Problem Set 1 "Working with the Solow model"

Let's define the following exogenous variables:

s = savings rate
δ = depreciation rate of physical capital
n = population growth rate
g = rate of growth of labor augmenting technology.
L = effective labor supply = $e^{(g+n)t}$ (Normalizing standard labor supply to unity)

and the following endogenous:
$\tilde{k}$ = capital stock per effective labor unit (K/L).
$\tilde{y}$ = output per effective labor unit (Y/L).
k = capital stock per capita
y = output per capita

1.

a) If $\tilde{k}$ grows at rate g (i.e., $\frac{\dot{\tilde{k}}}{\tilde{k}} = g$), what is the growth rate of k and K. What is the growth rate of L.

b) Provide some reasonable values for s, δ, n and g (consult Maddison and Romer, 1989)

c) The gross resource constraint for a closed economy is:

$$F(K,L) = C + \delta K + \dot{K} \tag{1.1}$$

Using Cobb-Douglas technology we easily find that:

$$\dot{K} = sK^{\alpha}L^{1-\alpha} - \delta K \qquad (1.2)$$

Make sure you understand that this is reasonable. Derive, step by step, the resource constraint in terms of $\dot{\tilde{k}}$. (Hint: Divide by L, then calculate the relation between $\dot{\tilde{k}}$ and $\dot{K}$ noting that $\dot{\tilde{k}} = \frac{\partial\left(\frac{K}{L}\right)}{\partial t}$, substitute and your done).

d) Using your parameter values from c and a reasonable value for α, find the steady state level of $\tilde{k}$. In steady state what is the gross capital output ratio K/Y?

2.

The differential equation for $\dot{\tilde{k}}$ is:

$$\dot{\tilde{k}}(t) = s\tilde{k}(t)^{\alpha} - (g+n+\delta)\tilde{k} \tag{1.3}$$

a) Is this a standard differential equation, that you have seen in Matfu?

Barro and Sala-i-Martin make a log-linear approximation to the growth rate of $\tilde{k}$ around it's steady state (equation 1.21). They write:

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \lambda\left(\log(\tilde{k}) - \log(\tilde{k}^*)\right) \tag{1.4}$$

b) Give an explanation in words for λ that my mother (who is not an economist) would understand.

c) Calculate this derivative from equation (1.3) at $\tilde{k} = \tilde{k}^*$. What is its numerical value using your parameter values.

d) Assume that from an initial steady state, a devastating war leads to a reduction in $\tilde{k}$ to one eighth of its steady state value. Calculate the output gap ($Y(t)/Y^*(t)$) immediately after this shock. What is now $\dot{\tilde{k}}$?. Calculate the change in interest rate (marginal productivity of capital) before and after the war. How big is the output gap after one year and after 10 years? Do you think this recovery is fast or slow?. Can you calculate the time it takes until half the output gap is gone? When is it totally gone?

e) Compare a country that start out with a capital stock k equal to the steady state you calculated above but that has half the savings rate you assumed. Calculate the steady state level of $\tilde{k}$ for this country and the initial growth rate of $\tilde{k}$ and Y.

The excellent program Mathematica helped me find a solution to the differential equation $\dot{\tilde{k}}(t) = s\tilde{k}(t)^{\alpha} - (g+n+\delta)\tilde{k}$:

$$\tilde{k}(t) = \left(\frac{s}{g+n+\delta} + ce^{-(1-\alpha)(g+n+\delta)t}\right)^{1/(1-\alpha)} \tag{1.6}$$

a) Calculate $\lim_{t\to\infty} \tilde{k}(t)$

b) Setting t to 0 gives you $\tilde{k}(0)$ as a function of the integration constant c. Use this to substitute for c in the expression for $\tilde{k}(t)$.

4. Voluntary bonus question.

a) Compute $\dot{\tilde{k}}$ from equation (1.6). Compare the value of $\left.\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)}\right|_{\tilde{k}(t)=\frac{\tilde{k}^*}{2}}$ with the linear approximation of Barro and Sala-i-Martin. Is their approximation reasonably accurate?

Problem Set 2
"The Ramsey Model for a Small Open Economy"[1]

This problem extends the standard Ramsey model you saw in class. Before you do the problem, compare this model to the Solow model in terms of model assumptions. Do they address the same questions? When is which model more useful?

Assumptions

We use the same notation as in Problem Set 1. For simplicity we set g and n to 0. This means that we don't have to worry about the differences between per capita, per effective labor unit and aggregate units. We are now looking at an open economy. In order not to get "bang-bang" adjustments we introduce an investment cost function so that it becomes relatively more costly to do heavy investments for a short period than smaller investment for a long period.

