Problem Set 1 "Working with the Solow model"

Let's define the following exogenous variables:

s = savings rate

δ = depreciation rate of physical capital

n = population growth rate

L = labor supply = e^{nt} (Normalizing labor supply per capita and time 0 population to unity)

x = rate of growth of labor augmenting technology.

$\tilde{L}$ = effective labor supply = $e^{(n+x)t}$

and the following endogenous:

$\tilde{k}$ = capital stock per effective labor unit ($K/\tilde{L}$).

$\tilde{y}$ = output per effective labor unit ($Y/\tilde{L}$).

k = capital stock per capita

y = output per capita

1.

a)What is the growth rate of $\tilde{L}$. If $\tilde{k}$ grows at rate γ (i.e., $\dot{\tilde{k}}/\tilde{k}=\gamma$), what is the growth rate of k, K and $\tilde{k}^{\alpha}$. What is a proper unit of γ ?

b) The gross resource constraint for a closed economy can be written

$$sF(K,\tilde{L}) = \text{Gross Investment} \qquad (1.1)$$

How should we define gross investment? Substitute your definition into (1.1) and then substitute a Cobb-Douglas production function $\left(Y = K^{\alpha}\tilde{L}^{1-\alpha}\right)$ for F. Use this to derive, step by step, an expression for the growth rate of $\tilde{k}$. (Hint: Divide by $\tilde{L}$, then calculate $\dot{\tilde{k}}$ by noting that $\dot{\tilde{k}} = \frac{\partial}{\partial t}\left(\frac{K}{\tilde{L}}\right)$, and you get and expression containing $\frac{\dot{K}}{\tilde{L}}$. Then substitute and your done).

c) Provide some reasonable values for s, δ, n and x.

d) Find the steady state level of $\tilde{k}$, $\equiv \tilde{k}^*$. Use your parameter values from c) to find the steady state capital output ratio K/Y? If this ratio is constant what does it imply for the growth rates of K and Y. Is the labor output ratio also constant?

e) In steady state what is the growth rate of GDP and of wages? What about the interest rate, i.e., the marginal product of capital?

Barro and Sala-i-Martin make a log-linear approximation to the growth rate of $\tilde{k}$ around it's steady state (equation 1.31). They write:

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = -\beta \log\left(\frac{\tilde{k}}{\tilde{k}^*}\right) \tag{1.2}$$

where

$$\beta = (1-\alpha)(x+n+\delta) \tag{1.3}$$

a) Starting from steady state, if you decrease the capital stock by 1%, what is a good approximation (no exact answers please) for the growth rate of $\tilde{k}$. How much is it given the parameter values in your answer to the previous question?

b) Assume that from an initial steady state, a devastating war leads to a reduction in $\tilde{k}$ to one half of its steady state value. Calculate the output gap (≡Y/Y^*-1) immediately after this shock. What is now the growth rate of $\tilde{k}$ and of K, given the approximation in (1.2)? Calculate the change in interest rate. Is it reasonable?

c) How big is the output gap after one year and after 10 years? Can you calculate the time it takes until half the output gap is gone? Is this slow or fast, do you think? When is the gap totally gone. (Hint, define the variable $d \equiv \log(\tilde{k} / \tilde{k}^*)$. What is the interpretation of d? Compute $\dot{d}$ from its definition. Substitute into (1.2) and you get a very simple differential equation that you can solve explicitly.)

d) If you increase α what happens to the rate of convergence and to the steady state capital stock. What do you think happens as α approaches 1.

Voluntary extra question

Use the exact solution to the differential equation for $\tilde{k}$ given in footnote 15 in Barro and Sala-i-Martin to see how far off the log-linear approximation is for some reasonable parameter values.

Problem Set 2
"The Ramsey Model for an Open Economy with Imperfect Markets"

This problem extends the standard Ramsey model you saw in class. Before you do the problem, compare this model to the Solow model in terms of model assumptions. Do they address the same questions? When is which model more useful?

Assumptions

We use the same notation as in Problem Set 1. For simplicity we set x and n to 0. This means that we don't have to worry about the differences between per capita, per effective labor unit and aggregate units. We are now looking at an open economy. The firm sells its good on the world market but it faces a downward sloping demand curve. The price P of the firm's output is a negative function of how much it produces. The good that it produces cannot be stored so everything the firm produces has to be sold immediately. The firm buys capital from abroad at a price P_k. There is, however, an imperfection in the international transportation sector. The more capital the firm buys per unit of time, the more expensive transportation becomes. Households buy consumption goods a a perfect world market at price 1.

