Macro II

John Hassler

Spring 2020

Solving the model

- What does solving the model mean?
- With i) log consumption utility, ii) Cobb-Douglas production, and iii) full depreciation, we can solve the model analytically.
- Set $U(C_t, L_t) = \ln C_t + v(L_t)$ and $K_{t+1} + C_t = Z_t K_t^{\alpha} (1 - L_t)^{1-\alpha}$.
- Defining the savings ratio $s_t \equiv K_{t+1} / \left(Z_t K_t^{\alpha} (1 - L_t)^{1-\alpha}\right)$, we have $C_t = (1 - s_t) Z_t K_t^{\alpha} (1 - L_t)^{1-\alpha}$.

Euler equation in terms of savings

- The Euler equation is $1 =$
  $= \beta E_t \left(\frac{C_t}{C_{t+1}} \right) R_{t+1}$
- $= \beta E_t \left(\frac{(1-s_t) Z_t K_t^{\alpha} (1-L_t)^{1-\alpha}}{(1-s_{t+1}) Z_{t+1} K_{t+1}^{\alpha} (1-L_{t+1})^{1-\alpha}} \right) Z_{t+1} \alpha K_{t+1}^{\alpha-1} \left(1 - L_{t+1}\right)^{1-\alpha}$
- $= \beta E_t \left(\frac{1-s_t}{1-s_{t+1}} Z_t K_t^{\alpha} \left(1 - L_t\right)^{1-\alpha} \right) \alpha \frac{1}{K_{t+1}}$
- $= \beta E_t \left(\frac{1-s_t}{1-s_{t+1}} Z_t K_t^{\alpha} \left(1 - L_t\right)^{1-\alpha} \right) \alpha \frac{1}{s_t Z_t K_t^{\alpha} (1-L_t)^{1-\alpha}}$
- $= \beta E_t \left(\frac{1-s_t}{1-s_{t+1}} \right) \alpha \frac{1}{s_t}$

Solving the model:2

- $\beta E_t \left(\frac{1-s_t}{1-s_{t+1}} \right) \alpha \frac{1}{s_t} = 1$ is a non-linear difference equation. It has a non-stochastic solution at $s_t = \alpha\beta \; \forall t$, i.e., a steady state.
- What about other solutions? Solving $\beta \left(\frac{1-s_t}{1-s_{t+1}} \right) \alpha \frac{1}{s_t} = 1$ for s_{t+1} yields $s_{t+1} = 1 - \frac{\alpha\beta(1-s_t)}{s_t}$ with a derivative $\frac{ds_{t+1}}{ds_t} = \frac{\alpha\beta}{s_t^2}$.
- Linearize the difference equation around the steady state yields $\hat{s}_{t+1} = \frac{1}{\alpha\beta}\hat{s}_t$ where $\hat{s}_t \equiv s_t - \alpha\beta$,i.e., the deviation from the steady state. Thus we see that that any initial deviation from the steady state explodes over time. Cannot be optimal! If $s_t > \alpha\beta$, the savings rate converges to 1 over time. Leads to overaccumulation of capital.
- Formally: use concavity of problem. Show that only $s_t = \alpha\beta \; \forall t$ satisfy transversality condition $\lim_{T\to\infty} \beta^T U_C K_{T+1} = 0$. Check growth courses for more on this.

- The intratemporal FOC says

$$
\begin{aligned}
U_L\left(C_t, L_t\right) &= U_C\left(C_t, L_t\right) w_t \\
v'\left(L_t\right) &= \frac{Z_t\left(1-\alpha\right) K_t^{\alpha}\left(1-L_t\right)^{1-\alpha}}{\left(1-L_t\right) C_t} \\
v'\left(L_t\right) &= \frac{Z_t\left(1-\alpha\right) K_t^{\alpha}\left(1-L_t\right)^{1-\alpha}}{\left(1-L_t\right)\left(1-s\right) Z_t K_t^{\alpha}\left(1-L_t\right)^{1-\alpha}} \\
\rightarrow v'\left(L_t\right)\left(1-L_t\right) &= \frac{1-\alpha}{1-s}
\end{aligned}
$$

- So, labor supply is constant. If, for example, $v\left(L_t\right) = \phi \ln L_t$, we get $L_t = \frac{1}{1+\frac{1-\alpha}{\phi(1-\alpha\beta)}} \forall t$. This is not particularly useful, right?

No propagation

Why is there no dependence on current and expected productivity?

