5. Dynamic optimization in continuous time

5.1 Calculus of variation

Look at the following simple dynamic problem

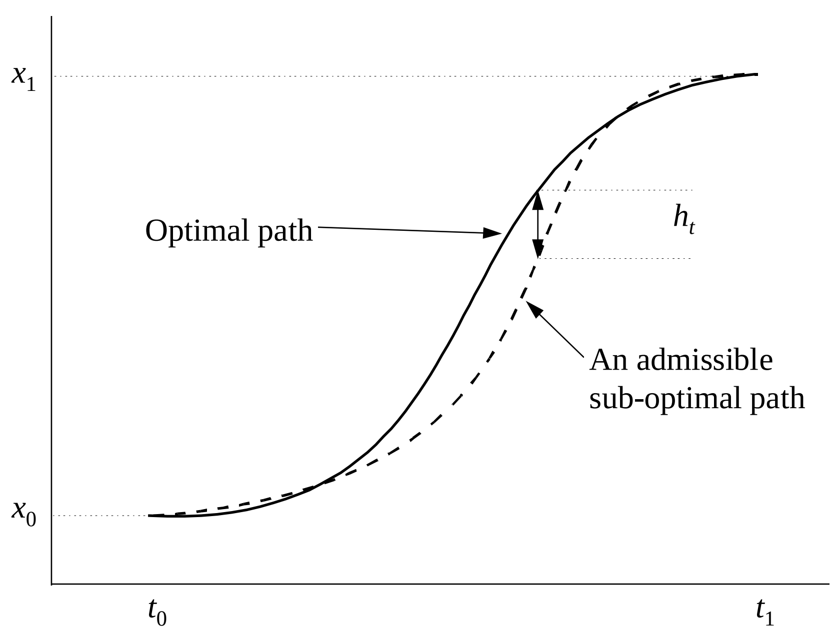
$$\max_{\{x(t)\}_{t_0}^{t_1}} \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t))dt$$
$$s.t. \qquad x(t_0) = x_0, \; x(t_1) = x_1 \tag{5.1}$$

where F is continuous in its arguments. The problem is dynamic since $\dot{x}(t)$ is included. Otherwise we could maximize point by point in time.

An economic example could be that F represents profits from a firm that make output by employing labor (x). Time enters the profit function since the firm discounts future profits. If changes in the number of persons employed is costly $\dot{x}(t)$ also enters the profit function. The firm can then not just in each moment hire the number of persons that maximize current profits.

A solution to the problem is a function $x^*(t)$ (with a continuous derivative). To find it we try to find some characteristics of it that can help us to search. We will in particular now derive some *necessary* conditions that the solution must satisfy. From them we *may* find the solution.

The trick is to define *admissible* deviations h. These are the differences between the optimal path and an admissible but sub-optimal path, i.e. $h_t \equiv x_t^* - x_t$. The constraints in (5.1) imply that $h_{t0} \equiv h_{t1} \equiv 0$ which in this case are the only admissibility constraints (together with differentiability).



Now look at a linear combination of the optimal path and an admissible deviation. For any constant a let

$$y(a) = x^* + ah \tag{5.2}$$

which is clearly admissible. Note that $y(a)$ is a *one parameter family of admissible paths,* i.e., the parameter a pins down a particular path of the variable y over the whole interval t_0 to t_1..

Define the value of the program if we use $y(a)$

$$J(a) \equiv \int_{t_0}^{t_1} F\left(t, \overbrace{x^* + ah}^{y(a)}, \overbrace{\dot{x}^* + a\dot{h}}^{\dot{y}(a)}\right) dt \tag{5.3}$$

This value must by assumption be maximized when a=0. We also find a standard necessary first order condition for a maximum

$$J'(0) = \frac{\partial}{\partial a}\left(\int_{t_0}^{t_1} F\left(t, x^* + ah, \dot{x}^* + a\dot{h}\right)dt\right)\Bigg|_{a=0} = 0 \tag{5.4}$$

$$\begin{aligned}\Rightarrow 0 &= \left(\int_{t_0}^{t_1} \frac{\partial}{\partial a} F\left(t, x^* + ah, \dot{x}^* + a\dot{h}\right)dt\right)\Bigg|_{a=0} \\ &= \int_{t_0}^{t_1}\left(F_x h + F_{\dot{x}}\dot{h}\right)dt.\end{aligned} \tag{5.5}$$

We can rewrite this by integrating $F_{\dot{x}}\dot{h}$ by parts

$$\begin{aligned}\int_{t_0}^{t_1} F_{\dot{x}}\dot{h}dt &= \left[F_{\dot{x}}h\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} h\dot{F}_{\dot{x}}dt \\ &= F_{\dot{x}_{t_1}}\overbrace{h_{t_1}}^{=0} - F_{\dot{x}_{t_0}}\overbrace{h_{t_0}}^{=0} - \int_{t_0}^{t_1} h\dot{F}_{\dot{x}}dt.\end{aligned} \tag{5.6}$$

Substituting this into (5.5) we find that the necessary condition is that along the optimal path

$$\int_{t_0}^{t_1}\left(F_x - \dot{F}_{\dot{x}}\right)hdt = 0 \tag{5.7}$$

But this must be true for all the infinitely many different admissible deviations h. This requires that the value within parenthesis in (5.7) is zero for all t within the planning horizon.

Result 17 A necessary condition for $x*$ to be an optimal path for the problem (5.1) is

$$\begin{aligned} &F_x\left(t, x*, \dot{x}*\right) = \frac{dF_{\dot{x}}\left(t, x*, \dot{x}*\right)}{dt}, \\ &\forall t \subseteq \left[t_0, t_1\right]. \end{aligned} \tag{5.8}$$

This is the *Euler Equation* for the problem. We will see that this can be interpreted as a an arbitrage condition between different points in time. Sometimes we may be able to solve for the function $x^*(t)$. At least we can derive some properties of it.