Firms

Technology and demand are given by

$$\begin{aligned} y_t &= \alpha k_t \\ P_t &= P(y_t). \end{aligned} \tag{2.1}$$

To invest at the flow i the firm has to pay $iP_k(1+T(\mathrm{i}))$ per unit of time. $T(i)$ is the transportation cost and $T'(i)>0$ for $i>0$. The firm also has a fixed number of employees and pays w in wages to them. The firm pays its profits to the household in the form of dividends, d_t. It maximizes the present value of dividends and thus solves

$$\begin{aligned} &\max_{\{i_t\}_0^\infty} \int_0^\infty e^{-rt}\left(P(y_t)y_t - i_t P_k\left(1+T(i_t)\right) - w\right)dt, \\ &\text{s.t.} \quad \begin{cases} y_t = \alpha k_t \\ \dot{k}_t = i_t - \delta k_t \\ k_0 = \bar{k} \end{cases} \end{aligned} \tag{2.2}$$

Since the country is small and open the interest rate, *r*, is given and constant. There is also access to a perfect capital market for both firms and households.

Households

Households own the firms and supply one unit of labor inelastically. The representative household solves:

$$\max_{\{c_t\}_0^\infty} \int_0^\infty e^{-\rho t}\left(\frac{c_t^{1-1/\sigma}}{1-1/\sigma}\right)dt,$$

$$\text{s.t.} \quad \begin{cases} \dot{a}_t = w + d_t + ra_t - c_t \\ a_0 = \bar{a} \\ \lim_{t\to\infty} e^{-rt} a_t = 0 \end{cases} \tag{2.3}$$

Questions

1.

a) Is the total cost of investing, $iP_k(1+T(\mathrm{i}))$, convex or concave in i for positive i? What does that imply for the relative cost of volatile and smooth investment plans?

Solve the maximization problem of the firm in the following steps.

b) Set up the current value Hamiltonian. Represent the solution as two differential equations, one for k_t and one for q_t, the current value shadow price of capital and an investment function $i=i(q,P_k)$.

c) Make a linearisation of the system around the steady state. For what could you use this linearisation? When can it be dangerous?

d) From now on assume that the price functions have the following simple structure

$$\begin{aligned} P(y) &= 1 - \frac{\beta y}{2} \\ T(i) &= \frac{\gamma i}{2} \end{aligned} \tag{2.4}$$

Solve for the steady states of k_t and q_t, (k* and q*) and draw a phase diagram.

2

Solve the problem of the household.

a) Start by setting up the Hamiltonian, now in present values for a change. Take time derivatives of H_c=0 and substitute in from the equation for H_a. This gives you a nice differential equation for c_t.

b) What happens if $\rho \neq r$? Is that reasonable? What is the interpretation of σ? How does things change with changes in σ.

c) Solve for the level of c_t using the transversality equation in (2.3) (Hint; present value of consumption equals present value of income plus starting wealth).

3.

Suppose r=ρ and that the economy is in steady state. Use the phase diagram to see what happens dynamically if we get a permanent negative shock to the demand for export goods (β increases). What happens to consumption and the current account over time. (Hint; use the following version of the inter temporal budget constraint:

$$\begin{aligned} c^p &= w^p + d^p + ra_0 \Rightarrow \\ c^p &= P(y^p)y^p - i^p P_k(1+T(i)) + ra_0 \end{aligned} \tag{2.14}$$

where superscript p denotes permanent values, defined by:

$$\int_0^\infty (x^p)e^{-rt}dt = \frac{x^p}{r} \equiv \int_0^\infty (x_t)e^{-rt}dt \tag{2.15}$$

The present discounted value is the same as for the actual path of x_t. Now write the equation for the current account, where b equals the stock of debt to the rest of the world.

$$-\dot{b}_t = P(y_t)y_t - i_t P_k(1+T(i)) - c_t - rb_t \tag{2.16}$$

Substitute (2.14) into the equation for the current account and note that a=-b, (the household cannot sell its ownership of the firm and the value of the firm is not included in a).Using this for t=0 we can then write the useful equation:

$$-\dot{b} = \left(P(y_t)y_t - P(y^p)y^p\right) - \left(i_t P_k(1+T(i)) - i^p P_k\left(1+T(i^p)\right)\right) - (c_0 - c^p) \tag{2.17}$$

Equation (2.17) can be used to intuitively derive what happens dynamically after the shock.

4.

What happens if the shock is *i*) temporary (happens at t=0 and ends at t=s) and; *ii*) anticipated permanent (the information about the shock arrives at t=0 and the actual shock happens at t=s).

(Hint. Draw the phase diagrams and note that k can newer jump (why?) and that q can jump only when new information (like information about a future productivity shock) arrives. In the case of the anticipated shock note that at the time of the shock the phase diagram shifts and that at exactly the time of the shift we have to be at the stable saddle path for the system not to explode. So when the information arrives at t=0, the "old" phase diagram continues to hold. Now q jumps so that $\{q,k\}$ hits the new stable path at exactly t=s).)

