II.B.Asset Pricing

In the previous sections we used the Euler equation to derive optimal consumption and investment decisions. Now note that the Euler equation defines a relation between consumption (or other real variables) and prices.

$$\text{Path of prices} \leftrightarrow \text{Path of real variables} \qquad (2.42)$$

Previously we took the prices as given and derived the optimal path of consumption. We may, however, use the Euler relation in the other direction. Take the path of real variables, e.g., consumption as given and derive what the prices have to be. A straightforward way to do this is to assume that output is exogenous, like manna from heaven, and cannot be stored. In that environment we may introduce markets for capital and production facilities. This is the setup in the seminal Lucas (*Econometrica*, 1978) paper.

We will later relax the assumption about exogenous and non-storable output. This will give us the basic stochastic growth (or RBC) model.

1. Asset Pricing in the Lucas Tree Model and other CAPM

. Asset Pricing in the Lucas Tree Model

1) Large number of identical agents.
2) Equal number of trees with stochastic crop d_t The distribution of d_t is Markov. Distribution is $F(d_t|d_{t-1}) = F(d_t|d_{t-1}, d_{t-2}, \ldots)$. The process known by all agents.
3) Purpose: find p_t – the price of a tree as a function of the state of the economy (d_t).
4) The gross return on a tree is $\frac{p_{t+1} + d_{t+1}}{p_t}$ (per capita)
5) No safe asset.
6) Perfect market in ownership of trees. All equal so no trade in equilibrium.
7) No storage or foreign trade so consumption $c_t = d_t$

Substitute for $\tilde{z}_{t+1}$ from the Euler equation noting that ω =0 gives

$$U'(c_t) = E_t\left[(1+\rho)^{-1}U'(c_{t+1})\frac{(p_{t+1}+d_{t+1})}{p_t}\right]$$

or (2.43)

$$p_t = E_t\left[\beta\frac{U'(c_{t+1})}{U'(c_t)}(p_{t+1}+d_{t+1})\right]$$

By 7)

$$p_t = E_t\left[\beta\frac{U'(d_{t+1})}{U'(d_t)}(p_{t+1}+d_{t+1})\right] \tag{2.44}$$

For any type of expectations in (2.44) based on d_t we can compute a price today as a function of d_t Let the individuals subjective expectations of d_{t+1} be described by $F^s(d_{t+1}|d_t)$ and the expectations about the relation between p_{t+1} and d_{t+1} be given by the function $p^s(d_{t+1})$. We then have

$$\begin{aligned} p_t &= \int_{-\infty}^{\infty}\beta\frac{U'(d_{t+1})}{U'(d_t)}\left(p^s(d_{t+1})+d_{t+1}\right)dF^s(d_{t+1}|d_t) \\ &\equiv p(d_t). \end{aligned} \tag{2.45}$$

Now Lucas defined the very powerful concept of *rational expectations*. Let's require that $p(\cdot) \equiv p^s(\cdot)$ and $F(d_{t+1}|d_t) \equiv F^s(d_{t+1}|d_t)$. Lucas proves that this together with (2.45) defines a *unique* and constant pricing function $p(\cdot)$.

Use recursions on (2.44)

$$p_t = E_t\left[\beta\frac{U'(d_{t+1})}{U'(d_t)}\left(E_{t+1}\left[\beta\frac{U'(d_{t+2})}{U'(d_{t+1})}\left(E_{t+2}\left[\beta\frac{U'(d_{t+3})}{U'(d_{t+2})}(\ldots+d_{t+3})\right]+d_{t+2}\right)\right]+d_{t+1}\right)\right]$$

$$= E_t\left[\beta\frac{U'(d_{t+1})}{U'(d_t)}\left(\left[\beta\frac{U'(d_{t+2})}{U'(d_{t+1})}\left(\left[\beta\frac{U'(d_{t+3})}{U'(d_{t+1})}(\ldots+d_{t+3})\right]+d_{t+2}\right)\right]+d_{t+1}\right)\right] \tag{2.46}$$

$$= E_t\sum_{j=1}^{\infty}\beta^j\frac{U'(d_{t+j})d_{t+j}}{U'(d_t)}+\lim_{j\to\infty}\beta^j\frac{U'(d_{t+j})p_{t+j}}{U'(d_t)}$$

A discounted sum of dividends. Stochastic discount rates unless marginal utility is constant.

