
1. Growth

1.B.3. The Samuelson-Diamond-Blanchard OLG model

. Households

Live for two periods and solve

$$\begin{aligned} \max_{C_{1,t}, C_{2,t+1}} \quad & U(C_{1,t}) + \frac{1}{1+\rho} U(C_{2,t+1}) \\ \text{s.t.} \quad & \frac{C_{2,t+1}}{1+r_{t+1}} + C_{1,t} = \text{labor income in } t. \end{aligned} \tag{1}$$

. Firms

Hire labor and capital on competitive market and combine them in a CRS production function to produce the only good. This is sold on a competitive market to the households.

Firms solve

$$\max_{K_t, L_t} F(K_t, A_t L_t) - (w_t A_t L_t + r_t K_t) \tag{2}$$

where A_t is a productivity index that grows geometrically at rate g . We can think of AL as the number of *effective* units of labor. Also L grows at a geometric rate n . So

$$\begin{aligned} A_{t+1} &= (1+g)A_t \\ L_{t+1} &= (1+n)L_t \end{aligned} \tag{3}$$

The competitive market imply that factors are paid their marginal products. $r = F_K$ and $w = F_{AL}$.

Since F is H(1) we have

$$\frac{1}{A_t L_t} F(K_t, A_t L_t) = F\left(\frac{K_t}{A_t L_t}, 1\right) \equiv f(k_t) \tag{4}$$

So f is production per effective labor unit and k is capital per effective labor unit. Note that

$$F(K_t, A_t L_t) \equiv A_t L_t f\left(\frac{K_t}{A_t L_t}\right) \quad (5)$$

$$r_t = F_{K_t} = A_t L_t f'(k_t) \frac{1}{A_t L_t} = f'(k_t)$$

Since we CRS and competitive markets firms make zero profits. We can write this as

$$w_t A_t L_t = F(K_t, A_t L_t) - r_t K_t \quad (6)$$

$$w_t = f(k_t) - r_t k_t$$

Note that we now see that both wages and interest rates are determined by the current capital to effective labor ratio. We thus write

$$w_t = w(k_t) \quad (7)$$

$$r_t = r(k_t)$$

. Capital market

The labor income of a young generation in t equals $w_t A_t L_t$. Part of this is consumed ($c_{1,t}$) and the rest is saved. Let s_t denote the share of labor income that is saved. This will, in general depend on labor income and the interest rate. The savings in period t is next periods capital stock

$$s_t w_t A_t L_t = K_{t+1}$$

$$s_t w_t = \frac{K_{t+1}}{A_t L_t} = \frac{K_{t+1}}{A_{t+1} L_{t+1} / (1+n)(1+g)} \quad (8)$$

$$k_{t+1} = \frac{s_t w_t}{(1+n)(1+g)}$$

Now assume that s_t only depends on r_{t+1} . (Which class of utility functions produce this result? What are the effects of higher r ?) Then we can write the last line of (8) as

$$k_{t+1} = \frac{s(r_{t+1}(k_{t+1}))w(k_t)}{(1+n)(1+g)} \quad (9)$$

This (implicitly) defines a difference equation for k , i.e., a relation between k_t and k_{t+1} that has to be satisfied in this model. Note that if we made the Solow assumption of a constant savings rate, the difference equation takes an explicit form. Since in this case the RHS contains no k_{t+1} . Fortunately there is a utility function for which the income and substitution effects of higher interest rates cancel so that the household will choose a constant savings rate regardless of the interest rate.

. Functional specification

Let us look at a particularly simple specification. Assume U is the log function and that that production is (Wicksell-) Cobb-Douglas, $f(k) = k^\alpha$.

Now we can write the consumption decision of the consumer as follows.

$$\max_{s_t} = \ln(1-s_t)w_t A_t L_t + \frac{1}{1+\rho} \ln s_t (1+r_{t+1})w_t A_t L_t \quad (10)$$