Problem Set 3
Endogenous Growth

Problem 1 Learning by Doing in a Two Sector Model

Consider a closed economy with a fixed number of people - normalized to one. The people get utility from two sources; by being taken care of by people employed in a service sector and from a good produced in a manufacturing sector. The instantaneuos utility function is

$$U_t = \log(s_t) + \log(c_t) \tag{3.1}$$

Where s_t is the number of hours of service per unit of time and c_t is the consumption of manufactured goods per unit of time. The individuals maximize future utility under a budget constraint;

$$\begin{aligned} &\int_0^\infty U_t e^{-\rho t} dt \\ &s.t. \\ &P_t c_t + s_t = w_t , \end{aligned} \tag{3.2}$$

where P_t is the price of manufactured goods, w_t is the wage and we fix individual labor supply to unity and normalize the price of services to unity. People can choose to work either in the service sector or in the manufacturing sector. Labor markets and consumption markets are competitive. Production in the manufacturing sector equals consumption and is given by

$$c_t = A_t l_c \tag{3.3}$$

where A_t is the level of productivity and l_c is the share of people working in the manufacturing sector. Each firm takes the level of productivity as given. There is learning by doing in the manufacturing sector, however, so A evolves over time according to:

$$\dot{A}_t = \upsilon c_t \tag{3.4}$$

Questions

a) Before doing any number crunching think about the following. In this economy, what will happen to the future consumption possibility set if all people decide to consume more manufactured goods today? If only one person consumes more manufactured goods? What do you think of the specification of learning by doing, does it make sense?

b) When we normalize the price of services to unity what is the wage rates in the two sectors?

c) With perfect competition in the manufacturing sector what is the price of manufactured goods?

d) Solve the maximization problem of the consumer. (Hint: All individuals are equal. This implies that if there is a capital market the interest rate will have to adjust so that no one will borrow. We can thus as well assume that there is no capital market. The maximization problem of the consumer is the static.)

What is the growth rate in this sector. What happens to the growth rate if more people work in the manufacturing sector?

f) Now let's find the optimal amount of people in the manufacturing sector. First, do you think it should be higher or lower than the market solution?

To find the solution, follow these steps:

1. Let there be a benevolent central planner who maximizes the individuals' utility but takes into account the learning by doing effect. Assume there is no possibility to store the manufactured good.

2. Write the Hamiltonian and the optimality conditions. Note how the Hamiltonian differs from the static problem of the individual. Guess that in an optimal steady state, the labor shares are constant and then differentiate the first optimality condition (H_l) w.r.t. time. What is then the relationship between the growth rate of the shadow value of the constraint and the growth rate of *A*? Eliminate the growth rate of the shadow value and the shadow value from the second optimality condition (H_A) and you get an expression for the growth rate of *A* involving υ, l_c and ρ

4. There is another expression for the growth rate of A_t coming from the learning by doing constraint. Use this to eliminate the growth rate of A_t.

(3.4)**Problem 2. The "School is Never Out" – model**

There is a continuum of workers-consumers on the interval [0,1], i.e. an infinite number of agents indexed by the real numbers *i* between 0 and 1. Each individual *i* has one unit of labor to spend per day (rather per instant). Since all individual are alike we can drop the *i* superscript where we don't need it. Her decision is how much of the day to spend in school (1-a_i) and how much to work (a_i). She can, however, never quit shool. When she goes to school she accumulates human capital (h_i) at a rate proportional to the time spent in school.

$$\dot{h}_t = \beta(1-a)h_t \qquad (3.12)$$

The interesting thing with the model is that the productivity of individual *i* depends on the human capital of everybody else. By working with smart people your productivity increases. The production of a single individual is thus given by:

$$y_t = \left(ah_t\right)^{\alpha}\left(AH_t\right)^{1-\alpha} \tag{3.13}$$

Where *A* and *H* denotes aggregate levels of *a* and *h*.

$$\begin{aligned} H_t &= \int_0^1 h_t^i\,di \\ A_t &= \int_0^1 a_t^i\,di \end{aligned} \tag{3.14}$$

The production cannot be stored, so the individual consumer-producer consumes what she produces each moment in time so she solves

$$\begin{aligned} &\max_a \int_0^\infty \log(y_t)e^{-\rho t}\,dt \\ &s.t. \\ &y_t = \left(ah_t\right)^{\alpha}\left(AH_t\right)^{1-\alpha} \\ &\dot{h}_t = \beta(1-a)h_t \\ &h_0 = \bar{h} \end{aligned} \tag{3.15}$$

Questions

a) Is there decreasing or constant returns to human capital in this model? How much of the externality is taken into account by this small selfish agents? This can be found be evaluating $\frac{\partial H}{\partial h^i}$.