5.1.1 A simple consumption example

$$\begin{aligned} &\max_{c_t} \int_0^T e^{-rt} U(c_t) dt \\ &s.t. \qquad \dot{K}_t = iK_t + v_t - c_t \\ &\qquad\qquad K_0 = k_0, \qquad\qquad K_T = k_T. \end{aligned} \tag{5.9}$$

$$\begin{aligned} &F\left(t, K, \dot{K}\right) = e^{-rt} U\left(\underbrace{iK_t + v_t - \dot{K}_t}_{c_t}\right), \\ &F_K = e^{-rt} i U'\left(c_t\right), \\ &F_{\dot{K}} = -e^{-rt} U'\left(c_t\right). \end{aligned} \tag{5.10}$$

The Euler equation is

$$\begin{aligned} &\dot{F}_{\dot{K}} = F_K \\ &\Rightarrow -d\left(e^{-rt} U'(c_t)\right) \big/ dt = i e^{-rt} U'(c_t). \end{aligned} \tag{5.11}$$

Note that we express the Euler equation in terms of c rather than in K. This will make it easier to interpret and gives us a first order differential equation in c instead of a second order in k.

Before solving we want to interpret the Euler equation by showing that it is an arbitrage condition between successive points in time. Integrate (5.11)

$$\begin{aligned} &\frac{-de^{-rt}\left(U'(c_t)\right)}{dt} = ie^{-rt} U'(c_t) \\ &\Rightarrow -\int_t^{t+dt} de^{-rt}\left(U'(c_t)\right) = \int_t^{t+dt} ie^{-rs} U'(c_t) ds. \\ &e^{-rt}\left(U'(c_t)\right) - e^{-r(t+dt)}\left(U'(c_{t+dt})\right) = \int_t^{t+dt} ie^{-rs} U'(c_t) ds. \end{aligned} \tag{5.12}$$

or

$$-U'(c_t)+e^{-rdt}\left(U'(c_{t+dt})\right)+\int_t^{t+dt} ie^{-r(s-t)}U'(c_t)ds=0 \tag{5.13}$$

(5.13) can be interpreted; Consider a small deviation from the optimal plan, namely a marginal decrease in consumption in t and increase in $t+dt$. This is the first two terms in (5.13). By doing this we earn more interest over the interval t to dt. The value of this is the third term. Overall, such a marginal change should, according to the FOC yield zero change in the value function, if it is done from the optimal path.

Now we use (5.11) to try to find a solution

$$\begin{aligned}
&-de^{-rt}\left(U'(c_t)\right)/dt=re^{-rt}\left(U'(c_t)\right)-e^{-rt}\left(U''(c_t)\right)\dot{c}_t\\
&=ie^{-rt}U'(c_t)\\
&\Rightarrow \dot{c}_t=\frac{U'(c_t)}{-U''(c_t)}(i-r)
\end{aligned} \tag{5.14}$$

(5.14) tells us a lot of the optimal path although we may be unable to solve for the level of consumption.

Note that (5.14) must hold for the solution to be optimal also for a non-constant interest rate. This is intuitive in light of (5.13).

By specifying a utility function we can go further.

In the CARA utility (exponential) case

$$U=\frac{-e^{-\lambda c}}{\lambda},\qquad U'=e^{-\lambda c},\qquad U''=-\lambda e^{-\lambda c} \tag{5.15}$$

so

$$\dot{c}_t=\frac{i-r}{\lambda},\qquad c_t=c_0+\frac{i-r}{\lambda}t. \tag{5.16}$$

This means that the absolute growth of consumption is constant. Note that this just defines the slope of the optimal path, the level is determined from the dynamic budget constraint.

$$\dot{K}_t-iK_t=v_t-c_t=v_t-c_0-\frac{i-r}{\lambda}t \tag{5.17}$$

Multiplying by the integration factor and integrating we have

$$\begin{aligned}
&e^{-it}\left(\dot{K}_t-iK_t\right)=\frac{de^{-it}K_t}{dt}=v_t-c_0-\frac{i-r}{\lambda}t\\
&e^{-iT}k_T-k_0=\int_0^T e^{-it}v_t dt-\int_0^T e^{-it}c_t dt.
\end{aligned} \tag{5.18}$$

Now define

$$\int_0^T e^{-it} v_t dt + k_0 - e^{-iT} k_T \equiv W_0 = \int_0^T e^{-it} c_t dt. \tag{5.19}$$

This is the *intertemporal* budget constraint. Solving this

$$\begin{aligned}
&\int_0^T e^{-it}\left(c_0 + \frac{i-r}{\lambda}t\right)dt = -\left[c_0 \frac{e^{-it}}{i}\right]_0^T + \int_0^T \overbrace{\left(\frac{i-r}{\lambda}t\right)}^{u} \overbrace{e^{-it}dt}^{dv} \\
&= -\left[c_0 \frac{e^{-it}}{i}\right]_0^T - \left[\frac{i-r}{\lambda}t\frac{e^{-it}}{i}\right]_0^T - \int_0^T -\frac{i-r}{\lambda}\frac{e^{-it}}{i}dt \\
&= -\left[c_0 \frac{e^{-it}}{i}\right]_0^T - \left[\frac{i-r}{\lambda}t\frac{e^{-it}}{i}\right]_0^T - \left[\frac{i-r}{\lambda}\frac{e^{-it}}{i^2}\right]_0^T \\
&= c_0(\frac{1-e^{-iT}}{i}) + \frac{i-r}{\lambda i^2} - e^{-iT}\left(\frac{i-r}{\lambda i}T + \frac{i-r}{\lambda i^2}\right) = W_0.
\end{aligned} \tag{5.20}$$

Note that if $i=r$ consumption is simply a fraction of wealth, that decreases with the length of the planning horizon.

$$c_0 = \frac{i}{(1-e^{-iT})}W_0 \tag{5.21}$$

So with an infinite horizon $c_t = iW_t$.

