Macro I Gothenburg

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Linearization

- Cannot solve model with $\delta < 1$ or most other specifications analytically, A way to go – linearize optimality conditions around steady state. Find equilibrium law-of-motion, i.e., choices as functions of state variables Z_t and K_t (in deviations from steady state).
- Need to specify shock process. Assume $\hat{Z}_t = \rho \hat{Z}_{t-1} + \varepsilon_t$ where $\hat{Z}_t \equiv \frac{Z_t - \bar{Z}}{\bar{Z}}, \varepsilon_t$,i.i.d. and $\bar{Z}$ is the average value of Z.
- Why first-order Markov? Why stationary?
- Markov so expectations about future realizations only depend on current shock. At some cost of complication, this could be relaxed, but it doesn't seem necessary empirically.

Non-stochastic steady state

- We first have to find the non-stochastic steady states. Going to linearize around it. Use three equations.
  1. The Euler equation is $1 = \beta E_t \left(\frac{C_t}{C_{t+1}} \right) R_{t+1}$ so in a non-stochastic steady state $1 = \beta \left(F_k \left(K_s, 1 - L_s, \bar{Z} \right) + 1 - \delta \right)$.
  2. Resource constraint $C_s + K_s = F \left(K_s, 1 - L_s, \bar{Z} \right) + (1 - \delta) K_s$
  3. The intratemporal condition in steady state

$$\frac{U_L(F\left(K_s, 1 - L_s, \bar{Z}\right) - \delta K_s, L_s)}{U_C\left(F\left(K_s, 1 - L_s, \bar{Z}\right) - \delta K_s\right)} = F_N\left(K_s, 1 - L_s, \bar{Z}\right)$$

- Gives steady state of C_s, K_s, L_s (uniquely under standard conditions).

Steady State in Cobb-Douglas, log utility case

- Set $\bar{Z} = 1$ and $U = \ln C + \phi \ln L$
  1. Intertemporal $\beta \left(\alpha \left(\frac{1-L_s}{K_s} \right)^{1-\alpha} + 1 - \delta \right) = 1$
  2. Resource constraint $C_s + K_s = K_s^{\alpha} \left(1 - L_s\right)^{1-\alpha} + (1-\delta) K_s$
  3. $\frac{\phi C_s}{L_s} = (1-\alpha) \left(\frac{K_s}{1-L_s} \right)^{\alpha}$
- This can be solved analytically.

Euler equation

- The Euler equation is $0 = \beta E_t U_C\left(C_{t+1}, L_{t+1}\right) R_{t+1} - U_C\left(C_t, L_t\right)$
- Now use the resource constraint to eliminate consumption $C_t = Y_t + (1-\delta) K_t - K_{t+1}$.
- For example, if $U = \ln C_t + \phi \ln L_t$ and $Y_t = Z_t K_t^{\alpha} \left(1 - L_t\right)^{1-\alpha}$ The Euler equation is

$$0 = E_t \beta \frac{1}{C_{t+1}} \left(\alpha Z_{t+1} K_{t+1}^{\alpha-1} \left(1 - L_{t+1}\right)^{1-\alpha} + (1-\delta)\right) - \frac{1}{C_t}$$

- Replace C_t and C_{t+1} by use of the resource constraint

$$\begin{aligned} 0 &= E_t \beta \frac{\alpha Z_{t+1} K_{t+1}^{\alpha-1} \left(1 - L_{t+1}\right)^{1-\alpha} + (1-\delta)}{Z_{t+1} K_{t+1}^{\alpha} \left(1 - L_{t+1}\right)^{1-\alpha} + (1-\delta) K_{t+1} - K_{t+2}} \\ &\quad - \frac{1}{Z_t K_t^{\alpha} \left(1 - L_t\right)^{1-\alpha} + (1-\delta) K_t - K_{t+1}} \end{aligned}$$

- Thus, we can write the Euler equation as

$$E_t v^K \left(K_{t+2}, K_{t+1}, K_t, L_{t+1}, L_t, Z_{t+1}, Z_t\right) = 0$$

Leisure condition

- Now, do the same with the intratemporal condition

$$0 = F_N\left(K_{t,}1 - L_t, Z_t\right) U_C\left(C_t, L_t\right) - U_L\left(C_t, L_t\right)$$

- Replacing C_t by $Y_t + (1-\delta)K_t - K_{t+1}$ this condition depends on Z_t, K_t and K_{t+1}.We can then write

$$v^L\left(K_{t+1}, K_t, L_t, Z_t\right) = 0$$

- In the example, this is

$$0 = \frac{(1-\alpha) Z_t K_t^{\alpha}\left(1 - L_t\right)^{-\alpha}}{Z_t K_t^{\alpha}\left(1 - L_t\right)^{1-\alpha} + (1-\delta) K_t - K_{t+1}} - \frac{\phi}{L_t}$$