A simple example with log utility

$$p_t = E_t\sum_{j=1}^{\infty}\beta^j\frac{U'(d_{t+j})d_{t+j}}{U'(d_t)} = E_t\sum_{j=1}^{\infty}\beta^j\frac{d_t d_{t+j}}{d_{t+j}} = d_t\sum_{j=1}^{\infty}\beta^j = d_t\frac{\beta}{1-\beta} \tag{2.47}$$

With d_t i.i.d. so $E_tU'(d_{t+j})d_{t+j} = E_tU'(d_{t+s})d_{t+s}\forall j,s>0$

$$p_t = E_t\sum_{j=1}^{\infty}\beta^j\frac{U'(d_{t+j})d_{t+j}}{U'(d_t)} = \frac{\beta}{(1-\beta)U'(d_t)}\overbrace{EU'(d_{t+1})d_{t+1}}^{\text{constant}} \tag{2.48}$$

Note that the price increases in d_t (if U is concave).

Assume an autocorrelation in d_t then $E_t\sum_{j=1}^{\infty}\beta^jU'(d_{t+j})d_{t+j}$ depends on d_t Both income and substitution effect, with log utility they cancel.

The Consumption and market CAPM

Previously we found that

$$U'(c_t) = E_t\left[(1+\rho)^{-1}U'(c_{t+1})\left(\left(R_{t+1}\omega_t + \tilde{Z}_{t+1}(1-\omega_t)\right)\right)\right]$$
$$E_t\left[U'(c_{t+1}))\left(R_{t+1} - \tilde{Z}_{t+1}\right)\right] = 0. \tag{2.49}$$

The second equation in (2.49) implies

$$E_t\left[U'(c_{t+1})\tilde{Z}_{t+1}\right] = E_t\left[U'(c_{t+1}))R_{t+1}\right]. \tag{2.50}$$

Substituting into the first equation in (2.49) yields

$$U'(c_t) = E_t\left[\frac{R_{t+1}}{1+\rho}U'(c_{t+1})\right]$$
$$U'(c_t) = E_t\left[\frac{\tilde{Z}_{t+1}}{1+\rho}U'(c_{t+1})\right] \tag{2.51}$$

Consider a similar problem but with *n* risky assets each with a stochastic net return of $\tilde{z}_i$ ($\equiv \tilde{Z}_i$-1). FOC for each risk asset *i* yields after the same substitution as above

$$r_{t+1}E_t\left[U'(c_{t+1})\right] = E_t\left[U'(c_{t+1})\tilde{z}_{i,t+1}\right]$$
$$= E_t\left[U'(c_{t+1})\right]E_t\left[\tilde{z}_{i,t+1}\right] + \text{cov}\left[U'(c_{t+1}), \tilde{z}_{i,t+1}\right]. \tag{2.52}$$

So

$$E_t\left[\tilde{z}_{i,t+1}\right] = r_{t+1} - \frac{\text{cov}\left[U'(c_{t+1}), \tilde{z}_{i,t+1}\right]}{E_t\left[U'(c_{t+1})\right]} \tag{2.53}$$

This is the Consumption CAPM.

So a risky asset can have an expected rate of return that is larger or smaller than the safe return. Note that a positive covariance between consumption and the risky return implies a negative covariance between marginal utility and $\tilde{z}_{i,t+1}$ so that asset will have a risk premium.

Note that if we have CRRA ($U' = c^{-\alpha}$) then we have

$$
\begin{aligned}
E_t\left[\tilde{z}_{i,t+1}\right] &= r_{t+1} - \frac{\operatorname{cov}\left[c_{t+1}^{-\alpha}, \tilde{z}_{i,t+1}\right]}{E_t\left[c_{t+1}^{-\alpha}\right]} \\
&= r_{t+1} - \frac{\operatorname{cov}\left[c_{t+1}^{-\alpha} / c_t^{-\alpha}, \tilde{z}_{i,t+1}\right]}{E_t\left[c_{t+1}^{-\alpha} / c_t^{-\alpha}\right]} \\
&\approx r_{t+1} + \alpha \frac{\operatorname{cov}\left[c_{t+1} / c_t, \tilde{z}_{i,t+1}\right]}{E_t\left[c_{t+1}^{-\alpha} / c_t^{-\alpha}\right]}
\end{aligned}
\tag{2.54}
$$

where I have used that $\operatorname{cov}(f(x), y) \approx f'(\bar{x}) \operatorname{cov}(x, y)$.

Now assume there exists an asset m which return is perfectly negatively correlated with marginal utility. So $U'(c_{t+1}) = -\gamma \tilde{z}_{m,t+1}$ and thus $\operatorname{cov}\left[U'(c_{t+1}), \tilde{z}_{m,t+1}\right]$ $= -\gamma \operatorname{cov}\left[\tilde{z}_{m,t+1}, z_{i,t+1}\right]$.