Firms

Technology is given by the production function

$$f(k_t) = \alpha k_t - \beta k_t^2, \tag{2.1}$$

but capital can not be installed freely. Instead an investment cost

$$g(i_t) = \frac{\gamma i_t}{2}, \tag{2.2}$$

has to be paid. All investment is financed out of earnings and the representative firm maximizes the present value of dividends:

$$\max_{\{i_t\}_0^\infty} \int_0^\infty e^{-rt}\left(f(k_t) - i_t\left(1+g(i_t)\right) - w_t\right)dt, \quad \text{s.t.} \quad \begin{cases} \dot{k}_t = i_t - \delta k_t \\ k_0 = \bar{k} \end{cases} \tag{2.3}$$

Since the country is small and open the interest rate, r, is given and constant.

Households

Household own the firms and supply one unit of labor inelastically. The representative household solves:

[1] After Gottfries, Persson and Lundvik 1989

$$\max_{\{c_t\}_0^\infty} \int_0^\infty e^{-\rho t}\left(\log(c_t)\right)dt,$$

$$\text{s.t.} \quad \begin{cases} \dot{a}_t = w_t + d_t + ra_t - c_t \\ a_0 = \bar{a} \\ \lim\limits_{t\to\infty} e^{-rt} a_t = 0 \end{cases} \tag{2.4}$$

Questions

1.

Think of the production function. Is it reasonable (for all values of *k*)? Solve the maximization problem of the firm. (Hint; set up the current value Hamiltonian. Set up the three optimality equations you have learnt. Two of them involves time differentials). Represent the solution as an investment function $i=i(q)$ and two differential equations, one for k_t and one for q_t, the current value shadow price of capital. Explain in words what q_t represents. Solve for the steady states of k_t and q_t, (k^* and q^*) and draw a phase diagram.

2.

Lets study the rate of convergence to the steady state. First we solve the differential equation for k_t. Remember that the solution for k_t can be written as[2]:

$$k_t = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + k^*, \tag{2.9}$$

where the λ's are the eigenvalues of the coefficients matrix in the differential equation system for *k* and *q*. Find expressions for these eigenvalues and find there signs. What does the signs imply for the value of one of the integration coefficients c_i. (Hint think of what happens as *t* goes to infinity). Solve for the other integration coefficient.

$\lambda_i = \frac{r}{2} \pm \frac{1}{2}\sqrt{\frac{\gamma(2\delta + r)^2 + 8\beta}{\gamma}}$ Solve the problem of the household. Start by setting up the Hamiltonian, now in present values for a change. Take time derivatives of H_c=0 and substitute in from the equation for H_a. This gives you a nice differential equation for c_t. What happens if $\rho \neq r$? Is that reasonable? Then solve for the level of c_t using the transversality equation in (2.4) (Hint; present value of consumption equals present value of income plus starting wealth).

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[2] Provided that a steady state k^* exists.

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Suppose $\rho=r$ and that the economy is in steady state. Use the phase diagram to see what happens dynamically if we get a permanent increase in productivity (α increases). What happens to consumption and the current account over time. (Hint; use the following version of the inter temporal budget constraint:

$$\begin{aligned} c^p &= w^p + d^p + ra_0 \Rightarrow \\ c^p &= f(k)^p - \left(i(1+g(i))\right)^p + ra_0 \end{aligned} \tag{2.13}$$

where superscript p denotes permanent values, defined by:

$$\int_0^\infty \left(x^p\right)e^{-rt}dt = \frac{x^p}{r} \equiv \int_0^\infty \left(x_t\right)e^{-rt}dt \tag{2.14}$$

Substitute (2.13) into the equation for the current account (2.15) and note that a=-b, (the household cannot sell its ownership of the firm and the value of the firm is not included in a).

$$-\dot{b}_t = f(k_t) - c_t - i_t(1+g(i_t)) - rb_t \tag{2.15}$$

Using this for t=0 we can then write the useful equation:

$$-\dot{b} = \left(f(k_0) - f(k)^p\right) - \left(\left(i_0(1+g(i_0)\right) - \left(i(1+g(i)\right)^p\right) - (c_0 - c^p) \tag{2.16}$$

Intuitively the permanent values are defined so that if x is kept at x^p for all t. The present discounted value is the same as for the actual path of x_t.

5.

What happens if the productivity shock is *i*) temporary (happens at t=0 and ends at t=s) and; *ii*) anticipated permanent (the information about the shock arrives at t=0 and the actual shock happens at t=s).

(Hint. Draw the phase diagrams and note that k can newer jump (why?) and that q can jump only when new information (like information about a future productivity shock) arrives. In the case of the anticipated shock note that at the time of the shock the phase diagram shifts and that at exactly the time of the shift we have to be at the stable saddle path for the system not to explode. So when the information arrives at t=0, the "old" phase diagram continues to hold. Now q jumps so that {q,k] hits the new stable path at exactly t=s).)

Problem Set 3
Endogenous Growth

These two problems come from the 1989 mid-term exam in the advanced macro course at MIT (Don't let that discourage you, they are easier than what you have done so far.)