Linearization

- Now use a linear approximization of the two optimality condition around the steady state

$$\begin{aligned}
&E_t v^K \left(K_{t+2}, K_{t+1}, K_t, L_{t+1}, L_t, Z_{t+1}, Z_t\right) \\
&\approx v_1^K E_t \left(K_{t+2} - K_S\right) + v_2^K \left(K_{t+1} - K_S\right) + v_3^K \left(K_t - K_S\right) \\
&+ v_4^K E_t \left(L_{t+1} - L_S\right) + v_5^K \left(L_t - L_S\right) \\
&+ v_6^K E_t \left(Z_{t+1} - 1\right) + v_7^K \left(Z_t - 1\right)
\end{aligned}$$

- Note that the $v_i^{K\prime} s$ are derivatives evaluated at the known steady states, i.e., known numbers (given parameters)!
- Similarily,

$$\begin{aligned}
v^L \left(K_{t+1}, K_t, L_t, Z_t\right) \approx & v_1^L \left(K_{t+1} - K_S\right) + v_2^L \left(K_t - K_S\right) \\
& + v_3^L \left(L_t - L_S\right) + v_4^L \left(Z_t - 1\right)
\end{aligned}$$

Example:1

- With the specifications of preferences and production function used above,

$$
\begin{aligned}
v_1^K &= \left[\frac{\partial\left(\beta\frac{\alpha K_s^{\alpha-1}(1-L_s)^{1-\alpha}+(1-\delta)}{K_s^{\alpha}(1-L_s)^{1-\alpha}+(1-\delta)K_s-K_{t+2}}\right)}{\partial K_{t+2}}\right]_{K_{t+2}=K_s} \\
&= \left(\beta\frac{\alpha K_s^{\alpha-1}\left(1-L_s\right)^{1-\alpha}+\left(1-\delta\right)}{\left(K_s^{\alpha}\left(1-L_s\right)^{1-\alpha}+\left(1-\delta\right)K_s-K_s\right)^2}\right),
\end{aligned}
$$

Example:2

and

$$
\begin{aligned}
v_1^L &= \left[\left(\frac{\partial\left(\frac{(1-\alpha)K_s^{\alpha}(1-L_s)^{-\alpha}}{K_s^{\alpha}(1-L_s)^{1-\alpha}+(1-\delta)K_s-K_{t+1}}\right)}{\partial K_{t+1}}\right)\right]_{K_{t+1}=K_s} \\
&= \frac{(1-\alpha)\,K_s^{\alpha}\,(1-L_s)^{-\alpha}}{\left(K_s^{\alpha}\,(1-L_s)^{1-\alpha}+(1-\delta)\,K_s-K_s\right)^2}
\end{aligned}
$$

Vector notation

- Define relative deviations from steady states, as $\hat{K}_t \equiv \frac{K_t - K_s}{K_s}$, $\hat{L}_t \equiv \frac{L_t - L_s}{L_s}$, $\hat{Z}_t \equiv Z_t - 1$ and stack the endogenous variables in a vector

$$X_t \equiv \begin{bmatrix} \hat{K}_{t+1} \\ \hat{L}_t \end{bmatrix}.$$

- We can then write the two optimality conditions as $E_t \left[\alpha_0 X_{t+1} + \alpha_1 X_t + \alpha_2 X_{t-1} + \beta_0 \hat{Z}_{t+1} + \beta_1 \hat{Z}_t\right] = 0$ with

$$\alpha_0 = \begin{bmatrix} v_1^K K_s & v_4^K L_s \\ 0 & 0 \end{bmatrix}, \alpha_1 = \begin{bmatrix} v_2^K K_s & v_5^K L_s \\ v_1^L K_s & v_3^L L_s \end{bmatrix}$$

$$\alpha_2 = \begin{bmatrix} v_3^K K_s & 0 \\ v_2^L K_s & 0 \end{bmatrix}, \beta_0 = \begin{bmatrix} v_6^K \\ 0 \end{bmatrix}, \beta_1 = \begin{bmatrix} v_7^K \\ v_4^L \end{bmatrix}$$

- Note again, that all elements in the $\alpha' s$ are numbers that we know (after specifying parameters of utility and production).

Decision rules

- Finally, we conjecture that a linear decision rule solves the linearized optimality conditions. That is, we postulate

$$\begin{bmatrix} \hat{K}_{t+1} \\ \hat{L}_t \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \hat{K}_t \\ \hat{L}_{t-1} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \hat{Z}_t$$

- Clearly, we don't expect the rule to depend on $\hat{L}_{t-1}$ so we set $a_{12} = a_{22} = 0$. We verify our conjecture by finding the a's and b's that solve the optimality conditions. Let $A \equiv \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix}$ and $B \equiv \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.
- Then, since $X_t \equiv \begin{bmatrix} \hat{K}_{t+1} \\ \hat{L}_t \end{bmatrix}$, we can write $X_t = AX_{t-1} + B\hat{Z}_t$ and $X_{t+1} = A^2 X_{t-1} + AB\hat{Z}_t + B\hat{Z}_{t+1}$