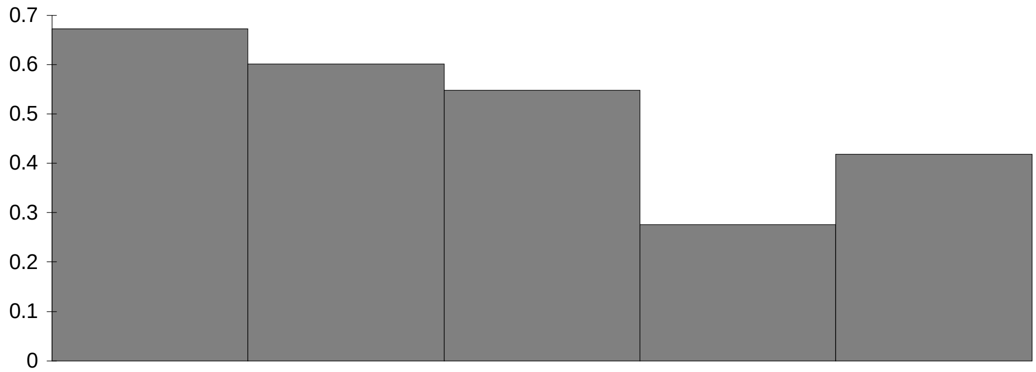
b) Assume a common value of *a* for all agents, (is this reasonable?), then compute the growth rate of the economy as a function of *a*.

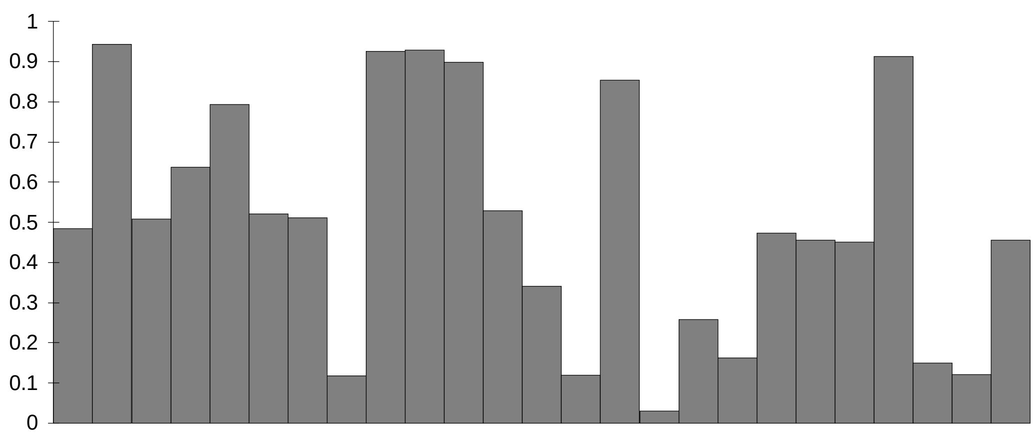
c) Derive the equilibrium growth rate of the economy by letting the agents choose *a* optimally. (Hint; Set up the maximization problem of the consumer-producer and substitute for *c* in the utility function. Set up the Hamiltonian with its optimality condition and the solve for the growth rate of human capital assuming a steady state where *a* is constant.

d) Derive the optimal growth rate of the economy. Can you explain your results?

Appendix to P-Set 3 Regarding continuous populations

A continuous population is surely an abstraction. If we index people with all rational numbers between 0 and 1 each have to be infinitely small, e.g., have zero human capital in order for aggregate human capital to be finite. Nevertheless this may be a useful abstraction. To try to give some intuition, look at a population consisting of N (where N is very big) individuals. To calculated their aggregate human capital we can divide the population into I *gr*oups of equal size, indexed by $i = \{1,...I\}$. Then we figure out the average level of human capital *per capita* in each group, $\bar{h}_i^I$. For higher I we get a finer division of the population and for each I we choose we get a sequence of $\bar{h}_i^I$ for $i = \{1,...I\}$. I have drawn two figures of $\bar{h}_i^I$ for $I = 5$ and 25 below. Note that if we increase I, $\bar{h}_i^I$ will change for each i but it will keep its approximate order of magnitude constant.[1]





[1] In the sense that the average over all i remains constant when I increase but the variance increases.

The amount of human capital in each group is then:

$$\frac{N}{I}\bar{h}_i^{\,I}\,, \tag{3.23}$$

So to get total human capital we sum over the groups:

$$H=\sum_{i=1}^{I}\frac{N}{I}\bar{h}_i^{\,I} \tag{3.24}$$

This is equal to the are of all the bars in the figures, noting that the base of each bar is *N/I*.