Problem 1 Learning by Doing

Consider a closed economy with a fixed number of people - normalized to one - with the following preferences.

$$U_0 = \int_0^{\infty} \left(\log(c_t) + \log(u_t) \right) e^{-\rho t} dt \tag{3.1}$$

Where *c* is consumption and *u* is leisure. The output cannot be stored and the production function is given by:

$$y_t = H_t l_t \tag{3.2}$$

H is the level of human capital and as can be seen also equal to the productivity level and *l* is the labor input. Markets are competitive and each firm takes the level of productivity given. There is learning by doing, however, so *H* evolves over time according to:

$$\dot{H}_t = \upsilon y_t \tag{3.3}$$

The consumer has one unit of time to divide between work and leisure each point in time.

Questions

1.

Before doing any number crunching think about the following. In this economy, what will happen to the future consumption possibility set if people decide to consume more today? Assume that the individual consumer is to small to consider this. What do you think of the specification of learning by doing, does it make sense?

2.

Set up the maximization problem of the consumer and solve it. (Hint: Normalize the wage to one then find the income of the consumer and the price of the consumption good. The economy is closed and all households are equal. This implies that if there is a capital market the interest rate will have to adjust so that no one will borrow. We can thus as well assume that there is no capital market. Then you can write a budget constraint for the consumer in each time point. With the assumption under 1., is the problem static or dynamic?).

3.

Now solve for the equilibrium growth rate of the economy.

4.

Now let's find the optimal growth rate of the economy. First, do you think it should be higher or lower than the market solution?

To find the solution, follow these steps:

1. Let there be a benevolent central planner who maximizes U_0 but takes into account the learning by doing effect.

2. Write down the central planning problem, substituting for c_t so the only constrain is the learning by doing constraint and the choice variable is l_t.

3. Write the Hamiltonian and the optimality conditions. Guess that in an optimal steady state, the labor supply is constant and then differentiate the first optimality condition (H_l) w.r.t. time. What is then the relationship between the growth rate of the shadow value of the constraint and the growth rate of H? Eliminate the growth rate of the shadow value and the shadow value from the second optimality condition (H_A) and you get an expression for the growth rate of H involving υ, l and ρ

4. There is another expression for the growth rate of H_t coming from the learning by doing constraint. Use this to eliminate l_t from the growth rate of H_t from the other expression.

5. Now you can solve for the optimal growth rate of H_t Lastly, note the relation between the growth rate of c and the growth rate of H and you're done.

(3.3)**Problem 2. Education and growth**

There is a continuum of workers-consumers on the interval [0,1], i.e. an infinite number of agents indexed by the real numbers i between 0 and 1. Each individual has one unit of labor and has a human capital level of h^i. Each agent decides that she allocates a^i share a to work and $(1-a^i)$ to study. Since all individual are alike we can drop the *i* superscript where we don't need it. This leads to the individual increasing her stock of human capital according to:

$$\dot{h}_t = \beta(1-a)h_t \tag{3.10}$$

The production of a single individual is then given by:

$$y_t = \left(ah_t\right)^{\alpha}\left(AH_t\right)^{1-\alpha} \tag{3.11}$$

Where A and H denotes aggregate levels of a and h.

$$H_t = \int_0^1 h_t^i di \qquad A_t = \int_0^1 a_t^i di \tag{3.12}$$

Note the externality here, the individual is more productive if the other individuals together have more human capital. The production cannot be stored, so the individual consumer-producer consumes what he produces each moment in time.

Individuals have logarithmic utility and a subjective discount rate of ρ.

Questions

1.

Is there decreasing or constant returns to human capital in this model? How much of the externality is taken into account by this small selfish agents? This can be found be evaluating $\frac{\partial H}{\partial h^i}$.

2.

Assume a common value of a for all agents, (is this reasonable?), then compute the growth rate of the economy as a function of a.

3.

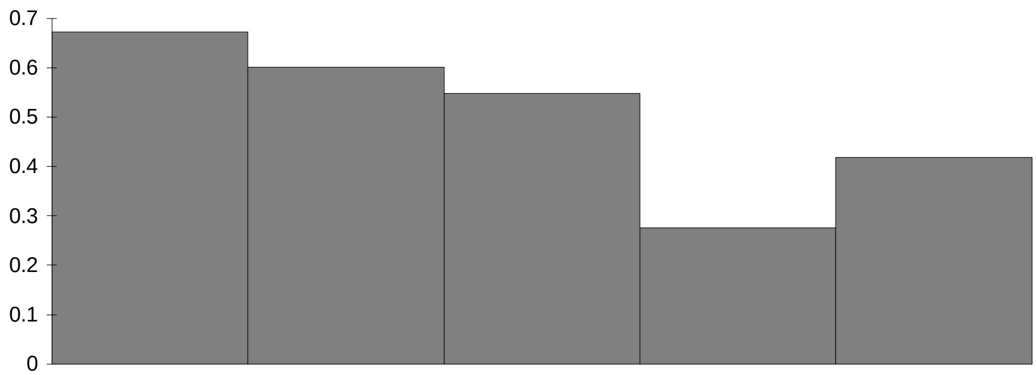
Derive the equilibrium growth rate of the economy by letting the agents choose a optimally. (Hint; Set up the maximization problem of the consumer-producer and substitute for c in the utility function. Set up the Hamiltonian with its optimality condition and the solve for the growth rate of human capital assuming a steady state where a is constant.