Similarly on the case of CRRA utility

$$U = \frac{c^{1-1/\sigma}}{1-1/\sigma}, \qquad U' = c^{-1/\sigma}, U'' = -\frac{c^{-1/\sigma-1}}{\sigma} \tag{5.22}$$

the Euler equation (5.14) becomes

$$\begin{aligned}
&\dot{c}_t / c_t = \sigma(i-r) \\
&\Rightarrow c_t = c_0 e^{\sigma(i-r)t}.
\end{aligned} \tag{5.23}$$

Here the growth *rate* of consumption is constant and proportional to the difference between market and subjective discount rates. Compare this to (5.16).

Using the intertemporal budget constraint

$$\begin{aligned}
&\int_0^T e^{-it} c_0 e^{\sigma(i-r)t} dt = \int_0^T c_0 e^{((\sigma-1)i-\sigma r)t} dt = W_0 \\
&\Rightarrow c_0 = \left(\frac{(\sigma r - (\sigma-1)i)}{1-e^{((\sigma-1)i-\sigma r)T}}\right) W_0
\end{aligned} \tag{5.24}$$

Note the results when σ=1 and when $T \to \infty$.

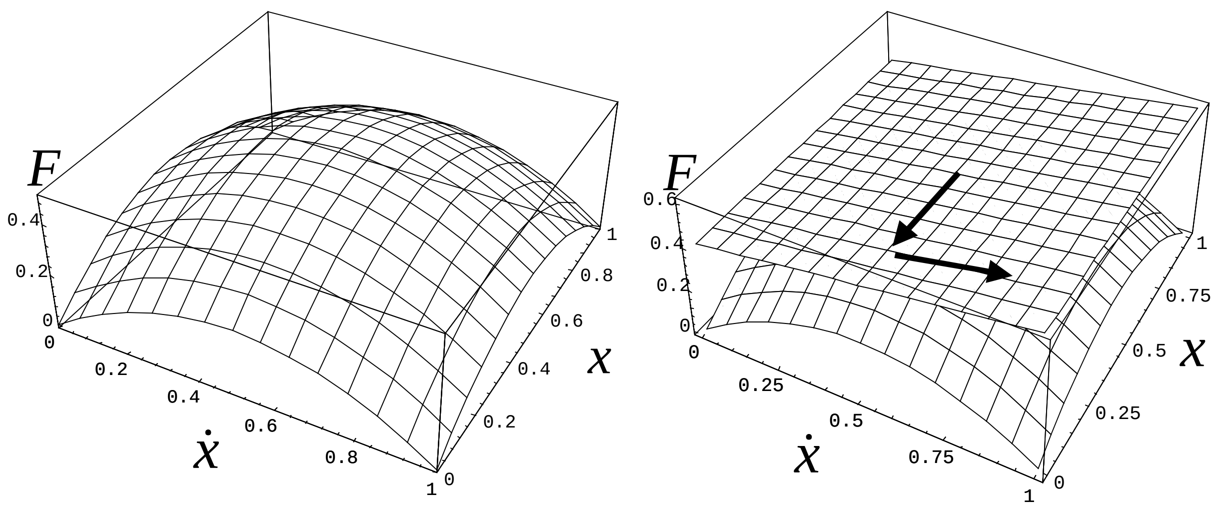
5.1.2 A sufficient condition

The Euler condition is necessary but not sufficient. It is however also sufficient for a maximum if $F(t,x,\dot{x})$ is concave in $x,\dot{x}$.

Recall that if $F(t,x,\dot{x})$ is concave in $x,\dot{x}$ then

$$\begin{aligned}&F(t,x,\dot{x})\\&\leq F(t,x_*,\dot{x}_*)+(x-x_*)F_x(t,x_*,\dot{x}_*)+(\dot{x}-\dot{x}_*)F_{\dot{x}}(t,x_*,\dot{x}_*).\end{aligned}\tag{5.25}$$

To see this



Assume that $F(t,x_t^*,\dot{x}_t^*)\equiv F_*$ satisfies the Euler equation and F is concave in $x,\dot{x}$. We then want to show that $F(t,x_t^*,\dot{x}_t^*)$ is optimal, i.e., that

$$\int_0^T F(t,x_t,\dot{x}_t)dt \leq \int_0^T F(t,x_t^*,\dot{x}_t^*)dt \tag{5.26}$$

for all admissible paths. Admissible deviations are defined $h(t)\equiv x(t)-x_*(t)$ with $\dot{h}(t)\equiv \dot{x}(t)-\dot{x}_*(t)$.

Now using (5.25) we have that

$$\begin{aligned}&F\leq F_*+\overbrace{(x-x_*)}^{h}F_x^*+\overbrace{(\dot{x}-\dot{x}_*)}^{\dot{h}}F_{\dot{x}}^*\\&\Rightarrow \int_0^T Fdt \leq \int_0^T F_*dt+\int_0^T\left(hF_x^*+\dot{h}F_{\dot{x}}^*\right)dt\end{aligned}\tag{5.27}$$

By integrating by parts we find that

$$
\begin{aligned}
\int_0^T \left(hF_x^* + \dot{h}F_{\dot{x}}^*\right)dt &= \int_0^T \left(hF_x^*\right)dt + \left[F_{\dot{x}}^* h\right]_0^T - \int_0^T h\dot{F}_{\dot{x}}^* dt \\
&\qquad F_{\dot{x}_T} \overbrace{h_T}^{=0} - F_{x_0} \overbrace{h_0}^{=0} + \int_0^T h\left(F_x^* - \dot{F}_{\dot{x}}^*\right)dt \qquad (5.28) \\
&= 0.
\end{aligned}
$$

This shows that concavity, which gave us the inequality in (5.27), makes the Euler equation sufficient for optimality.