We can also compute how much an increase in the average human capital level for one group *i* affects aggregate human capital. It is going to be:

$$\frac{dH}{dh_i^I}=\frac{N}{I} \tag{3.25}$$

Thus we see that:

$$\frac{dH}{dh_i^I}=\frac{N}{I} \tag{3.26}$$

decreases when *I* increases (the partition becomes finer). Suppose we increase *I* very much, sooner or later then *N=I*. After that we cannot go on in reality since then we have to start splitting people (we could in fact divide people into parts and let the parts share equally the human capital of the individual, in this case it doesn't, however, make much sense to think of increasing the human capital of a part of an individual). On the other hand, if *N* is large we can increase *I* very much before we run into this problem and we may be forgiven if we forget that constraint. If we the let *I* go to infinity we have:

$$\begin{gathered}\lim_{I\to\infty}\left(N\sum_{i=1}^{I}\frac{1}{I}\bar{h}_i^{\,I}\right)=N\int_0^1\bar{h}_i^{\,I}\,di\\ \lim_{I\to\infty}\frac{dH}{dh_i^I}=\lim_{I\to\infty}\frac{N}{I}=0\end{gathered} \tag{3.27}$$

In some cases, like the one in P-set 3, it is much easier to do calculations with *I* going to infinity. Then we may formally disregard the externality when optimizing for the individual. For *I* large but finite this is only on approximation. Nota bene the connection to small firms and perfect markets.

Problem Set 4
The Overlapping Generations Model

Question 1.

Consider an overlapping generations model in discrete time, where people work and earn a wage w_t in the first period and are retired consuming their savings s_t plus interest in the second. A generation born at time t consumes c_t when young and d_{t+1} when old. They thus solve the problem:

$$\begin{aligned} &\max_{c_t, d_{t+1}} \left(U(c_t) + (1+\theta)^{-1} U(d_{t+1}) \right) \\ &s.t. \quad c_t + s_t = w_t \\ &\qquad\quad d_{t+1} = s_t (1 + r_{t+1}) \end{aligned} \tag{4.1}$$

All variables are per capita which is taken to mean that we divide with the size of the relevant generation. There is production with constant returns to scale. Perfect markets ensures that:

$$\begin{aligned} r_t &= f'(k_t) \\ w_t &= f(k_t) - r_t k_t \end{aligned} \tag{4.2}$$

The savings in period t constitutes the capital for the next generation to work with in period t+1. The population grows at rate n so

$$k_{t+1}(1+n) = s_t \tag{4.3}$$

1.

a) Show that indirect utility increases in w_t and r_{t+1}.

b) Solve the problem of the consumer for a general utility function. Express your result as a savings function only depending on k_t, k_{t+1} and parameters. Give me an equation that implicitly defines a steady state.

c) Can we say anything about the stability of the steady state?

2.

Now assume logarithmic utility. Find a savings function. Show that it does not depend on k_{t+1} and explain why.

3.

Now also assume Cobb-Douglas production so

$$f(k) = k^{\alpha}. \tag{4.8}$$

Use the savings function to get a difference equation which you solve for a steady state and find a condition for stability.

4.

Now let's use this model to study the effect of a budget deficit used to finance transfers to the current tax payers, i.e., the current situation in Sweden. According to the Ricardian Equivalence Theorem by Barro, such a policy should have no effect. But what is the predictions of this model? Let's formalize the question as follows. The government borrows b_t at time t from the currently young. The receipts are immediately paid back to the currently young as transfers. The government is not allowed to go bankrupt so in t+1 the old get back what the government borrowed plus interest.

a) First assume that next generation restores the government's financial position by paying back $b_t(1+r_{t+1})$ to the old. Write down the budget constraints for the young in t and t+1. Find the sign of the effects on capital stocks, wages and utility for the generations born in t to t+2 for a small b_t and starting from a steady state.

b) Now assume that the generation born in t+1 and onwards pays interest but rolls over the principal by letting the government borrow to finance paying back the principal to the old. Write down the budget constraints for the young in t and t+1. Find the sign of the effects on capital stocks, wages and utility for the generations born in t to t+2 for a small b_t and starting from a steady state.

c) Assume that the generations born from t+1 and onwards roll over all debt including interest rates. How will b_s evolve for $s>t$? What happens if $r_s<n$? Explain.

Appendix to P-Set 4

A Simple Example with Death Probability

Here is an example that shows that it is not the chance of death but rather the birth of new generations that breaks Ricardian Equivalence in an overlapping generations model. Let's assume that there is a probability to die which is 1-*p*, that $r=\phi=0$, and that there is a perfect capital market so if one saves and don't die one gets a return of $1/p$ on the savings. In addition *n* individuals are born in period 2. The individual in period 1 then maximizes

$$\max_{c_1,c_2}\left(U\left(c_1\right)+pU\left(c_2\right)\right)$$
$$s.t.\ c_2=\frac{w-c_1}{p}-\frac{\tau_1}{p}-\tau_2$$