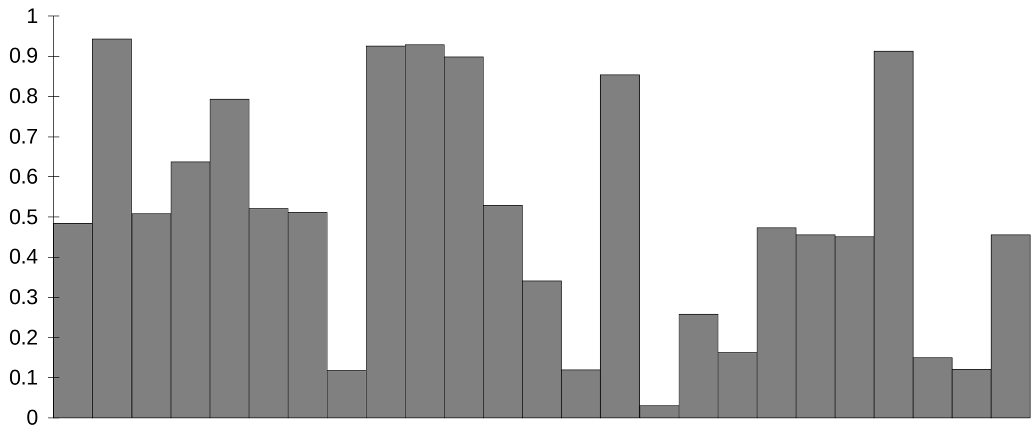
4.

Derive the optimal growth rate of the economy. Can you explain your results?

Appendix to P-Set 3 Regarding continuous populations

A continuous population is surely an abstraction. If we index people with all rational numbers between 0 and 1 each have to be infinitely small, e.g., have zero human capital in order for aggregate human capital to be finite. Nevertheless this may be a useful abstraction. To try to give some intuition, look at a population consisting of *N* (where *N* is very big) individuals. To calculated their aggregate human capital we can divide the population into *I groups* of equal size, indexed by $i = \{1,...I\}$. Then we figure out the average level of human capital *per capita* in each group, $\bar{h}_i^I$. For higher *I* we get a finer division of the population and for each *I* we choose we get a sequence of $\bar{h}_i^I$ for $i = \{1,...I\}$. I have drawn two figures of $\bar{h}_i^I$ for I = 5 and 25 below. Note that if we increase *I*, $\bar{h}_i^I$ will change for each *i* but it will keep its approximate order of magnitude constant.[3]





[3] In the sense that the average over all *i* remains constant when *I* increase but the variance increases.

The amount of human capital in each group is then:

$$\frac{N}{I}\bar{h}_i^I, \tag{3.20}$$

So to get total human capital we sum over the groups:

$$H=\sum_{i=1}^{I}\frac{N}{I}\bar{h}_i^I \tag{3.21}$$

This is equal to the are of all the bars in the figures, noting that the base of each bar is *N/I*.

We can also compute how much an increase in the average human capital level for one group *i* affects aggregate human capital. It is going to be:

$$\frac{dH}{dh_i^I}=\frac{N}{I} \tag{3.22}$$

Thus we see that:

$$\frac{dH}{dh_i^I}=\frac{N}{I} \tag{3.23}$$

decreases when *I* increases (the partition becomes finer). Suppose we increase *I* very much, sooner or later then *N=I*. After that we cannot go on in reality since then we have to start splitting people (we could in fact divide people into parts and let the parts share equally the human capital of the individual, in this case it doesn't, however, make much sense to think of increasing the human capital of a part of an individual). On the other hand, if *N* is large we can increase *I* very much before we run into this problem and we may be forgiven if we forget that constraint. If we the let *I* go to infinity we have:

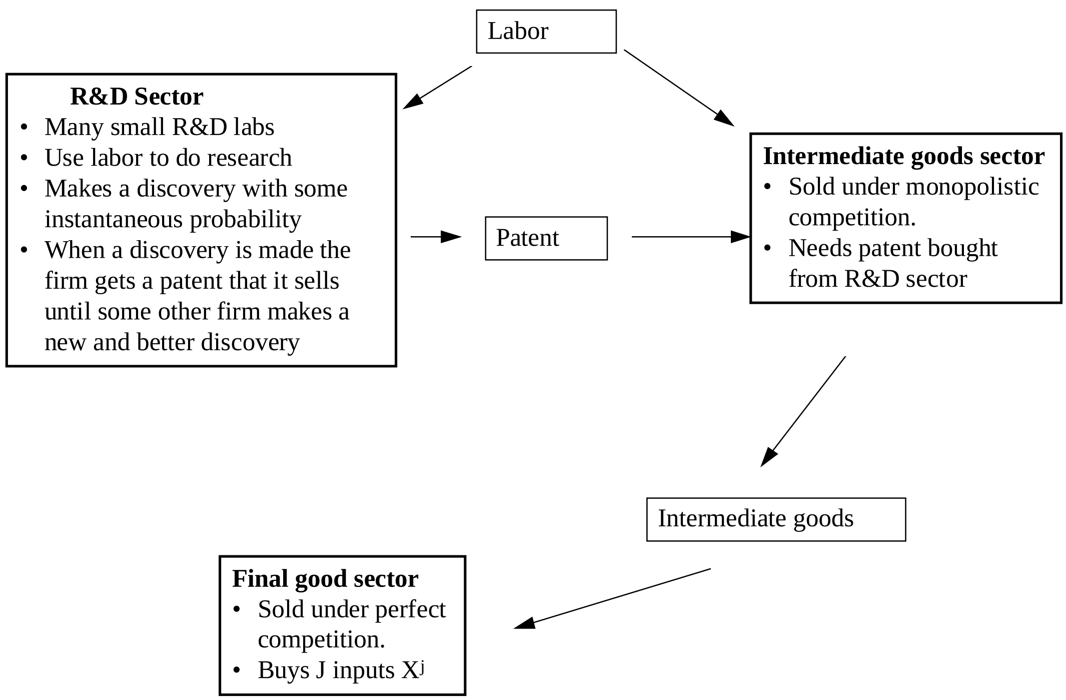
$$\begin{aligned}&\lim_{I\to\infty}\left(N\sum_{i=1}^{I}\frac{1}{I}\bar{h}_i^I\right)=N\int_0^1\bar{h}_i^I\,di\\&\lim_{I\to\infty}\frac{dH}{dh_i^I}=\lim_{I\to\infty}\frac{N}{I}=0\end{aligned} \tag{3.24}$$

In some cases, like the one in P-set 3, it is much easier to do calculations with *I* going to infinity. Then we may formally disregard the externality when optimizing for the individual. For *I* large but finite this is only on approximation. Nota bene the connection to small firms and perfect markets.

Problem Set 4
Endogenous Growth

Problem 1 Growth from R&D[4]

Consider the following version of an R&D model, similar to what you saw in class. It consists of the following parts.



The production in the final goods sector is done according to:

$$Y_t = \sum_{j=1}^{J} A_t X_j^{1-\alpha} , \tag{4.1}$$

where we have normalized the number of perfectly competing firms to one (sic!). A_t is the productivity level which will increase over time due to the work of the R&D-sector. The final goods sector buys intermediate goods X_j from a fixed number J monopolistically competing firms producing according to:

$$X_j = L_j \tag{4.2}$$

In order to be able to produce, the intermediate goods producing firms must buy a patent (the same for all firms) from an R&D sector. In the latter we have a lot of small

[4] Adapted from Aghion and Howitt, and Helpman.

R&D labs who do research. When they do research each firm makes a new discovery with instantaneous probability λZ, where λ is a parameter and Z is the amount of researcher (labor) engaged. The wage in both sectors using labor is the same. When a new discovery is made the inventing firm gets a patent that it can sell to all intermediate goods producing firms. It thus has a monopoly which lasts until some other firm makes a new discovery. A new patent increases the productivity A_t to γA_t. The time index will from here on denote the number of new discoveries from the start of the world. Thus;

$$A_{t+1} = \gamma A_t \tag{4.3}$$

Questions

1.

The final good producing firm pays the marginal product value for the intermediate goods. Normalizing the final goods price to one. Calculate the demand (price function) the monopolistic intermediate goods producers face.

2

Solve the static profit maximization problem of the intermediate goods firms for a given wage *w*. Express your result in a function given the price of the intermediate good as a markup on the good. Invert this function to get *w* as a function of parameters and X_j. Find a function for the profit π as a function of parameters and X_j.

The R&D firm that has the current patent is able to get all profits in the intermediate goods sector. The value of a patent is thus the discounted present value of these profits until a new patents is found. Lets find this using the following steps:

1. Simplify by normalizing J to unity. Using r to denote the discount rate and π_t as the flow of profits until the new patent comes what is the value of the patent if the spell length of a patent actually is *T?*

2. New patents comes with Poison probability λZ_t. This means that the probability of a spell of length *T* (the time until a new patent is found) is $\lambda Z_t e^{-\lambda Z_t \tau}$ Set up the formula for the sum (integral) of the discounted values of the profits for all possible spell lengths .