Result 18 If F is concave in $x, \dot{x}$, the Euler equation, given in (5.8) is both necessary and sufficient for an optimum.

5.1.3 Transversality conditions

Assume now that k_{t_1} is free. Before we used the terminal condition for k_{t1} to find one integration constant. Now we need some other condition to do this – the *transversality condition.*

An admissible deviation h is now *not* required to satisfy $h(t_1)=0$. The necessary condition (5.4), (5.5) are still valid but (5.6) is changed slightly.

$$
\begin{aligned}
\int_{t_0}^{t_1} F_{\dot{x}}\dot{h}dt &= \left[F_{\dot{x}}h\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} h\dot{F}_{\dot{x}}dt \\
&= F_{\dot{x}_T} \overbrace{h_T}^{\neq 0} - F_{x_0} \overbrace{h_0}^{=0} - \int_{t_0}^{t_1} h\dot{F}_{\dot{x}}dt.
\end{aligned}
\qquad (5.29)
$$

So the necessary condition becomes that along the optimal path

$$
\int_{t_0}^{t_1} \left(F_x - \dot{F}_{\dot{x}}\right)hdt + F_{\dot{x}}(t_1, x_{t_1}^*, \dot{x}_{t_1}^*)h(t_1) = 0. \qquad (5.30)
$$

So we see that the Euler equation is still valid. In addition to the Euler equation we have the added condition

$$
F_{\dot{x}}(t_1, x_{t_1}^*, \dot{x}_{t_1}^*) = 0 \qquad (5.31)
$$

This is the transversality condition.

5.1.4 Example

In the consumption example (5.9) we expect that if no end condition is set for K it must be optimal to consume so much that marginal utility goes to zero. Let's verify that.

$$
F_{\dot{K}}(t_1, K_{t_1}, \dot{K}_{t_1}) = -e^{-rt_1}U'(c_{t_1}). \qquad (5.32)
$$

5.1.5 Infinite horizon

The intuition for the Euler equation above as an arbitrage between successive time points still suggests that it is valid also in infinite horizon problems. This is the case for properly specified economic problems where the objective function converge to something finite for all admissible deviations. If this is not the case optimality becomes ambiguous. General transversality conditions are, however, not known.

In economic infinite horizon models we often want to find a steady state solution x^{ss} (s.t. $\dot{x} = \ddot{x} = 0$) for some properly detrended variable. This works if time does not enter F or just as an exponential discounting. The problem is then (time) autonomous. If we take this steady state to be the boundary condition

$$\lim_{t\to\infty} x_t = x^{ss} \tag{5.33}$$

or

$$\lim_{t\to\infty} \dot{x}_t = 0 \tag{5.34}$$

we have the necessary information to find the solution with integration constants. Take the Ramsey problem as an example.

$$\begin{aligned} &\max_k \int_{t_0}^{\infty} e^{-\theta t} U(c_t) dt \\ &s.t. \qquad \dot{k}_t = f(k_t) - c_t, \end{aligned} \tag{5.35}$$

and the initial and boundary conditions

$$\begin{aligned} &k_0 = \bar{k} \\ &\lim_{t\to\infty} \dot{k}_t = 0. \end{aligned} \tag{36}$$

We then have

$$\begin{aligned} &F(t,k,\dot{k}) = e^{-\theta t} U\left(f(k) - \dot{k}\right) \\ &F_k = e^{-\theta t} U' f'(k) \\ &F_{\dot{k}} = -e^{-\theta t} U'\left(f(k) - \dot{k}\right) = -e^{-\theta t} U'(c) \\ &\frac{dF_{\dot{k}}}{dt} = \theta e^{-\theta t} U' - e^{-\theta t} \frac{dU'}{dt} = \theta e^{-\theta t} U' - e^{-\theta t} U'' \dot{c} \end{aligned} \tag{5.37}$$

Note that in the following I find expressions for c rather than for k. Note also that the discounting terms cancels. So the Euler equation can be written

$$
\begin{aligned}
&U'f'(k) = \theta U' - U''\dot{c} \\
&\Rightarrow \dot{c} = \frac{U'}{-U''}\left(f'(k) - \theta\right)
\end{aligned} \tag{5.38}
$$