Then from (2.53)

$$
E_t\left[\tilde{z}_{m,t+1}\right] = r_{t+1} - \frac{\operatorname{cov}_t\left[U'(c_{t+1}), \tilde{z}_{m,t+1}\right]}{E_t\left[U(c_{t+1})\right]}
\tag{2.55}
$$

$$
E_t\left[U'(c_{t+1})\right] = \frac{\gamma \operatorname{var}\left[\tilde{z}_{m,t+1}\right]}{E_t\left[\tilde{z}_{m,t+1}\right] - r_{t+1}}
\tag{2.56}
$$

Substitute (2.56) into (2.53)

$$
\begin{aligned}
E_t\left[\tilde{z}_{i,t+1}\right] &= r_{t+1} + \frac{\gamma \operatorname{cov}\left[\tilde{z}_{m,t+1}, z_{i,t+1}\right]}{\gamma \operatorname{var}\left[\tilde{z}_{m,t+1}\right]} \left(E_t\left[\tilde{z}_{m,t+1}\right] - r_{t+1}\right) \\
&= r_{t+1} + \beta_{i,t+1}\left(E_t\left[\tilde{z}_{m,t+1}\right] - r_{t+1}\right).
\end{aligned}
\tag{2.57}
$$

This is the *market or traditional CAPM*. Note that β_i is the (true) regression coefficient in a regression of asset *i* on *m*. The term $\left(E_t\left[\tilde{z}_{m,t+1}\right] - r_{t+1}\right)$ can be interpreted as the price of aggregate or systematic risk.

2. Empirics

The Mehra – Prescott Puzzle

Consider a representative household that "maximizes"

$$E_1 \sum_{t=1}^{T} \beta^t \frac{c^{1-\alpha}}{1-\alpha}$$
$$s.t. \quad c_t = y_t, \quad y_t = \lambda_t y_{t-1} \tag{2.58}$$

λ is a stochastic growth rate that can take n different values $\{\lambda_1, \ldots, \lambda_n\}$ all >0. The probability of a specific growth rate depends only on last periods growth rate.

$$\Pr\left\{\lambda_t = \lambda_j \middle| \lambda_{t-1} = \lambda_i \middle|\right\} = \phi_{ij}. \tag{2.59}$$

Assume there is a share that entitles the owner to the entire output the next period. From (2.46) we have that the price of this share is

$$\begin{aligned} p_t &= E_t \sum_{j=1}^{\infty} \beta^j \frac{U'(y_{t+j}) y_{t+j}}{U'(y_t)} = E_t \sum_{j=1}^{\infty} \beta^j \frac{y_{t+j}^{-\alpha} y_{t+j}}{(y_t)^{-\alpha}} \\ &= E_t \sum_{j=1}^{\infty} \beta^j \frac{\left(\Pi_{k=1}^{j} \lambda_{t+k}\right)^{1-\alpha} \cancel{y_t^{-\alpha}} y_t}{\cancel{y_t^{-\alpha}}} \\ &= p(y_t, \lambda_t) = w(\lambda_t) y_t \equiv w_i y_t = w_i c_t \end{aligned} \tag{2.60}$$

where $i \in \{1,...,n\}$. So the price is H(1) in c and y.

We can also use (2.44) to get

$$
\begin{aligned}
w_i y_t &= E_t\left[\beta \frac{U'(y_{t+1})}{U'(y_t)}\left(w y_{t+1} + y_{t+1}\right)\right] \\
&= E_t\left[\beta\left(\frac{\lambda y_t}{y_t}\right)^{-\alpha}\left(w\lambda y_t + \lambda y_t\right)\right] \\
&= E_t\left[\beta(\lambda)^{1-\alpha}\left(w y_t + y_t\right)\right] \\
\Rightarrow w_i &= \beta \sum_{j=1}^{n} \phi_{ij}\left(\lambda_j^{1-\alpha}\left(w_j + 1\right)\right)
\end{aligned}
\tag{2.61}
$$

This is a linear equation system in n unknowns so we can solve it for the price of the share in all states of the world. Now we can calculate the net return on the asset

$$
\begin{aligned}
\tilde{r}_{ij} &= \frac{p(\lambda_j y_t, \lambda_j) + \lambda_j y_t - p(y_t, \lambda_i)}{p(y_t, \lambda_i)} \\
&= \frac{w_j \lambda_j y_t + \lambda_j y_t - w_i y_t}{w_i y_t} = \frac{\lambda_j (w_j + 1)}{w_i} - 1
\end{aligned}
\tag{2.62}
$$

and expected return is

$$
\tilde{r}_i^e = \sum_{i=1}^{n} \phi_{ij} r_{ij} \tag{2.63}
$$

We can also compute the price of a safe asset in this economy.

$$
\begin{aligned}
p^f(y_t, \lambda_t) &= E_t\left[\beta \frac{U'(y_{t+1})}{U'(y_t)}\right] = E_t\left[\beta\left(\frac{\lambda y_t}{y_t}\right)^{-\alpha}\right] \\
&= E_t\left[\beta \lambda^{-\alpha}\right] = \beta \sum_{j=1}^{n} \phi_{ij} \lambda_j^{-\alpha} \equiv p_i^f.
\end{aligned}
\tag{2.64}
$$

with a return of $r_i^f = 1/p_i^f - 1$.