3. Show that this can be written as:

$$\frac{\pi_t}{r + \lambda Z_t}. \tag{4.6}$$

4.

Now we should look for an equilibrium. There is free entry to the R&D-sector so expected aggregate flows of profit is zero in this sector. The flow of profits is $\lambda Z_t \dfrac{\pi_{t+1}}{r+\lambda Z_{t+1}}$ and the cost is $w_t Z_t$. Use the resource constraint that $Z+L = 1$. Now you have enough information to set up the zero profit condition as an equation only parameters, X^t and X^{t+1}. Set up this difference equation.

5.

Show that the difference equation has a steady state at:

$$X^* = \frac{(1-\alpha)(r+\lambda)}{\lambda(1-\alpha)+\gamma\alpha)} \qquad (4.11)$$

Do comparative statics (see what happens to average growth when parameters are changed) and interpret with intuition.

Problem 2 Growth and Government Spending

Here there is a input G to the production function that is only provided by the government, like infra-structure services. The production function of the economy is thus taken to be:

$$y = k^{1-\alpha} g^{\alpha}. \qquad (4.15)$$

Government spending on G is financed via a flat income tax so:

$$g = \tau y. \qquad (4.16)$$

The representative consumer maximize a CES utility function:

$$\begin{aligned} &\max_c \left(\int_0^\infty \left(\frac{c_t^{1-\sigma}}{1-\sigma} \right) e^{-\rho t} dt \right) \quad \sigma > 0 \\ &s.t \qquad \dot{k} = (1-\tau)k^{1-\alpha} g^{\alpha} - c \\ &\qquad\qquad \lim_{T\to\infty} k_T e^{-rT} = 0 \end{aligned} \qquad (4.17)$$

Questions

1.

What is the marginal product of capital faced by the consumer.

2.

For a given and constant tax rate solve the consumer problem and get a growth rate of the economy γ. Write γ as a function of σ, τ, α and ρ.

(4.15)(4.16)**3.**

Find an expression for $\frac{d\gamma}{d\tau}$ and try to sign it. And try to give some intuition for your results.

4.

Now let there be a central benevolent planner. Assume that we can find an optimal path where g/k is constant (is this reasonable?). Show that the problem of the central planner can be written as:

$$\begin{aligned} &\max_{c,\tau}\left(\int_0^\infty \left(\frac{c_t^{1-\sigma}}{1-\sigma}\right)e^{-\rho t}dt\right) \quad \sigma > 0 \\ &s.t \qquad \dot{k} = (1-\tau)\tau^{\alpha/(1-\alpha)}k - c \\ &\qquad\quad \lim_{T\to\infty} k_T e^{-rT} = 0 \end{aligned} \tag{4.22}$$

Solve the problem and see if the centrally planned economy grows faster or slower than the market economy. What is the intuitive reason for your result.

(4.22)**5.**

Discuss the relevancy of the model.

Problem Set 5
The Overlapping Generations Model

Question 1.

Consider an overlapping generations model in discrete time, where people work and earn a wage w_t in the first period and are retired consuming their savings s_t plus interest in the second. A generation born at time t consumes c_t when young and d_{t+1} when old. The thus solve the problem:

$$\begin{aligned} &\max_{c_t, d_{t+1}} \left(U(c_t) + (1+\theta)^{-1} U(d_{t+1}) \right) \\ &s.t. \quad c_t + s_t = w_t \\ &\qquad\; d_{t+1} = s_t (1 + r_{t+1}) \end{aligned} \tag{5.1}$$

There is production with constant returns to scale. Perfect markets ensures that:

$$\begin{aligned} r_t &= f'(k_t) \\ w_t &= f(k_t) - r_t k_t \end{aligned} \tag{5.2}$$

The savings in period t constitutes the capital for the next generation to work with in period t+1. Since the population grows at rate n we have:

$$k_{t+1}(1+n) = s_t \tag{5.3}$$

1.

Solve the problem of the consumer for a general utility function. Express your result as a savings function only depending on k_t, k_{t+1} and parameters. Give me an equation that implicitly defines a steady state. Can we say anything about the stability of this steady state?

2.

Now assume log utility. Find a savings function. Show that it does not depend on k_{t+1} and explain why.

3.

Now assume Cobb-Douglas production so

$$f(k) = k^{\alpha}. \tag{5.7}$$

Use the savings function to get a difference equation which you solve for a steady state and find a condition for stability.

4.

Now let's use this model to compare two pension systems, the fully funded and the pay as you go. First we take the pay as you go system. Let the government tax the young at rate τ and immediately pay the receipts to the currently old. Assuming log utility, perfect foresight and Cobb-Douglas technology solve for the steady state capital stock k^* as a function of τ. What is the sign of the derivative of k^* with respect to τ at τ=0?