This gives us the system of differential equations

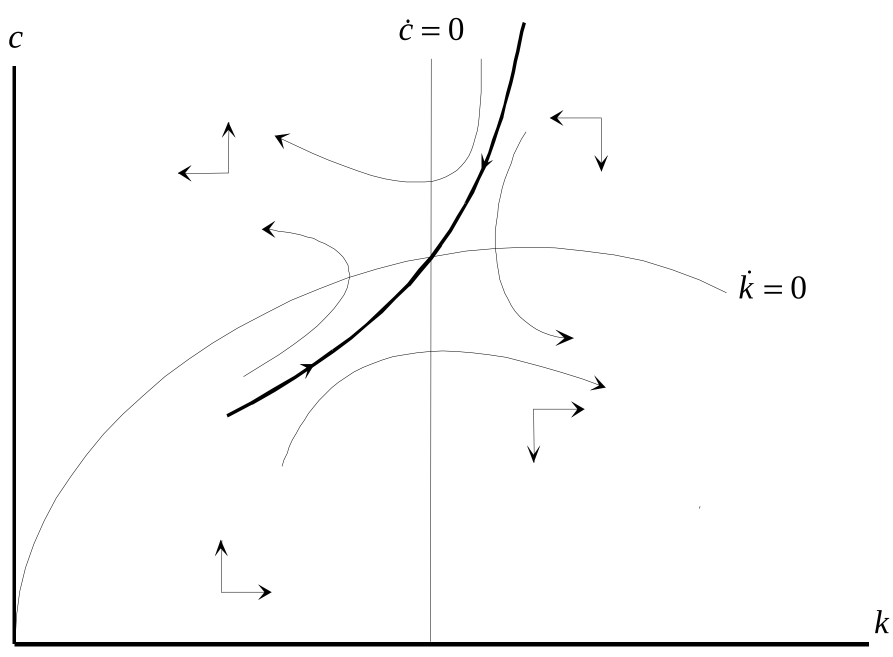
$$
\begin{aligned}
\dot{c}_t &= \frac{U'}{-U''}\left(f'(k_t) - \theta\right) \\
\dot{k}_t &= f(k_t) - c_t .
\end{aligned} \tag{5.39}
$$

Note that in terms of k the Euler equation is

$$
\begin{aligned}
&e^{-\theta t}U'f'(k) = \theta e^{-\theta t}U' - e^{-\theta t}U''f'\dot{k} + e^{-\theta t}U''\ddot{k} \\
&U'f'(k) = \theta U' - U''f'\dot{k} + U''\ddot{k},
\end{aligned} \tag{5.40}
$$

i.e., a second order differential equation. Note the equivalence between a second order differential equations and a two equation system of differential equations.

The system (5.39) has variable coefficients and may be difficult to solve analytically. But we have already seen that it can be analyzed qualitatively. First we clearly have steady state. Secondly we can draw its phase diagram.



Note that as always all paths (arrows) in the figure satisfy the Euler equation. The initial and boundary conditions together pick just one path. Only the saddle path satisfies the boundary condition and by knowledge of k_0 we can then find c_0. We have thus pinned down just one path and (implicitly) found the two integration constants.

5.2 Optimal control

Optimal control can be seen as an extension of calculus of variation and is often more convenient when there is restrictions on the way the system can controlled. To facilitate this, we make a distinction between control variables (e.g., consumption or investments) and state variables (e.g., capital stocks or debt) that are governed by a differential equation (*transition* equation) and thus given in each point in time.

$$\begin{aligned} &\max_{\{u_t\}_{t0}^{t1}} \int_{t_0}^{t_1} f(t, x_t, u_t) dt \\ &s.t. \qquad \dot{x}_t = g(t, x_t, u_t), \qquad x_{t_0} = x_0. \end{aligned} \tag{5.41}$$

Here, x is the state variable. The way it changes over time, can be affected by the control variable u. As a means to finding a solution we define a multiplier function λ_t for the transition equation.

For a combination of x,u to be admissible it must be that $\forall\ t \subseteq [t_{0,} t_1]\ \ g(t, x_t, u_t) - \dot{x}_t = 0$. Adding this zero to the maximand yields,

$$\int_{t_0}^{t_1} f(t, x_t, u_t) + \lambda_t \left(g(t, x_t, u_t) - \dot{x}_t \right) dt \tag{5.42}$$

Now we want to get rid $\dot{x}_t$. So integrate by parts

$$\int_{t_0}^{t_1} \lambda_t \dot{x}_t dt = \left[\lambda_t x_t \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \dot{\lambda}_t x_t dt \tag{5.43}$$

giving

$$\int_{t_0}^{t_1} \left(f(t, x_t, u_t) + \lambda_t g(t, x_t, u_t) + \dot{\lambda}_t x_t \right) dt - \lambda_{t_1} x_{t_1} + \lambda_{t_0} x_{t_0} \tag{5.44}$$

Now we use the same procedure as when deriving necessary conditions for the Calculus of Variation problem. Instead of looking at admissible variations of x we look at admissible variations of the control variable u. Let $u*$ represent the optimal control and u some other admissible control and define $h= u*-u$ Let $y(a)$ denote the state variable generated by using the control $u*+ah$. Let $J(a)$ denote the value of the program (5.44) when using the control $u*+ah$. Clearly $J(0)$ is the maximum of J by definition and $J'(0)=0$. As before we will use this necessary condition to find necessary properties of the solution.

$$\begin{aligned} J(a) \equiv & \int_{t_0}^{t_1} \left(f\left(t, y_t(a), u_t^* + ah_t\right) + \lambda g\left(t, y_t(a), u_t^* + ah_t\right) + \dot{\lambda}_t y_t(a) \right) dt \\ & -\lambda_{t_1} y_{t_1}(a) \ + \lambda_{t_0} y_{t_0}(a) \ , \end{aligned} \tag{5.45}$$

$$
\begin{aligned}
J'(0) = &\int_{t_0}^{t_1}\left(f_x y_t'(0) + f_u h_t + \lambda_t g_x y_t'(0) + \lambda_t g_u h_t + \dot{\lambda}_t y_t'(0) \right)dt \\
&-\lambda_{t_1} y_{t_1}'(0) \ + \lambda_{t_0} y_{t_0}'(0) \\
= &\int_{t_0}^{t_1}\left(\left(f_x + \lambda_t g_x + \dot{\lambda}_t \right) y_t'(0) + (f_u + \lambda g_u) h_t \right)dt \\
&-\lambda_{t_1} y_{t_1}'(0) \ + \lambda_{t_0} \overbrace{y_{t_0}'(0)}^{=0}, \\
= &\, 0.
\end{aligned}
\tag{5.46}
$$