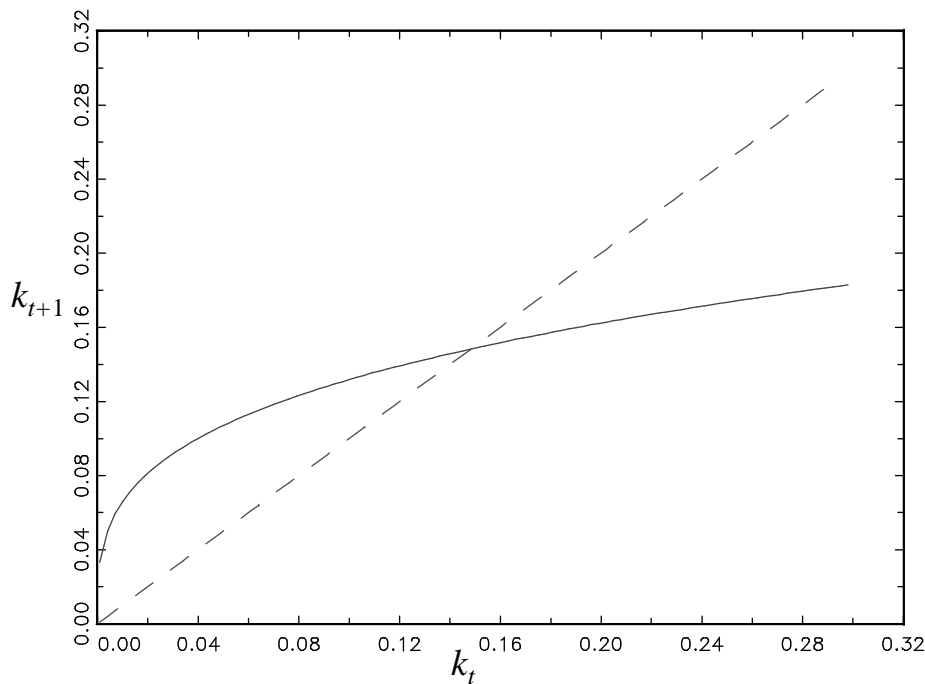
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$$\begin{aligned} \frac{1}{1-s_t} &= \frac{1}{(1+\rho)s_t} \\ \Rightarrow s_t &= \frac{1}{2+\rho} \end{aligned} \quad (11)$$

Now we only need to specify the wage function to have an explicit version of (9). We see immediately that $w = (1-\alpha)k^\alpha$. So (9) becomes

$$\begin{aligned} k_{t+1} &= \frac{(1-\alpha)k_t^\alpha}{(2+\rho)(1+n)(1+g)} \\ \ln k_{t+1} &= \ln \frac{1-\alpha}{(2+\rho)(1+n)(1+g)} + \alpha \ln k_t \end{aligned} \quad (12)$$

Can you solve this difference equation and is it stable? Here is an example of a plot of (12).



The steady state of (12) is its fixed point

$$k^* = \left(\frac{1 - \alpha}{(2 + \rho)(1 + g)(1 + n)} \right)^{\frac{1}{1 - \alpha}} \quad (13)$$

The interest rate in steady state is

$$r^* = \alpha (k^*)^{\alpha - 1} = \left(\frac{\alpha(2 + \rho)(1 + g)(1 + n)}{1 - \alpha} \right) \quad (14)$$

• Dynamic Inefficiency

In each time period the amount of resources is $K_t + F(K_t, A_t L_t)$. This has to be split into aggregate consumption and next periods capital stock. We can than write

$$K_t + F(K_t, A_t L_t) = K_{t+1} + C_t \quad (15)$$

where C_t is total consumption of young and old in period t . Now divide by $A_t L_t$

$$\begin{aligned}
k_t + f(k_t) &= \frac{K_{t+1}}{A_t L_t} + c_t \\
&= k_{t+1}(1+g)(1+n) + c_t.
\end{aligned}
\tag{16}$$

c_t is aggregate consumption per unit of current effective unit of labor.

Now consider an economy in a steady state. We then have

$$\begin{aligned}
k^* + f(k^*) &= k^*(1+g)(1+n) + c^* \\
\Rightarrow c^* &= f(k^*) - k^*(g+n+gn)
\end{aligned}
\tag{17}$$

Let us find the value of the steady state capital stock that maximizes aggregate consumption per effective unit of labor. The FOC for this problem is

$$f'(k^{gr}) = n + g + ng \tag{18}$$

This is a variant of the Ramsey Golden Rule. A steady state capital stock above k^{gr} is not Pareto efficient. If we are above k^{gr} we could increase aggregate consumption in all periods now and in the future by reducing forcing the steady state capital stock to be lower.

Now compare (18) and (14)

$$\begin{aligned}
r^* &= \left(\frac{\alpha(2+\rho)(1+g)(1+n)}{1-\alpha} \right) \lesseqgtr n + g + ng \\
\left(\frac{\alpha(2+\rho)}{1-\alpha} \right) &\lesseqgtr 1 - \frac{1}{(1+g)(1+n)}
\end{aligned}
\tag{19}$$

So it is possible that the economy is not on the Pareto frontier and thus dynamically inefficient. What goes wrong here and why does not the first welfare theorem hold? One way of understanding this is to realize that the only way a young generation can reduce the capital stock is by saving less but then consumption would have to be lower for them in the second period. If, however, there was a mechanism that specified that young people saved less but in return got some transfers from the young next period, everybody could be made better off. What does this remind you off?

. More general savings functions

In a more general case, the difference equation given by (9) can be less nice. It can have no or multiple steady states. With multiple steady states, some will be non-stable. Multiple steady states could occur if, for example, the savings rate increase as interest rate falls (the income effect dominates).

