MATFUII

Class Notes

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1. Introduction

This course is about dynamic systems, i.e., systems that evolve over time. The first task will be to simply describe how the system evolves, given some law of motion. Let us take a simple example.

1.1. A Simple Differential Equation

Consider Patrik Sjöberg when he jumps. Define the height he has over the ground at time t as $s(t)$. When he is in the air the gravity affects him so he accelerates towards the ground. This gives us a very simple second order differential equation:

$$\begin{aligned} &\ddot{s}(t) = g, \\ &g \equiv -9.81\frac{m}{s^2} \end{aligned} \tag{1.1}$$

We could immediately see that the following function satisfies the differential equation

$$s(t) = \frac{gt^2}{2} + c_1 t + c_2 \tag{1.2}$$

for any constants c_1 and c_2.

It is clear that in order to now where Patrik is at some time t we need to know more. For example, if we know that he leaves the ground at time 0 and at an upward speed of 5 *m/s*, this should be enough to know where he is at all times. So we have

$$\begin{aligned} &\dot{s}(0) = g \cdot 0 + c_1 = 5 \Rightarrow c_1 = 5 \\ &s(0) = \frac{g \cdot 0^2}{2} + c_1 0 + c_2 = 0 \Rightarrow c_2 = 0 \end{aligned} \tag{1.3}$$

So Patriks height over the ground is given by

$$s(t) = \frac{gt^2}{2} + 5t \tag{1.4}$$

Note the units.

(1.4) is the solution to the problem. It can be used to, for example, calculating how high he will jump.

$$\begin{aligned} &\max_t s(t) = \frac{gt^2}{2} + 5t \\ &FOC;\ gt + 5 = 0 \Rightarrow t = \frac{-5}{g} \approx 0.5 \\ &\Rightarrow \max_t s(t) \approx \frac{g \cdot 0.5^2}{2} + 5 \cdot 0.5 \approx 1.25m \end{aligned} \tag{1.5}$$

Solving problems of this type, in both continuous and discrete time, is the first topic of the course.

When we have learnt how to describe how a system evolves, we want to know how to optimize in some dimension given the restrictions given by the law of motion of the system. This is the topic of the second part of the course.

1.2. Approaches to Dynamic Optimization

The issue is to choose from a set of admissible paths or functions $x(t)$ for $t \in [t_0,T]$ the one that maximizes a given objective *functional* which associates a particular value to each admissible path $V[x(t)]$.

Example: resource (oil) extraction. Choose an extraction plan $x(t)$ stating how much is left in the well. To be admissible $x(t)$ has to satisfy:

$$\begin{aligned} &x(0) = K, \\ &\dot{x}(t) \le 0, \forall t, \\ &x(T) \ge 0._t \end{aligned} \tag{1.6}$$

The value (or objective) function associates a number to each path. For example;

$$\begin{aligned} V[x(t)] &= \int_0^T e^{-rt}\left(p(\dot{x}(t)) - c\left(x(t),\dot{x}(t)\right)\right)dt \\ &= \int_0^T F\left(t, x(t), \dot{x}(t)\right)dt. \end{aligned} \tag{1.7}$$

Three approaches in this course:

1. Calculus of Variation (Newton, Bernouilli)

$$\begin{aligned} &\max_{x(t)} \int_0^T F\left(t, x(t), \dot{x}(t)\right)dt \\ &s.t.\ x(t) \text{ is admissible.} \end{aligned} \tag{1.8}$$

2. Optimal Control (Pontryagin)

$$\begin{aligned} &\max_{u(t)} \int_0^T F\left(t, x(t), u(t)\right)dt \\ &s.t. \\ &\dot{x}(t) = g\left(t, x(t), u(t)\right) \\ &x(0) = A, \\ &x(T) = Z, \\ &u(t) \text{ admissible for all } t. \end{aligned} \tag{1.9}$$

3. Dynamic Programming (Bellman).

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\begin{aligned}
&V(\underline{x}) = \max \sum_{t=1}^{T} F(t, x_t, u_t) \\
&s.t. \\
&x_{t+1} = g(t, x_t, u_t) \\
&x_1 = \underline{x}, \\
&x_T = \overline{x}.
\end{aligned}
\tag{1.10}
$$

Example:

$$
\begin{aligned}
&\max_{c,k} \sum_{t=0}^{T} \beta^t U(c_t) \\
&s.t. \quad k_0 = \overline{k}, \\
&\qquad\;\; k_{t+1} = f(k_t) - c_t,\ t = 0, \ldots, T, \\
&\qquad\;\; k_{T+1} \geq 0
\end{aligned}
\tag{1.11}
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Before starting, we need to establish some basic concepts.