1. A change in next periods expected productivity changes the return to saving between t and $t+1$ and next periods marginal utility in opposite directions.
2. With log consumption utility and full depreciation the effects exactly cancel. Future does not matter for current consumption. Z_{t+1} cancels.
3. Similarly, a shock today, Z_t increases the wage and consumption proportionally since the savings rate is constant.The ratio of wages and marginal utility is thus not affected and marginal utility of leisure does not need to be changed.
4. Seems fine for long-run changes, but not for business cycle.
5. A fix: relax full depreciation. Intuitively, an additional resource in the budget constraint (stock of non-depreciated capital) that does not change one-for-one with productivity makes consumption respond less than one-for-one with wage. Income effcts are smaller. Income and substitution effects don't cancel.

Linearization

- Cannot solve model with $\delta < 1$ analytically. A way to go – linearize model around steady state. Find equilibrium law-of-motion, i.e., choices as functions of state variables Z_t and K_t (in deviations from steady state).
- Strategy;
  1. Find non-stochastic steady state.
  2. Linearize optimality conditions around steady state.
  3. Guess that decision rules are linear functions of state variables (K_t and Z_t).
  4. Verify by finding coefficients in decision rules such that linearized optimality conditions are satisfied.
- Now done by software packages like DYNARE.
- Need to specify shock process. Assume $\hat{Z}_t = \rho \hat{Z}_{t-1} + \varepsilon_t$ where $\hat{Z}_t \equiv \frac{Z_t - \bar{Z}}{\bar{Z}}$, ε_t,i.i.d. and $\bar{Z}$ is the average value of Z.

Non-stochastic steady state

- We first have to find the non-stochastic steady states. Going to linearize around it. Use three equations.
  1. The Euler equation is $1 = \beta E_t \left(\frac{U_C(t+1)}{U_C(t)} \right) R_{t+1}$ so in a non-stochastic steady state $1 = \beta \left(F_k \left(K_s, \bar{Z} \left(1 - L_s\right)\right) + 1 - \delta \right)$.
  2. Resource constraint $C_s + K_s = F\left(K_s, \bar{Z}\left(1 - L_s\right)\right) + \left(1 - \delta\right) K_s$
  3. The intratemporal condition in steady state

$$\frac{U_L\left(F\left(K_s, \bar{Z}\left(1 - L_s\right)\right) - \delta K_s, L_s\right)}{U_C\left(F\left(K_s, \bar{Z}\left(1 - L_s\right)\right) - \delta K_s, L_s\right)} = \bar{Z} F_N\left(K_s, \bar{Z}\left(1 - L_s\right)\right)$$

- Gives steady state of C_s, K_s, L_s.

Steady State in Cobb-Douglas, log utility case

- Set $\bar{Z} = 1$ and $U = \ln C + \phi \ln L$
  1. Intertemporal $\beta \left(\alpha \left(\frac{1-L_s}{K_s} \right)^{1-\alpha} + 1 - \delta \right) = 1$
  2. Resource constraint $C_s + K_s = K_s^{\alpha} \left(1 - L_s\right)^{1-\alpha} + \left(1 - \delta\right) K_s$
  3. $\frac{\phi C_s}{L_s} = (1-\alpha) \left(\frac{K_s}{1-L_s} \right)^{\alpha}$
- Three equations in three unknowns that can easily be solved (even analytically).

Euler and Leisure

- The Euler equation is $0 = \beta E_t \left[U_C \left(C_{t+1}, L_{t+1} \right) R_{t+1} \right] - U_C \left(C_t, L_t \right)$
- Now use the resource constraint to eliminate consumption $C_t = F\left(K_{t,} Z_t \left(1 - L_t\right)\right) + \left(1 - \delta\right) K_t - K_{t+1}$ implying that the the Euler equation depends on $K_{t+2}, K_{t+1}, K_t, Z_{t+1}, Z_t$ as well as on L_{t+1} and L_t.

$$\beta E_t \left[U_C \left(F\left(K_{t+1,} Z_{t+1} \left(1 - L_{t+1}\right)\right) + \left(1 - \delta\right) K_{t+1} - K_{t+2}, L_{t+1} \right) R_t \right.$$
$$- U_C \left(F\left(K_{t,} Z_t \left(1 - L_t\right)\right) + \left(1 - \delta\right) K_t - K_{t+1}, L_t \right)$$