So far we have not put any restrictions on λ. It will soon be clear that it is very convenient to let it follow the differential equation

$$
\begin{aligned}
\dot{\lambda} &= - f_x - \lambda g_x \\
\lambda_{t_1} &= 0.
\end{aligned}
\tag{5.47}
$$

For (5.46) and (5.47)to hold we thus require that along the optimal path

$$
\int_{t_0}^{t_1} (f_u + \lambda g_u) h dt = 0, \tag{5.48}
$$

for all admissible deviations h. For this to hold for all such deviations we need that

$$
f_u + \lambda g_u = 0, \qquad \forall t \subseteq [t_0, t_1]. \tag{5.49}
$$

We have now found that if we define λ_t according to (5.47) the necessary condition for optimality can be written as (5.49). To remember this and the we construct the *Hamiltonian*.

$$
H(t, x_t, u_t, \lambda_t) = f(t, x_t, u_t) + \lambda_t g(t, x_t, u_t), \tag{5.50}
$$

from which we can derive the necessary conditions for optimality.

Result 19 Necessary conditions for a solution to (5.41), i.e., an optimal control, are

$$
\begin{aligned}
H_u &= f_u\left(t, x_t, u_t^*\right) + \lambda_t g_u\left(t, x_t, u_t^*\right) = 0 \\
-H_x &= - f_x\left(t, x_t, u_t^*\right) - \lambda_t g_x\left(t, x_t, u_t^*\right) = \dot{\lambda}_t \\
H_\lambda &= g_x\left(t, x_t, u_t^*\right) = \dot{x}_t
\end{aligned}
\tag{5.51}
$$

and the initial condition $x_{t_0} = x_0$ and the terminal condition $\lambda_{t_1} = 0$.

5.2.1 Sufficiency

As in the Calculus of Variation we get a sufficiency condition by imposing the right concavity condition. Assume that f and g are concave in x,u and $\lambda \geq 0$. This implies that the Hamiltonian is concave in x,u. Then since f is concave

$$f \le f_* + (x - x_*) f_x^* + (u - u_*) f_u^*$$
$$\int_{t_0}^{t_1} f dt \le \int_{t_0}^{t_1} f_* dt + \int_{t_0}^{t_1} \left((x - x_*) f_x^* + (u - u) f_u^* \right) dt. \tag{5.52}$$

So we want to show that the last integral in (5.52) is ≤0. Substitute for f_x from (5.51) and then integrate by parts the term involving $\dot{\lambda}$ to get rid of that.

$$\begin{aligned}
&\int_{t_0}^{t_1} \left((x - x_*) f_x^* + (u - u_*) f_u^* \right) dt \\
&= \int_{t_0}^{t_1} \left((x - x_*) \left(-\dot{\lambda} - \lambda g_x^* \right) + (u - u_*)(-\lambda g_u^*) \right) dt \\
&= -\overbrace{\left[\lambda (x - x_*) \right]_{t_0}^{t_1}}^{0} + \int_{t_0}^{t_1} \overbrace{(\dot{x} - \dot{x}_*)}^{g - g_*} \lambda dt \\
&+ \int_{t_0}^{t_1} \left((x - x_*) \left(-\lambda g_x^* \right) + (u - u_*)(-\lambda g_u^*) \right) dt \\
&= \int_{t_0}^{t_1} \lambda \left((g - g_*) - \left((x - x_*) g_x^* + (u - u_*) g_u^* \right) \right) dt \le 0.
\end{aligned} \tag{5.53}$$

If $\lambda \le 0$ we need that g is convex in x,u. Then H is still concave. If g is linear we see that its sufficient that f is concave in x,u.

5.2.2 Interpretations

To see the direct analogy of the Optimal Control approach and standard Lagrange approach, look at a discrete time version of (5.41)

$$\begin{aligned}
&\max_u \sum_{t=t_0}^{t_1} f(t, x_t, u_t) \\
&s.t. \qquad x_{t+1} - x_t = g(t, x_t, u_t), \\
&\qquad\qquad x_{t_0} = x_0,\ x_{t_1} = x_1
\end{aligned} \tag{5.54}$$

The Lagrangean is

$$\begin{aligned}
L = &\sum_{t=t_0}^{t_1} f(t, x_t, u_t) + \lambda_t \left(g(t, x_t, u_t) - x_{t+1} + x_t \right) \\
&+ \mu_0 \left(x_0 - x_{t_0} \right) + \mu_1 \left(x_1 - x_{t_1} \right)
\end{aligned} \tag{5.55}$$

First order conditions are

$$f_u(t, x_t, u_t) + \lambda_t g_u(t, x_t, u_t) = 0 \tag{5.56}$$

This is directly analogous to the optimality condition(5.49). It is straightforward to interpret. A marginal change in the control variable has two effects. It affects the current flow of value by an amount f_u. Furthermore, if changes the rate of accumulation of the state variable by an amount g_u. The value of this is λg_u.

Now, to shed some light on the second condition in (5.51), differentiate L with respect to x_t to get

$$f_x(t,x_t,u_t)+\lambda_t\big(g_x(t,x_t,u_t)+1\big)-\lambda_{t-1} \tag{5.57}$$

We know that the interpretation of λ_t in the discrete problem is the shadow value of capital.