Use undetermined decision rules in optimality condition

- Recall the optimality conditions $E_t\left[\alpha_0 X_{t+1} + \alpha_1 X_t + \alpha_2 X_{t-1} + \beta_0 \hat{Z}_{t+1} + \beta_1 \hat{Z}_t\right] = 0$ where the $\alpha' s$ and $\beta' s$ contain the known derivatives at non-stochastic steady state and use the yet unknown decision rule.

$$
\begin{aligned}
0 &= E_t\left[\alpha_0\left(A^2 X_{t-1} + AB\hat{Z}_t + B\hat{Z}_{t+1}\right) + \beta_0 \hat{Z}_{t+1}\right] \\
&\quad + \alpha_1\left(AX_{t-1} + B\hat{Z}_t\right) + \alpha_2 X_{t-1} + \beta_1 \hat{Z}_t
\end{aligned}
$$

- Use $\hat{Z}_t = \rho\hat{Z}_{t-1} + \varepsilon_t$,implying $E_t\hat{Z}_{t+1} = \rho\hat{Z}_t$ to get $0 = \left(\alpha_0 A^2 + \alpha_1 A + \alpha_2\right) X_{t-1} + \left(\alpha_0 B\left(A + \rho\right) + \alpha_1 B + \beta_0\rho + \beta_1\right)\hat{Z}_t$
- If we can find A and B so that this is true for all X_{t-1} and $\hat{Z}_t$ we have a solution. Stability? Multiplicity?

Solving undetermined coefficients

- Need
$\left(\alpha_0 A^2+\alpha_1 A+\alpha_2\right) X_{t-1}+\left(\alpha_0 B\left(A+\rho\right)+\alpha_1 B+\beta_0\rho+\beta_1\right)\hat{Z}_t=0$
for all $X_{t-1},\hat{Z}_t$ requires
  - $\left(\alpha_0 A^2+\alpha_1 A+\alpha_2\right)=0$
  - $\left(\alpha_0 B\left(A+\rho\right)+\alpha_1 B+\beta_0\rho+\beta_1\right)=0$
- Write out first equation,

$$
\begin{bmatrix} v_1^K K_s & v_4^K L_s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11}^2 & 0 \\ a_{21}a_{11} & 0 \end{bmatrix}
+ \begin{bmatrix} v_2^K K_s & v_5^K L_s \\ v_1^L K_s & v_3^L L_s \end{bmatrix} \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} + \begin{bmatrix} v_3^K K_s & 0 \\ v_2^L K_s & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

- Two equations in two unknowns.

Solution

- The equations on the previous page contained the quadratic term a_{11}^2. This means that there will be two sets of solutions.
- Here, they are I got, $a_{11} = -\frac{v_3^L L_s \xi + v_2^L K_s}{v_1^L K_s}$, $a_{21} = \xi$,where ξ is a root of the quadratic equation

$$
\begin{aligned}
0 &= \left(v_1^K K_s v_3^L L_s - v_4^K L_s v_1^L K_s \right) \xi^2 \\
&\quad + \left(v_3^L L_s v_2^K K_s - v_4^K L_s v_2^L K_s - v_5^K L_s v_1^L K_s \right) \xi \\
&\quad + v_3^K K_s v_3^L L_s - v_5^K L_s v_2^L K_s
\end{aligned}
$$

- One root is (should be) explosive. Recall what we did with savings.

Compare to data

- Doing the same for B, we have a system we can simulate

$$\begin{bmatrix} \hat{K}_{t+1} \\ \hat{L}_t \end{bmatrix} = \begin{bmatrix} -\frac{v_3^L L_s \xi + v_2^L K_s}{v_1^L K_s} & 0 \\ \xi & 0 \end{bmatrix} \begin{bmatrix} \hat{K}_t \\ \hat{L}_{t-1} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \hat{Z}_t$$

  for different values of the parameters (elastity of intertemporal substitution and labor supply, discount rate, depreciation, production function, shock persistence and depreciation + possibly others).
- Calibrate and compare (Test?)

Summary

1. Write optimality condition in terms of state variables, choice variables and shocks

$$E_t v^K \left(K_{t+2}, K_{t+1}, K_t, L_{t+1}, L_t, Z_{t+1}, Z_t\right)$$

2. Linearize around steady state

$$E_t v^K \approx v_1^K E_t \left(K_{t+2} - K_S\right) + v_2^K \left(K_{t+1} - K_S\right) + v_3^K \left(K_t - K_S\right)$$
$$+ v_4^K E_t \left(L_{t+1} - L_S\right) + v_5^K \left(L_t - L_S\right) + v_6^K E_t \left(Z_{t+1} - 1\right) + v_7^K \left(Z_t - 1\right)$$

3. Conjecture a linear decision rule for undetermined coefficients

$$\begin{bmatrix} \hat{K}_{t+1} \\ \hat{L}_t \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} \hat{K}_t \\ \hat{L}_{t-1} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \hat{Z}_t$$

4. Find coefficients in decision rule so that 2. is satisfied. Disregard explosive root.
5. Calibrate and compare to data.