- Thus, we can write the Euler equation as

$$E_t v^K \left(K_{t+2}, K_{t+1}, K_t, L_{t+1}, L_t, Z_{t+1}, Z_t \right) = 0$$

- Now, do the same with the intratemporal condition.$0 = Z_t F_N \left(K_{t,} 1 - L_t\right) U_C \left(C_t, L_t\right) - U_L \left(C_t, L_t\right)$ which after substitution depends on Z_t, K_t and K_{t+1} giving

$$v^L \left(K_{t+1}, K_t, L_t, Z_t \right) = 0$$

Linearization

- Now use a linear approximization of the two optimality condition around the steady state

$$
\begin{aligned}
&E_t v^K\left(K_{t+2}, K_{t+1}, K_t, L_{t+1}, L_t, Z_{t+1}, Z_t\right) \\
&\approx v_1^K E_t\left(K_{t+2}-K_S\right)+v_2^K\left(K_{t+1}-K_S\right)+v_3^K\left(K_t-K_S\right) \\
&+v_4^K E_t\left(L_{t+1}-L_S\right)+v_5^K\left(L_t-L_S\right) \\
&+v_6^K E_t\left(Z_{t+1}-1\right)+v_7^K\left(Z_t-1\right)
\end{aligned}
$$

- Note that the $v_i^{K\prime}s$ are derivatives evaluated at the known steady states, i.e., known numbers (given parameters)!
- Similarily,

$$
\begin{aligned}
v^L\left(K_{t+1}, K_t, L_t, Z_t\right) \approx & v_1^L\left(K_{t+1}-K_S\right)+v_2^L\left(K_t-K_S\right) \\
& +v_3^L\left(L_t-L_S\right)+v_4^L\left(Z_t-1\right)
\end{aligned}
$$

Vector notation

- Define relative deviations from steady states, as $\hat{K}_t \equiv \frac{K_t - K_s}{K_s}$, $\hat{L}_t \equiv \frac{L_t - L_s}{L_s}$, $\hat{Z}_t \equiv Z_t - 1$ and stack the endogenous variables in a vector

$$X_t \equiv \begin{bmatrix} \hat{K}_{t+1} \\ \hat{L}_t \end{bmatrix}.$$

- We can then write the two optimality conditions as $E_t \left[\alpha_0 X_{t+1} + \alpha_1 X_t + \alpha_2 X_{t-1} + \beta_0 \hat{Z}_{t+1} + \beta_1 \hat{Z}_t \right] = 0$ with

$$\alpha_0 = \begin{bmatrix} v_1^K K_s & v_4^K L_s \\ 0 & 0 \end{bmatrix}, \alpha_1 = \begin{bmatrix} v_2^K K_s & v_5^K L_s \\ v_1^L K_s & v_3^L L_s \end{bmatrix}$$

$$\alpha_2 = \begin{bmatrix} v_3^K K_s & 0 \\ v_2^L K_s & 0 \end{bmatrix}, \beta_0 = \begin{bmatrix} v_6^K \\ 0 \end{bmatrix}, \beta_1 = \begin{bmatrix} v_7^K \\ v_4^L \end{bmatrix}$$

- Note again, that all elements in the $\alpha' s$ are numbers that we know (after specifying parameters of utility and production).

Decision rules

- Finally, we conjecture that a linear decision rule solves the linearized optimality conditions. That is, we postulate

$$\begin{bmatrix} \hat{K}_{t+1} \\ \hat{L}_t \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \hat{K}_t \\ \hat{L}_{t-1} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \hat{Z}_t$$

- Clearly, we don't expect the rule to depend on $\hat{L}_{t-1}$ so we set $a_{12} = a_{22} = 0$. We verify our conjecture by finding the a's and b's that solve the optimality conditions. Let $A \equiv \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix}$ and $B \equiv \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.
- Then, since $X_t \equiv \begin{bmatrix} \hat{K}_{t+1} \\ \hat{L}_t \end{bmatrix}$, we can write $X_t = AX_{t-1} + B\hat{Z}_t$ and $X_{t+1} = A^2 X_{t-1} + AB\hat{Z}_t + B\hat{Z}_{t+1}$

Use undetermined decision rules in optimality condition

- Recall the optimality conditions $E_t\left[\alpha_0 X_{t+1} + \alpha_1 X_t + \alpha_2 X_{t-1} + \beta_0 \hat{Z}_{t+1} + \beta_1 \hat{Z}_t\right] = 0$ where the $\alpha' s$ and $\beta' s$ contain the known derivatives at the non-stochastic steady state. Use the yet unknown decision rule in these.

$$\begin{aligned} 0 &= E_t\left[\alpha_0\left(A^2 X_{t-1} + AB\hat{Z}_t + B\hat{Z}_{t+1}\right) + \beta_0 \hat{Z}_{t+1}\right] \\ &\quad + \alpha_1\left(AX_{t-1} + B\hat{Z}_t\right) + \alpha_2 X_{t-1} + \beta_1 \hat{Z}_t \end{aligned}$$

- Use $\hat{Z}_t = \rho\hat{Z}_{t-1} + \varepsilon_t$,implying $E_t\hat{Z}_{t+1} = \rho\hat{Z}_t$ to get $0 = \left(\alpha_0 A^2 + \alpha_1 A + \alpha_2\right) X_{t-1} + \left(\alpha_0 B\left(A + \rho\right) + \alpha_1 B + \beta_0\rho + \beta_1\right)\hat{Z}_t$
- If we can find A and B so that this is true for all X_{t-1} and $\hat{Z}_t$ we have a solution. Stability? Multiplicity?