Now, consider the following situation. Two individuals solve identical problems. At *t-1* one decides to buy one marginal unit of capital from the other and to sell it back at *t*. If prices of capital reflect shadow values they must be λ_t and λ_{t-1}. The consequences on the total value of the program is then given by (5.57) where the first part is the direct increase in value, the second the resell value and the third the price to pay. We conclude that (5.57) =0. Rewriting we see the direct analogy to (5.47).

$$\begin{aligned}
&-f_x(t,x_t,u_t)-\lambda_t g_x(t,x_t,u_t)=\lambda_t-\lambda_{t-1}\\
&-f_x(t,x_t,u_t)-\lambda_t g_x(t,x_t,u_t)=\dot{\lambda}_t
\end{aligned} \tag{5.58}$$

(5.47) is thus a consequence of a correct valuation of x_t (not an optimality condition) and the idea the λ is the shadow value of the state variable. The interpretation of λ_t is identical in the two problems.

Define the optimal value function to problem (5.41) as the value of the objective function using the optimal control. This obviously depends on the initial condition and the starting point.

$$V(x_t,t)=\int_t^{t_1} f\left(s,x^*,u^*\right)ds\,. \tag{5.59}$$

$$V_x(x,t)=\lambda_t\,. \tag{5.60}$$

This is also true at later points in time. Note that λ_{t+s} in (5.59) is the shadow value of the state variable at *t+s seen from the time of start of the program*.

5.2.3 Current value hamiltonian

Often we have problems where *t* only enters as an exponential discounting. E.g.,

$$\begin{aligned}
&\max_u \int_0^T e^{-rt} f\left(x_t,u_t\right)dt\\
&s.t. \qquad \dot{x}_t=g\left(x_t,u_t\right), \quad x_{t_0}=x_0.
\end{aligned} \tag{5.61}$$

The Hamiltonian with necessary conditions is

$$
\begin{aligned}
&H(t,x_t,u_t)=e^{-rt}f(x,u)+\lambda g(x_t,u_t)\\
&H_u=e^{-rt}f_u+\lambda g_u=0\\
&H_x=e^{-rt}f_x+\lambda g_x=-\dot{\lambda},\ \lambda_T=0.
\end{aligned}
\tag{5.62}
$$

It is often convenient to use a *current* shadow value defined as

$$
\begin{aligned}
&e^{-rt}\mu_t\equiv\lambda_t\\
&\Rightarrow\dot{\lambda}=-re^{-rt}\mu+e^{-rt}\dot{\mu}=e^{-rt}(\dot{\mu}-r\mu)
\end{aligned}
\tag{5.63}
$$

Substitute into (5.62)

$$
\begin{aligned}
&H(t,x_t,u_t)=e^{-rt}f(x,u)+e^{-rt}\mu g(x_t,u_t)\\
&H_u=e^{-rt}(f_u+\mu g_u)=0\\
&H_x=e^{-rt}(f_x+\mu g_x)=-e^{-rt}(\dot{\mu}-r\mu),\ e^{-rT}\mu_T=0.
\end{aligned}
\tag{5.64}
$$

We get rid of all the discounting factors by defining the *current value Hamiltonian* with associated necessary conditions.

Result 20 Defining th current value Hamiltonian as $\mathcal{H}(t,x_t,u_t)\equiv e^{rt}H(t,x_t,u_t)$. Necessary conditions of an optimal control are

$$
\begin{aligned}
&\mathcal{H}_u=f_u+\mu g_u=0\\
&\mathcal{H}_x=f_x+\mu g_x=-(\dot{\mu}-r\mu),\ e^{-rT}\mu_T=0.
\end{aligned}
\tag{5.65}
$$

If T is infinity we may not divide the transversality condition by the discount factor since it is zero.

5.2.4 Some infinite horizon results

The optimality condition and the differential equation for the shadow value and the state variable

$$
\begin{aligned}
H_u&=f_u(t,x_t,u_t^*)+\lambda_t g_u(t,x_t,u_t^*)=0\\
-H_x&=-f_x(t,x_t,u_t^*)-\lambda_t g_x(t,x_t,u_t^*)=\dot{\lambda}_t\\
H_\lambda&=g_x(t,x_t,u_t^*)=\dot{x}_t
\end{aligned}
\tag{5.66}
$$

are necessary also in the infinite horizon case. The transversality condition is the problem. If we are ready to assume that the optimal path eventually settles down to a steady state

$$
\lim_{t\to\infty}x_t=x^s \tag{5.67}
$$

we have enough information to solve for the optimal path. The phase diagram is one way to do this.

This would be the case in the Ramsey problem.

Another simple case if $H(t,x,u,\lambda)$ is concave in x,u. Then (5.66) together with the transversality condition