The budget constraint for the government is then

$$G=\tau_1+(p+n)\tau_2.$$

In the second period there are *p* people around, so the tax receipts are $p\tau_2$. Then a balanced tax shift satisfies

$$\Delta\tau_1+(p+n)\Delta\tau_2=0\Rightarrow\Delta\tau_1=-(p+n)\Delta\tau_2$$

Substitute this into the budget constraint of the individual and we get

$$\begin{aligned}c_2&=\frac{w-c_1}{p}-\frac{\tau_1+\Delta\tau_1}{p}-(\tau_2+\Delta\tau_2)\\&=\frac{w-c_1}{p}-\frac{\tau_1+(-(p+n)\Delta\tau_2)}{p}-(\tau_2+\Delta\tau_2)\\&=\frac{w-c_1}{p}-\frac{\tau_1}{p}-\tau_2+\frac{n}{p}\Delta\tau_2\end{aligned}$$

So the chance to get away with some tax payments is exactly balanced by that higher taxes have to be paid if one survives if $n=0$. I.e., expected tax payments are then independent of the death probability. If, on the other hand, $n>0$ the consumption possibility set is increased if taxes are shifted forward in time.

Problem Set 5
Consumption under uncertainty

Question 1.

Consider a person who lives for two periods and has the same utility function in both periods but discounts utility in the second with 1+θ. She faces a constant interest rate r and her income in period one is y_1. In period two her income $\tilde{y}_2$ is random with mean y_2. She maximizes expected utility seen from period 1.

1.

Define the maximization problem of the consumer and show that the first order condition is

$$U'(c_1) = \frac{1+r}{1+\theta} E\big(U'(c_2)\big) \qquad (5.1)$$

where E defines the expectations operator.

Will equation (5.1) be satisfies also in a multi-period model? (If you can, define a value function and use it to prove the answer. Otherwise make an informed guess and provide some heuristic arguments.)

2.

Now assume quadratic utility

$$U = \alpha c - \frac{\beta c^2}{2}. \qquad (5.6)$$

Find the FOCs. Solve for the consumption c_1 of the consumer. How does the consumption depend on the degree of income risk? Explain your result and discuss the usefulness of this utility specification.

3.

Using the quadratic utility function, let's do a Hall-type consumption time series regression.

$$c_t = \mu + \phi c_{t-1} + \varepsilon_t \qquad (5.8)$$

What is the true regression coefficient ϕ? Estimate the parameters of the regression from Swedish consumption data. Discuss potential problems and remedies in OLS estimation.

Include more RHS variables that are known in period *t-1*, e.g., c_{t-2}? Do your result accord with the theoretical predictions?

4.

Now let's look at some other utility functions. First let the individual have the following utility function.

$$U(c) = -\frac{e^{-\alpha c}}{\alpha} \tag{5.11}$$

Find the degree of absolute risk aversion of this individual.

Now, set for simplicity r and θ to zero. Let the second period risk be additive and normally distributed so that

$$\tilde{y}_2 = y_2 + \varepsilon, \qquad \varepsilon \overset{d}{=} N\left(0, \sigma^2\right) \tag{5.12}$$

Solve for c_1. Hint; Remember the following relation:

$$\begin{aligned} & x \overset{d}{=} N\left(\bar{x}, \sigma_x^2\right) \\ \Rightarrow\ & E\left[e^{\beta x}\right] = e^{\beta \bar{x} + \beta^2 \sigma_x^2 / 2} \end{aligned} \tag{5.13}$$

Define precautionary savings, $s(\sigma)$, as the difference between consumption with no risk and with risk.

$$s(\sigma) \equiv \left(c_1 \middle| \sigma^2 = 0\right) - c_1 . \tag{5.14}$$

Is precautionary savings increasing in the variance and α? Is it increasing in income?

5.

Now let

$$U(c) = \frac{c^{1-\alpha}}{1-\alpha} \tag{5.17}$$

Find the degree absolute risk aversion of this individual, how does it depend on consumption?

To get analytical solution to this problem we need multiplicative risk that is log-normally distributed, i.e., we may think of risk as a risky return and that the percentage return has a normal distribution. So

$$\begin{aligned} c_2 &= \left(y_1 + y_2 - c_1\right)\varepsilon \\ \log \varepsilon &\overset{d}{=} N\left(0, \sigma^2\right) \end{aligned} \qquad (5.18)$$

where y_2 is known in the first period.

Solve for c_1. Hint; log(c_2) is normal so you can use the hint to the previous question.

Now define relative precautionary savings as

$$s_r(\sigma) \equiv 1 - \frac{c_1}{c_1\big|\sigma^2 = 0} \qquad (5.19)$$

Is relative precautionary savings increasing in the variance and α? Is it increasing in income? Which of the two utility functions you have worked with do you think captures attitudes towards risk best?

Problem Set 6
Is there a Swedish Equity Premium Puzzle?*

In this problem set we are going to use a standard model of asset pricing. We are also going to assume a specific stochastic process for how consumption evolves over time. We will calibrate the parameters of the process is calibrated so as to replicate the mean, variance and autocorrelation of Swedish real consumption. Then we can compute the expected risky and risk free returns and compare them with actual returns in Sweden.