5.

Do the same as under **4.** but now assume that the tax is used to buy capital which is used as private capital in the production. The tax receipts plus interest is then given back to the old generation next period. The old get pensions equal to the tax they paid plus interest. Solve for the steady state and the sign of the derivative of k^* with respect to τ at τ=0.

6.

Without doing any calculations, compare the effects of the two pension systems in a small open economy so that the interest rate is given from abroad at r^*.

$f'(k_t) = r^* \Rightarrow k_t = f'^{-1}(r^*)$ **7.**

Go back to the closed economy without pensions and taxes. Now lets study the effects of a temporarily extra big generation ("fyrtiotalisterna"). Using the difference equation for k_t you found under **3.**, show the dynamic effects on k, w and r if n_t is extra big at t but then goes back to normal. Follow the economy at least a few periods after t.

Bonus question.

Analyze the welfare of the big generation compared to its followers.

Appendix to P-Set 5

A Simple Example with Death Probability

Here is an example that shows that it is not the chance of death but rather the birth of new generations that breaks Ricardian Equivalence in an overlapping generations model. Let's assume that there is a probability to die which is *p*, that $r=\theta=0$, and that there is a perfect capital market so if one saves and dont die one gets a return of $1/p$ on the savings. The individual then maximizes

$$\max_{c_1,c_2}\left(U(c_1)+pU(c_2)\right)$$
$$s.t.\ c_2=\frac{w-c_1}{p}-\frac{\tau_1}{p}-\tau_2$$

The budget constraint for the government is then

$$G=\tau_1+p\tau_2.$$

In the second period there are *p* people around, so the taxreciepts are $p\tau_2$. Then a balanced tax shift satisfies

$$\Delta\tau_1+p\Delta\tau_2=0\Rightarrow\Delta\tau_1=-p\Delta\tau_2$$

Substitute this into the budget constraint of the individual and we get

$$\begin{aligned}c_2&=\frac{w-c_1}{p}-\frac{\tau_1+\Delta\tau_1}{p}-(\tau_2+\Delta\tau_2)\\&=\frac{w-c_1}{p}-\frac{\tau_1+(-\Delta p\tau_2)}{p}-(\tau_2+\Delta\tau_2)\\&=\frac{w-c_1}{p}-\frac{\tau_1}{p}-\tau_2\end{aligned}$$

So the the chance to get away with some tax payments is exactly balanced by that higher taxes have to be paid if one survives. I.e., expected tax payments are independent of the death probability.

Why is the steady state capital stock lower in a Pay-as-You-Go System?

In the answer to Problem set 5 we found, as in Blanchard and Fisher, that the steady state capital stock was lower in the Pay-as-you-go system. They interpret this as a result of the transfer structure of the system. There is, however, another interpretation that does not stress the structural differences.[5]

In the experiment we looked at we introduced the Pay-as-You-Go-System by immediately starting the transfer scheme from young to old. This means that the generation that is old when the scheme begins gets an unearned pension. This is the clue to the effect.

In an actuarially fair system, the budget constraint of each generation is unchanged and thus also their consumption. The government doesn't consume so aggregate consumption and production is unchanged as well the steady state capital stock.

This shows that it is not the Pay-as-You-Go structure that is the clue to the effects but rather the actuarial unfairness. In the experiment in Problem Set 5 as well as in Blanchard and Fisher, the capital stock is lower with the Pay-as-You-Go system of the following reason; the introduction of the system means that the currently old take resources from their descendants. It is this inter-generational transfer that gives the result not the Pay-as-You-Go structure.

[5] See Lindbeck, Assar (1992), "Klarar vi pensionerna?", SNS, Stockholm.

Problem Set 6
Consumption under uncertainty

Question 1.

Consider an agent who lives for two periods and has the same utility function in both periods but discounts utility in the second with 1+θ. She faces a constant interest rate r and her income in period one is y_1. In period two her income $\tilde{y}_2$ is random with mean y_2. She maximizes expected utility seen from period 1.

1.

Define the maximization problem of the consumer and show that the first order condition is

$$U'(c_1) = \frac{1+r}{1+\theta}\mathbf{E}\left(U'(c_2)\right) \tag{6.1}$$

where **E** defines the expectations operator.

Will equation (6.1) be satisfies also in a multiperiod model? (If you can, define a value function and use the Bellman equation to prove the answer. Otherwise make an informed guess and provide some heuristic arguments.)

2.

Now assume quadratic utility

$$U = \alpha c - \frac{\beta c^2}{2}. \tag{6.6}$$

Find the FOCs. Solve for the consumption c_1 of the consumer. How does the consumption depend on the degree of income risk? Explain your result and discuss the usefulness of this utility specification.

3.