$$\lim_{t\to\infty} \lambda_t (x_t - x_t^*) \geq 0 \tag{5.68}$$

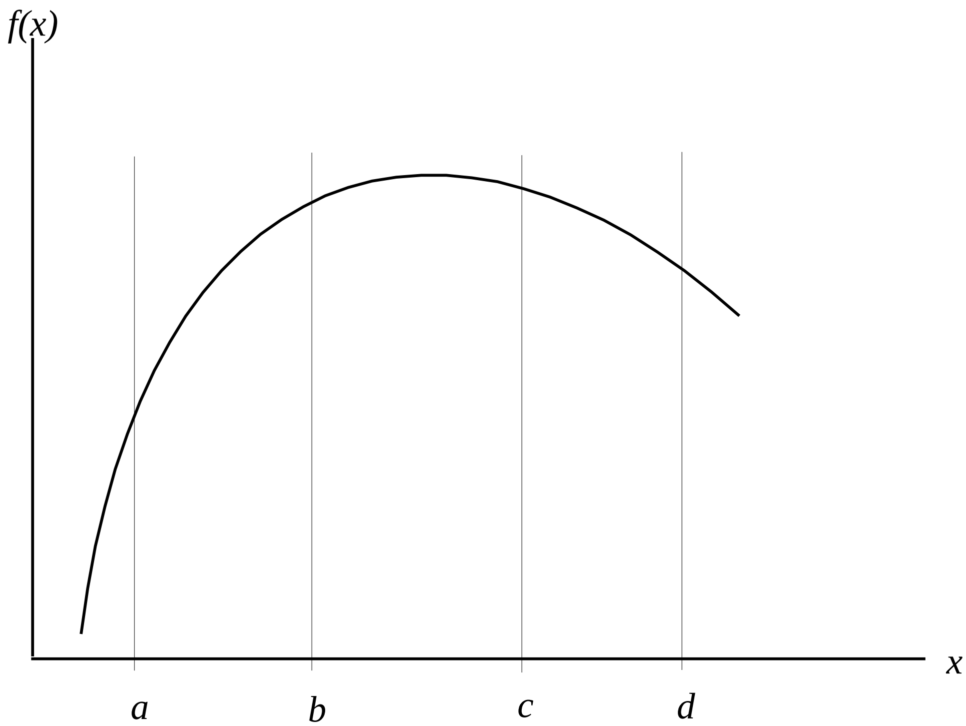
for admissible paths x provide *sufficient* conditions for optimality (Mangasarin). It seems economically reasonable to assume that $\lim_{t\to\infty} \lambda_t x_t^* = 0$, i.e. that the value seen from today of the capital stock infinitely far away into the future is zero. Then its enough to assume that the shadow value of capital and all admissible levels of x are positive for (5.68) to hold.

5.2.5 Bounded controls

For a control to be optimal it is necessary that it solves

$$\max_u H(t, x_t, u_t, \lambda_t) \tag{5.69}$$

If u is bounded $H_u = 0$ is not necessary for an optimum. As in standard maximization the first order conditions only holds for interior solutions.



If we maximize $f(x)$ over $[a,b]$ b is optimal and $f'(x^*)>0$. If the range is $[c,d]$ c is optimal with $f'(x^*)<0$. Also in optimal control we may use *Kuhn-Tucker* multipliers in this case. Assume we solve problem (5.41) but restrict u to the range [a,b]. We then form the appended Hamiltonian

$$\begin{aligned} H(t, x_t, u_t, \lambda_t) = {} & f(t, x_t, u_t) + \lambda_t g(t, x_t, u_t) \\ & + w_1(b-u) + w_2(u-a). \end{aligned} \tag{5.70}$$

The optimality condition now becomes

$$
\begin{aligned}
&f_u + \lambda g_u - w_1 + w_2 = 0 \\
&w_1, w_2 \geq 0, \\
&w_1(b - u^*) = w_2(u^* - a) = 0.
\end{aligned} \tag{5.71}
$$

Except for the knife-edge case we have that if w_1>0, $(b - u^*)$ =0 so the H_u>0 as in the figure

5.2.6 The Pontryagin maximum principle

Now we come to a more general formulation of the necessary conditions for a solution to the optimal control problem. We allow for a any finite number of discontinuity points in the control, n control and state variables and that the controls are restricted to a constant weak subset of R^n.

$$\begin{aligned}
&\max_{\mathbf{u}} \int_{\mathbf{0}}^{T} f(t,\mathbf{x},\mathbf{u})dt \\
&s.t. \quad \dot{x}_i = g_i(t,\mathbf{x},\mathbf{u}), \quad i = 1,\dots n, \\
&\qquad \mathbf{x}(0) = \overline{\mathbf{x}}, \\
&\qquad x_i(T) = x_{iT}, \qquad i = 1,\dots p \\
&\qquad x_i(T) \geq x_{iT}, \qquad i = p+1,\dots q \\
&\qquad x_i(T) \text{ free} \qquad i = q+1,\dots n. \\
&\qquad \mathbf{u} \in U \subseteq R^n
\end{aligned} \tag{5.72}$$

Result 21

For **u*** *and the resulting state vector* **x*** *to maximize* (5.72) *it is necessary that there exists a constant* λ_0 *and continuous functions* λ(t) (λ_i(t) i=1,...,n) *such that* $\forall t \in [0,T]$

$$\lambda_0 = 0 \text{ or } 1, \ \{\lambda_0, \lambda(t)\} \neq \{0,\mathbf{0}\}, \tag{5.73}$$

$$\mathbf{u}^* = \arg\max_{\mathbf{u}} H(t,\mathbf{x}^*,\mathbf{u},\lambda), \tag{5.74}$$

where

$$H(t,\mathbf{x},\mathbf{u},\lambda) = \lambda_0 f(t,\mathbf{x},\mathbf{u}) + \sum_{i=1}^{n} \lambda_i g_i(t,\mathbf{x},\mathbf{u}) \tag{5.75}$$

except at points of discontinuity of **u**

$$\dot{\lambda}_i = -H_{x_i} \tag{5.76}$$

and the transversality conditions

$$\begin{aligned}
&\lambda_i(T) \text{ free}, && i = 1,\dots p, \\
&\lambda_i(T) \geq 0, && i = p+1,\dots q, \\
&\lambda_i(T) = 0, \text{if } x_i(T) > x_{iT}, && i = p+1,\dots q, \\
&\lambda_i(T) = 0 && i = q+1,\dots n.
\end{aligned} \tag{5.77}$$

The strange shadow value on the objective function λ_0 may under some perverse circumstances be 0. I believe you can safely ignore this possibility for the coming courses in economics.

Note that by specifying the control region we may formulate *Kuhn-Tucker* first order conditions instead of (5.74).

At the points in time when the control jumps λ has a kink. It is, however, *always continuous.* Note also that H is always continuous. Kuhn-Tucker shadow values on the control constraints, μ_t, may be discontinuous.