Solving undetermined coefficients

- Need
  $\left(\alpha_0 A^2 + \alpha_1 A + \alpha_2\right) X_{t-1} + \left(\alpha_0 B\left(A+\rho\right) + \alpha_1 B + \beta_0\rho + \beta_1\right)\hat{Z}_t = 0$
  for all $X_{t-1}, \hat{Z}_t$ requires
  - $\left(\alpha_0 A^2 + \alpha_1 A + \alpha_2\right) = 0$
  - $\left(\alpha_0 B\left(A+\rho\right) + \alpha_1 B + \beta_0\rho + \beta_1\right) = 0$

Write out first equation,

$$\begin{bmatrix} v_1^K K_s & v_4^K L_s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11}^2 & 0 \\ a_{21} a_{11} & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} v_2^K K_s & v_5^K L_s \\ v_1^L K_s & v_3^L L_s \end{bmatrix} \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} + \begin{bmatrix} v_3^K K_s & 0 \\ v_2^L K_s & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} v_1^K K_s a_{11}^2 + v_4^K L_s a_{21} a_{11} + v_2^K K_s a_{11} + v_5^K L_s a_{21} + v_3^K K_s &= 0 \\ v_1^L K_s a_{11} + v_3^L L_s a_{21} + v_2^L K_s &= 0 \end{aligned}$$

- Two equations in two unknowns a_{21} and a_{11}.

Solution

- The equations on the previous page contained the quadratic term a_{11}^2. This means that there will be two sets of solutions.
- Here, they are, $a_{11} = -\frac{v_3^L L_s \xi + v_2^L K_s}{v_1^L K_s}$, $a_{21} = \xi$,where ξ is a root of the quadratic equation

$$
\begin{aligned}
0 &= \left(v_1^K K_s v_3^L L_s - v_4^K L_s v_1^L K_s \right) \xi^2 \\
&\quad + \left(v_3^L L_s v_2^K K_s - v_4^K L_s v_2^L K_s - v_5^K L_s v_1^L K_s \right) \xi \\
&\quad + v_3^K K_s v_3^L L_s - v_5^K L_s v_2^L K_s
\end{aligned}
$$

- One root is (should be) explosive. Recall what we did with savings.

Compare to data

- Doing the same for B, we have a system we can simulate

$$\begin{bmatrix} \hat{K}_{t+1} \\ \hat{L}_t \end{bmatrix} = \begin{bmatrix} -\frac{v_3^L L_s \xi + v_2^L K_s}{v_1^L K_s} & 0 \\ \xi & 0 \end{bmatrix} \begin{bmatrix} \hat{K}_t \\ \hat{L}_{t-1} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \hat{Z}_t$$

  for different values of the parameters (elastity of intertemporal substitution and labor supply, discount rate, depreciation, production function, shock persistence and depreciation + possibly others).

- Calibrate and compare (Test?)

Summary

1. Write optimality condition in terms of state variables, choice variables and shocks

$$E_t v^K \left(K_{t+2}, K_{t+1}, K_t, L_{t+1}, L_t, Z_{t+1}, Z_t\right)$$

2. Linearize around steady state

$$E_t v^K \approx v_1^K E_t \left(K_{t+2} - K_S\right) + v_2^K \left(K_{t+1} - K_S\right) + v_3^K \left(K_t - K_S\right)$$
$$+ v_4^K E_t \left(L_{t+1} - L_S\right) + v_5^K \left(L_t - L_S\right) + v_6^K E_t \left(Z_{t+1} - 1\right) + v_7^K \left(Z_t - 1\right)$$

3. Conjecture a linear decision rule for undetermined coefficients

$$\begin{bmatrix} \hat{K}_{t+1} \\ \hat{L}_t \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} \hat{K}_t \\ \hat{L}_{t-1} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \hat{Z}_t$$

4. Find coefficients in decision rule so that 2. is satisfied. Disregard explosive root.
5. Calibrate and compare to data.