Assumptions

The representative consumer maximizes

$$E_t \sum_{s=0}^{\infty} \beta \frac{c_t^{1-\alpha}}{1-\alpha} \tag{6.1}$$

The firm produces a good that has to be consumed immediately. The country is closed so by assuming a process for production we also know how consumption evolves. To simplify we assume that the growth rate of consumption follows a first order Markov process, i.e., it only depends on last periods growth. We simplify further by assuming that growth only can take two values; high or low. We assume that consumption growth is ruled by

$$\frac{c_t}{c_{t-1}} \equiv \lambda_t = \begin{cases} \lambda^1 \text{ with probability } \phi, & \text{if } \lambda_{t-1} = \lambda^1 \\ \lambda^2 \text{ with probability } 1-\phi, & \text{if } \lambda_{t-1} = \lambda^1 \\ \lambda^1 \text{ with probability } 1-\phi, & \text{if } \lambda_{t-1} = \lambda^2 \\ \lambda^2 \text{ with probability } \phi, & \text{if } \lambda_{t-1} = \lambda^2 \end{cases} \tag{6.2}$$

So the probability of a repetition of last periods growth is ϕ and of a change 1-ϕ. Note that λ is stationary, i.e., its unconditional moments are constant over time.

Questions

a) Show that the unconditional (ergodic) probabilities of $\lambda_t = \lambda^1$ and λ^2 are 0.5 and 0.5.
Hint; the probabilities can be found by solving the following system of equations.

$$\begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} \phi & 1-\phi \\ 1-\phi & \phi \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} \tag{6.3}$$

* The approach follows Mehra and Prescott, JME 1985.

where the π's are the wanted probabilities. Try to understand why these equations give you the unconditional probabilities.

b) Parameterize by setting

$$\begin{aligned} \lambda^1 &= 1+\mu-\delta \\ \lambda^2 &= 1+\mu+\delta \end{aligned} \tag{6.4}$$

Show that the unconditional mean, variance and autocorrelation of λ_t are $\{1+\mu, \delta^2, 2\phi-1\}$. Hint; Use the relation

$$E[m_t] = E[m_t | \lambda_{t-1} = \lambda^1]\pi_1 + E[m_t | \lambda_{t-1} = \lambda^2]\pi_2 \tag{6.5}$$

where m_t is the moment you are looking for.

c) Estimate the sample mean, variance and autocorrelation of λ_t from the Swedish data on consumption. Use the results calibrate the model, that is to assign numbers to $\{\mu, \delta, \phi\}$.

Now we can also compute the process for the price of the firm in the model. The price will generally depend on the current level of consumption and on expectations about future consumption growth. With the simple Markov process for consumption, expectations only depend on last periods growth. Furthermore the CRRA utility function implies that the price is homogeneous of degree 1 in the current level of consumption. This means that the price of the firm only depends on current consumption and on last periods growth, and that it is linear in current consumption. Let $P(c,\lambda)$ denote the price of the firm. We can then write standard pricing function for assets as

$$\begin{aligned} P(c_t,\lambda^1) \equiv p^1 c_t &= \beta E_t\left[\frac{U'(c_{t+1})}{U'(c_t)}(p^i c_{t+1} + c_{t+1}) \middle| \lambda_t = \lambda^1\right] \\ P(c_t,\lambda^2) \equiv p^2 c_t &= \beta E_t\left[\frac{U'(c_{t+1})}{U'(c_t)}(p^i c_{t+1} + c_{t+1}) \middle| \lambda_t = \lambda^2\right] \end{aligned} \tag{6.9}$$

p^i can now be interpreted as the price of the firm relative to the current level of consumption, given that the growth from last to current period was λ^i.

d) Divide both sides of the pricing function with c_t, substitute to get rid of c_t and c_{t+1}. Derive a two equation system that (implicitly) defines p^i in terms of $\{\beta, \phi, \lambda^1, \lambda^2, \alpha\}$.

e) If $\lambda_t = \lambda^i$ and λ_{t+1} happens to be λ^j, the return on equity from t to t+1 can be written

$$\tilde{r}_{ij} = \frac{p^j c_{t+1} + c_{t+1} - p^i c_t}{p^i c_t} \qquad (6.13)$$

Get rid of consumption in the last equation and derive an expression for the unconditional expected return on equity.

Plug in the estimated values of $\{\mu, \delta, \phi\}$ and set β = 0.95 and α = 1,5 and 10. Compute the unconditional expected return on equity.

f) Show that the unconditional risk less rate of return equals

$$E\bar{r} = \frac{1}{2\beta\left(\phi\left(\lambda^1\right)^{-\alpha} + (1-\phi)\left(\lambda^2\right)^{-\alpha}\right)} + \frac{1}{2\beta\left(\phi\left(\lambda^2\right)^{-\alpha} + (1-\phi)\left(\lambda^1\right)^{-\alpha}\right)} - 1 \qquad (6.16)$$

g) Compute the models predicted equity premium for the three different risk aversion coefficients.

h) Estimate average real return on equity and on bonds from the Swedish financial data. What is the average equity premium in the sample?

i) Compare the sample equity premium with the predictions of the model. What do you conclude?