Using the quadratic utility function, let's do a Hall-type consumption time series regression.

$$c_t = \mu + \phi c_{t-1} + \varepsilon_t \tag{6.8}$$

What is the true regression coefficient ϕ? Will OLS on (6.8) give you unbiased and consistent estimates of ϕ? What happens if you include more RHS variables that are known in period $t-1$, e.g., c_{t-2}?

4.

Now let's look at some other utility functions. First let the individual have the following utility function.

$$U(c) = -\frac{e^{-\alpha c}}{\alpha} \tag{6.11}$$

Find the degree of absolute risk aversion of this individual.

Now, set for simplicity r and θ to zero and let the second period income be distributed according to

$$\tilde{y}_2 = \begin{cases} y_2 + \varepsilon, \text{with probability } 0.5 \\ y_2 - \varepsilon, \text{with probability } 0.5 \end{cases} \tag{6.12}$$

Solve for c_1. Define $s(\varepsilon)$ as precautionary savings, i.e. the difference between the consumption with no risk and with risk.

$$s(\varepsilon) \equiv \left(c_1 \middle| \varepsilon = 0\right) - c_1. \tag{6.13}$$

Is precautionary savings increasing in the variance? (Hint; look at $s'(\varepsilon)$.) Is it increasing in income?

5.

Now let

$$U(c) = \ln(c) \tag{6.15}$$

Find the degree absolute risk aversion of this individual, how does it depend on income.

Solve for c_1.

Is precautionary savings increasing in the variance? Is it increasing in income?

Problem Set 7
Asset pricing and term structure in the Lucas (1978) model

Question 1.

Consider a large amount of identical agents who live for ever on a closed island with apple trees on it (not very appealing). Each individual owns one tree which every year gives y_t apples. y_t is stochastic but the same for all individuals a given year and nothing can be done to affect the harvest nor can apples be stored between years. The price of apples is normalized to 1. There is a perfect markets for risk free bonds of maturities from 1 to K periods. They give one SEK at maturity and $p_{k,t}$ is the price of a bond in time period t that matures at $t+k$. The agents have a time additive utility function with the consumption of apples each period, c_t, as the only argument, time is discrete and their time preference is $1+\theta$.

1.

Define the maximization problem of the consumer.

2.

Use the following arbitrage argument to derive an Euler equation for c_t and c_{t+k}: If the consumption path of the individual is optimal she should neither gain nor loose expected utility by taking dc of current consumption, investing it in the risk free bond with a maturity of k and then using the proceeds to buy apples in $t+k$. Formalize this argument and use it to derive the Euler equation; first for k=1 and then for any strictly positive k.

3.

Do you think that the return of holding a 1 year bond between t and t+1 and the return on holding a two year bond one year and the selling it, will be the same and/or the same in expectation? I.e., is

$$\frac{1}{p_{1,t}} = \begin{cases} \dfrac{p_{1,t+1}}{p_{2,t}} \\ \mathbf{E}_t \dfrac{p_{1,t+1}}{p_{2,t}} \end{cases} ?$$

Why, or why not?

4.

Let the utility function in each period be given by

$$U(c_t) = \frac{c_t^{1-\alpha}}{1-\alpha} \tag{7.4}$$

and assume the the harvests follow

$$\ln(y_t) = \rho \ln(y_{t-1}) + \varepsilon_t, \tag{7.5}$$

with ε_t i.i.d. N(0,σ^2) and -1<ρ<1.

If all agents are the same, all agents get the same harvest in a given period and apples are non-storable, what is the consumption of each individual? Follow the difference equation you then can set up for c_t forward and show that:

$$\ln(c_{t+k}) \overset{d}{=} \mathrm{N}\left(\rho^k \ln(c_t), \sigma^2\left(1+\cdots+\left(\rho^{k-1}\right)^2\right)\right) = \mathrm{N}\left(\rho^k \ln(c_t), \sigma^2 \frac{1-\rho^{2k}}{1-\rho^2}\right) \tag{7.6}$$

Now remember the properties of the log-normal distribution, in particular that if ln(x) is N(m,σ^2), then

$$\mathbf{E}\left(x^a\right) = e^{am+\frac{(a\sigma)^2}{2}} \tag{7.7}$$

Now find the expectation of the marginal utility at t+k if we know c_t.

(7.6)**5.**

Use your answers to calculate $p_{t,k}$. Is anyone going to buy or issue these bonds?

Then find the yield to maturity defined in a continuously compounding way as:

$$-\frac{\log(p_{t,k})}{k}. \tag{7.10}$$

Make sure you understand this way of defining yield to maturity.

Often we hear that the prices of financial assets follow random walks, do they here? Why, or why not?

6.

For a one year bond, how does the yield depend on α, σ and ρ. How does it depend on c_t, for positive, negative and zero values of ρ? Why?

7.

Assume that ρ is positive, use the expression for the yield to maturity to draw a figure with some approximate yield structures for normal, bad and good current harvests.