Now we want to find the unconditional (average) returns on the assets. First we need the unconditional probabilities of the states π.

Assume ergodic growth rates

$$\pi = \lim_{s\to\infty}\left(\phi^T\right)^s \pi_0 \;\forall \pi_0, \pi_0'\mathbf{1} = 1$$
$$\Rightarrow \begin{bmatrix} p(\lambda_1) \\ \vdots \\ p(\lambda_n) \end{bmatrix} \equiv \pi = \begin{bmatrix} \phi_{11} & \cdots & \phi_{n1} \\ \vdots & \ddots & \vdots \\ \phi_{1n} & \cdots & \phi_n \end{bmatrix} \pi \equiv \phi^T \pi \tag{2.65}$$

Then

$$\begin{aligned} \tilde{r}^e &= \pi' \tilde{r}_i^e \\ r^f &= \pi' r_i^f \end{aligned} \tag{2.66}$$

The risk premium is then defined as $\tilde{r}^e - r^f$.

Results

Now simplify and assumed there is two states and a common transition probability.

$$\begin{aligned} \lambda_1 &= 1 + \mu + \delta, \\ \lambda_2 &= 1 + \mu - \delta, \\ \phi^T &= \begin{bmatrix} \phi & 1-\phi \\ 1-\phi & \phi \end{bmatrix} \Rightarrow \pi = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}. \end{aligned} \tag{2.67}$$

Now we calibrate the model using the mean growth rate of GDP, its variance and autocorrelation.

$$\begin{aligned} & E(\lambda) = \mu = 0.018 \\ & Var(\lambda)^{0.5} = \delta = 0.036 \\ & Corr(\lambda_t, \lambda_{t-1}) = \frac{\delta(\phi\delta + (1-\phi)(-\delta))}{\delta^2} = 2\phi - 1 = -0.14 \\ & \Rightarrow \phi = 0.43. \end{aligned} \tag{2.68}$$

Using these value and some reasonable values for β and α <10 we find that the risk premium should be something in the order of 0 to 0.4%. But on average the US stock market has yielded 6% more on average than government.

Consumption versus market β (Mankiw & Shapiro 1986)

Mankiw and Shapiro first estimates (2.57) from 464 different stocks. They first estimate the market β, i.e., $\beta_{mi} \equiv \frac{\mathrm{cov}[m_{t+1}, z_{i,t+1}]}{\mathrm{var}[m_{t+1}]}$ assuming it is constant over time. Note that the coefficient on β should be equal to the equity premium (around 6%) and constant for all periods and assets. Then they use this variable to predict the return on the corresponding asset.

$$\tilde{z}_{i,t} = \alpha_0 + \alpha_1 \beta_{mi} + \varepsilon_t \tag{2.69}$$

The coefficient is significant and around 6 for most estimation methods.

Now rewrite (2.54)

$$\begin{aligned} \tilde{z}_{i,t+1} &\approx r_{t+1} + \alpha \frac{\mathrm{cov}\left[c_{t+1}/c_t, \tilde{z}_{i,t+1}\right]}{E_t\left[c_{t+1}^{-\alpha}/c_t^{-\alpha}\right]} + \varepsilon_{t+1} \\ &= r_{t+1} + \alpha \frac{\mathrm{cov}\left[c_{t+1}/c_t, \tilde{z}_{m,t+1}\right]}{E_t\left[c_{t+1}^{-\alpha}/c_t^{-\alpha}\right]} \frac{\mathrm{cov}\left[c_{t+1}/c_t, \tilde{z}_{i,t+1}\right]}{\mathrm{cov}\left[c_{t+1}/c_t, \tilde{z}_{m,t+1}\right]} + \varepsilon_{t+1} \end{aligned} \tag{2.70}$$

Assume that all relevant moments in (2.70) are constant over time. We then β_{ci} as sample moments

$$\beta_{ci} \equiv \frac{\mathrm{cov}\left[c_{t+1}/c_t, \tilde{z}_{i,t+1}\right]}{\mathrm{cov}\left[c_{t+1}/c_t, \tilde{z}_{m,t+1}\right]} \tag{2.71}$$

Then we can run the regression

$$\tilde{z}_{i,t+1} = \gamma_0 + \gamma_1 \beta_{ci} + \varepsilon_{t+1} \tag{2.72}$$

Now the estimate of γ_1 is insignificant and unstable. M&S also run a regression with both β, then only the coefficient on β_{mi} comes out significant.