Problem Set 7
A Simple Real Business Cycle Model*

In this problem set we are going to construct a small real business cycle model and use Swedish data to calibrate it. Since it is very simple it may not perform perfect but it will illustrate the main ideas and virtues of this kinds of models. RBC models have some typical features;

- They build on maximizing agents.
- A maintained hypothesis is that real shocks drive the business cycle.
- They try to model a propagation mechanism such that a simple shock, typically technological, can cause persistent business cycles.
- Parameter values are as far as possible taken from other "usually reliable sources" (which in this case will be my pure guesses).
- They choose the rest of the parameter values so that the model replicates some interesting moments of aggregate macro variables, typically relative standard deviations, correlations and autocorrelations. Often the models are analytically unsolvable so the model moments are generated by simulation.

Constructing the Model

Let there be representative consumer with log utility who owns a firm with CD technology. At time t the problem for the consumer is then

$$\begin{aligned} &\max E_t \sum_{s=0}^{\infty} \beta^s \ln\left(C_{s+t}\right) \\ &s.t. \quad C_t + K_{t+1} = Y_t \\ &\qquad\;\; Y_t = Z_t K_t^{1-\alpha} \end{aligned} \tag{7.1}$$

Where Z_t is a technological shock that we will specify later. Necessary conditions for an optimal solution is that

$$\begin{aligned} &\frac{1}{C_t} = \Lambda_t \\ &\Lambda_t = (1-\alpha)\beta E_t \Lambda_{t+1} Z_{t+1} K_{t+1}^{-\alpha} \\ &C_t + K_{t+1} = Z_t K_t^{1-\alpha} \end{aligned} \tag{7.2}$$

where Λ is a shadow value.

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* The approach follows McCallum, 1989, in *Modern Business Cycle Theory*.

a) Write down a verbal interpretation of each of the necessary conditions.

Now we will try to find one of the possibly infinite solutions and hope that one satisfies the transversality condition. If so, the solution also satisfies sufficient optimality conditions. First we start by guessing that consumption is proportional to production, with proportionality Π so that

$$C_t = \Pi Z_t K_t^{1-\alpha} \tag{7.3}$$

b) Find K_{t+1} as a function of Π, Z_t and K_t.

c) Show that if our guess is correct

$$\Pi = 1 - (1-\alpha)\beta \tag{7.4}$$

Hint. Use first the optimality condition to eliminate Λ in the second. Then use the guess and your answer to b).

From now on we use lower case letters to indicate logs and we assume that the log of the technology shock is white noise. We can now establish that

$$\begin{aligned} k_{t+1} &= \pi_k + (1-\alpha)k_t + z_t \\ c_t &= \pi_c + (1-\alpha)c_{t-1} + z_t \\ y_t &= \pi_y + (1-\alpha)y_{t-1} + z_t \end{aligned} \tag{7.5}$$

where

$$\begin{aligned} \pi_k &= \log(1-\Pi) \\ \pi_c &= \log \Pi \\ \pi_y &= (1-\alpha)\log \pi_k \end{aligned} \tag{7.6}$$

d) Is the solution satisfying the transversality condition that

$$\lim_{j\to\infty} E_t \beta^j \Lambda_{t+j} K_{t+j+1} = 0\,? \tag{7.7}$$

e) Using (7.5) it is possible to show that this very simple model gives us perfectly correlated series for k, c and y. Can you think of a way to break this?

Calibration

Take the following parameter as given from "usually reliable sources" $\beta = 0.95$. We now have two parameters left; the variance of z_t, defined as σ_z^2 and α. These are going to be chosen so as to match the variance and autocorrelation of consumption.

a) Detrend consumption data by regressing the log of c_t on a constant and a time trend. Define the residuals as the business cycle component. Calculate the variance and autocorrelation of the consumption business cycle, σ_c^2.

b) Use a computer to simulate the model. Generate a series of normally distributed shocks of suitable length. Generate simulated series for a few values of σ_z^2 in the range from $\sigma_c^2/5$ to σ_c^2 and for α in the range 0.1 to 0.7. Include at least the four combinations of endpoints. Set the first observations on simulated c to π_c / α. Then choose the set of parameters that gives you the best match between simulated and sample moments. This is the end of the calibration.

Simulation

Now we have all the necessary parameters so we can see if the model is able to generate simulated series with statistical properties similar to data.

a) Simulate the model. Plot the result including the shocks. Also plot the business cycle of c_t. See if you can find some similarities and differences.

c) Compute the autocorrelation for 1,2, and 3 lags in the sample and in the simulated data.

d) Discuss the model's performance and what you think could be done to improve it.