last periods growth rate.

$$
\Pr\left\{\lambda_t = \lambda_j \middle| \lambda_{t-1} = \lambda_i \middle|\right\} = \phi_{ij}\,. \tag{2.59}
$$

Assume there is a share that entitles the owner to the entire output the next period. From (2.46) we have that the price of this share is

$$
\begin{aligned}
p_t &= E_t \sum_{j=1}^{\infty} \beta^j \frac{U'\left(y_{t+j}\right) y_{t+j}}{U'\left(y_t\right)} = E_t \sum_{j=1}^{\infty} \beta^j \frac{y_{t+j}^{-\alpha} y_{t+j}}{(y_t)^{-\alpha}} \\
&= E_t \sum_{j=1}^{\infty} \beta^j \frac{\left(\prod_{k=1}^{j} \lambda_{t+k}\right)^{1-\alpha} \cancel{y_t^{-\alpha}} y_t}{\cancel{y_t^{-\alpha}}} \\
&= p(y_t, \lambda_t) = w(\lambda_t) y_t \equiv w_i y_t = w_i c_t
\end{aligned}
\tag{2.60}
$$

where $i \in \{1,\ldots,n\}$. So the price is H(1) in c and y.

We can also use (2.44) to get

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$$\begin{aligned} w_i y_t &= E_t\left[\beta \frac{U'(y_{t+1})}{U'(y_t)}(wy_{t+1}+y_{t+1})\right] \\ &= E_t\left[\beta\left(\frac{\lambda y_t}{y_t}\right)^{-\alpha}(w\lambda y_t+\lambda y_t)\right] \\ &= E_t\left[\beta(\lambda)^{1-\alpha}(wy_t+y_t)\right] \\ &\Rightarrow w_i = \beta\sum_{j=1}^{n}\phi_{ij}\left(\lambda_j^{1-\alpha}(w_j+1)\right) \end{aligned} \tag{2.61}$$

This is a linear equation system in *n* unknowns so we can solve it for the price of the share in all states of the world. Now we can calculate the net return on the asset

$$\begin{aligned} \tilde{r}_{ij} &= \frac{p(\lambda_j y_t,\lambda_j)+\lambda_j y_t - p(y_t,\lambda_i)}{p(y_t,\lambda_i)} \\ &= \frac{w_j\lambda_j y_t+\lambda_j y_t - w_i y_t}{w_i y_t} = \frac{\lambda_j(w_j+1)}{w_i} - 1 \end{aligned} \tag{2.62}$$

and expected return is

$$\tilde{r}_i^e = \sum_{i=1}^{n}\phi_{ij}r_{ij} \tag{2.63}$$

We can also compute the price of a safe asset in this economy.

$$\begin{aligned} p^f(y_t,\lambda_t) &= E_t\left[\beta\frac{U'(y_{t+1})}{U'(y_t)}\right] = E_t\left[\beta\left(\frac{\lambda y_t}{y_t}\right)^{-\alpha}\right] \\ &= E_t\left[\beta\lambda^{-\alpha}\right] = \beta\sum_{j=1}^{n}\phi_{ij}\lambda_j^{-\alpha} \equiv p_i^f. \end{aligned} \tag{2.64}$$

with a return of $r_i^f = 1/p_i^f - 1$.

Now we want to find the unconditional (average) returns on the assets. First we need the unconditional probabilities of the states π.

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Assume ergodic growth rates

$$\pi = \lim_{s\to\infty}\left(\phi^T\right)^s \pi_0 \;\forall \pi_0, \pi_0'\mathbf{1} = 1$$
$$\Rightarrow \begin{bmatrix} p(\lambda_1) \\ \vdots \\ p(\lambda_n) \end{bmatrix} \equiv \pi = \begin{bmatrix} \phi_{11} & \cdots & \phi_{n1} \\ \vdots & \ddots & \vdots \\ \phi_{1n} & \cdots & \phi_n \end{bmatrix} \pi \equiv \phi^T \pi \tag{2.65}$$

Then

$$\tilde{r}^e = \pi'\tilde{r}_i^e$$
$$r^f = \pi' r_i^f \tag{2.66}$$

The risk premium is then defined as $\tilde{r}^e - r^f$.

Results

Now simplify and assumed there is two states and a common transition probability.

$$\lambda_1 = 1 + \mu + \delta,$$
$$\lambda_2 = 1 + \mu - \delta, \tag{2.67}$$
$$\phi^T = \begin{bmatrix} \phi & 1-\phi \\ 1-\phi & \phi \end{bmatrix} \Rightarrow \pi = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.$$

Now we calibrate the model using the mean growth rate of GDP, its variance and autocorrelation.

$$E(\lambda) = \mu = 0.018$$
$$Var(\lambda)^{0.5} = \delta = 0.036$$
$$Corr(\lambda_t, \lambda_{t-1}) = \frac{\delta(\phi\delta + (1-\phi)(-\delta))}{\delta^2} = 2\phi - 1 = -0.14 \tag{2.68}$$
$$\Rightarrow \phi = 0.43.$$

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Using these value and some reasonable values for β and α <10 we find that the risk premium should be something in the order of 0 to 0.4%. But on average the US stock market has yielded 6% more on average than government.

Consumption versus market β (Mankiw & Shapiro 1986)

Mankiw and Shapiro first estimates (2.57) from 464 different stocks. They first estimate the market β, i.e., $\beta_{mi} \equiv \frac{\operatorname{cov}\left[m_{t+1}, z_{i,t+1}\right]}{\operatorname{var}\left[m_{t+1}\right]}$ assuming it is constant over time. Note that the coefficient on β should be equal to the equity premium (around 6%) and constant for all periods and assets. Then they use this variable to predict the return on the corresponding asset.

$$\tilde{z}_{i,t} = \alpha_0 + \alpha_1 \beta_{mi} + \varepsilon_t \tag{2.69}$$

The coefficient is significant and around 6 for most estimation methods.

Now rewrite (2.54)

$$\begin{aligned} \tilde{z}_{i,t+1} &\approx r_{t+1} + \alpha \frac{\operatorname{cov}\left[c_{t+1}/c_t, \tilde{z}_{i,t+1}\right]}{E_t\left[c_{t+1}^{-\alpha}/c_t^{-\alpha}\right]} + \varepsilon_{t+1} \\ &= r_{t+1} + \alpha \frac{\operatorname{cov}\left[c_{t+1}/c_t, \tilde{z}_{m,t+1}\right]}{E_t\left[c_{t+1}^{-\alpha}/c_t^{-\alpha}\right]} \frac{\operatorname{cov}\left[c_{t+1}/c_t, \tilde{z}_{i,t+1}\right]}{\operatorname{cov}\left[c_{t+1}/c_t, \tilde{z}_{m,t+1}\right]} + \varepsilon_{t+1} \end{aligned} \tag{2.70}$$

Assume that all relevant moments in (2.70) are constant over time. We then β_{ci} as sample moments

$$\beta_{ci} \equiv \frac{\operatorname{cov}\left[c_{t+1}/c_t, \tilde{z}_{i,t+1}\right]}{\operatorname{cov}\left[c_{t+1}/c_t, \tilde{z}_{m,t+1}\right]} \tag{2.71}$$

Then we can run the regression

$$\tilde{z}_{i,t+1} = \gamma_0 + \gamma_1 \beta_{ci} + \varepsilon_{t+1} \tag{2.72}$$

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Now the estimate of γ_1 is insignificant and unstable. M&S also run a regression with both β, then only the coefficient on β_{mi} comes out significant.

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