6. Dynamic Optimization in Discrete Time

6.1. Non-Stochastic Dynamic Programming

Consider the dynamic problem

$$\begin{aligned}
&\max_{c,k} \sum_{t=0}^{T} \beta^t U(c_t) \\
&s.t. \qquad k_0 = \bar{k}_0, \\
&\qquad\qquad k_{t+1} = f(k_t) - c_t,\ t = 0,\ldots,T, \\
&\qquad\qquad k_{T+1} = 0
\end{aligned} \tag{6.1}$$

The direct way to solve this would be to form the Lagrangean

$$L = \sum_{t=0}^{T} \beta^t U(c_t) + \sum_{t=0}^{T} \lambda_t \left(\left(f(k_t) - c_t \right) - k_{t+1} \right) \tag{6.2}$$

with first order conditions

$$\begin{aligned}
&\beta^t U'(c_t) - \lambda_t = 0, \\
&\lambda_t f'(k_t) - \lambda_{t-1} = 0.
\end{aligned} \tag{6.3}$$

An alternative way is to recognize the recursive structure of the problem. (6.1) can be written

$$\begin{aligned}
&\max_{c_0, k_1|_{k_0}} \left(U(c_0) + \beta \max_{c_1, k_2|_{k_1}} \left(U(c_1) + \ldots + \beta \max_{c_T, k_{T+1}|_{k_T}} \left(U(c_T) \right) \right) \right) \\
&s.t. \qquad k_0 = \bar{k}_0, \\
&\qquad\qquad k_{t+1} = f(k_t) - c_t,\ t = 0,\ldots,T, \\
&\qquad\qquad k_{T+1} = 0
\end{aligned} \tag{6.4}$$

We then solve the problem backwards starting from the last period. In period T-1 the remaining problem only depends on earlier actions through k_{T-1}. Substituting from the constraint we then want to solve

$$\max_{k_T} U\left(f(k_{T-1}) - k_T \right) + \beta U\left(f(k_T) \right) \tag{6.5}$$

We need to solve for the *function* $k_T = h(k_{T-1})$, i.e., for all possible values of k_{T-1}. Then we solve the problem for *T-2* given that we do what is optimal in *T-1*. So we solve

$$\max_{k_{T-1}} U\left(f(k_{T-2}) - k_{T-1} \right) + \beta \left(\max_{k_T} U\left(f(k_{T-1}) - k_T \right) + \beta U\left(f(k_T) \right) \right) \tag{6.6}$$

Define the current value function for the last periods problem as

$$V_1(k_{T-1}) = \max_{k_T} U(f(k_{T-1}) - k_T) + \beta U(f(k_T)). \tag{6.7}$$

Substitute into (6.6) and continue in the same iterative way to get

$$V_2(k_{T-2}) \equiv \max_{k_{T-1}} U(f(k_{T-2}) - k_{T-1}) + \beta V_1(k_{T-1}) \tag{6.8}$$

$$V_s(k_{T-s}) \equiv \max_{k_{T-s+1}} U(f(k_{T-s}) - k_{T-s+1}) + \beta V_{s-1}(k_{T-s+1}) \tag{6.9}$$

where *s* denote the number of periods left to termination.

The equations in (6.8) and (6.9) are called *Bellman equations*. The first order conditions implicitly defines difference equations for *k*.

$$-U'(f(k_{T-s}) - k_{T-s+1}) + \beta V'_{s-1}(k_{T-s+1}) = 0. \tag{6.10}$$

To find the policy functions $k_{T-s+1} = h_s(k_{T-s})$ we need to find the value function. In a finite horizon problem this is done as above by starting from the last period.

Infinite Horizon

In an infinite horizon problem we cannot use the method of starting from the last period. Instead we can use to different approaches. 1. Guess on a value function and make sure it satisfies the Bellman equation. 2. Iterate on the Bellman equation until it converges.

Guessing

Guessing is often feasible when the problem is autonomous. Then the value function is independent of time so we can write.

$$\begin{aligned} &V(k_t) = \max_{u_t} U(k_t, u_t) + \beta V(k_{t+1}) \\ &s.t. k_{t+1} = g(k_t, u_t). \end{aligned} \tag{6.11}$$

We can rewrite

$$V(k_t) = \max_{u_t} U(k_t, u_t) + \beta V(g(k_t, u_t)) \tag{6.12}$$

with first order conditions

$$U_u(k_t, u_t) + \beta V'(k_{t+1}) g_u(k_t, u_t) = 0. \tag{6.13}$$

Suppose we find a solution to (6.12) (and to (6.13) when relevant) has to be a function $u(k_t)$. Plugging that into (6.12) we get rid of the max so we have

$$V(k_t) = U(k_t, u_t(k_t)) + \beta V(g(k_t, u_t(k_t))) \tag{6.14}$$

Note that we can use the envelope result that we can evaluate $V'(k)$ as the partial derivative holding *u* constant, i.e.,

$$
\begin{aligned}
V'(k_t) &= U_k(k_t,u_t) + \frac{du_t}{dk_t}\left(\overbrace{U_u(k_t,u_t) + \beta V'(k_{t+1}) g(k_t,u_t)}^{=0}\right) \\
&\quad +\beta V'(g(k,u)) g_k(k,u) \\
&= U_k(k_t,u_t) + \beta V'(g(k,u)) g_k(k,u)
\end{aligned}
\quad . \tag{6.15}
$$

If (6.14) is satisfied we have a solution to the value function. On the other hand, if, for example, u depends on more variables than k, (6.14) is not satisfied and our guess was incorrect.

Note that the whole RHS of (6.11) is a functional of the unknown function V(.) for the given functions U and g. We can define this functional as $T(V)$. The Bellman equation then defines a *fixed point* for T in the space of *functions V*. The Bellman equation can thus be written

$$V(k) = T(V(k)) \equiv \max_u U(k,u) + \beta V(g(k,u)) \tag{6.16}$$

The solution to the Bellman equation is thus a fixed point *in the space of functions* we are looking for, not a fixed point for k. I.e., if we plug in some function of k in the RHS of (6.14) we must get out the same function on the LHS. We will return this when we analyse the conditions under which we know that one and just one such fixed point exists (Contraction mapping theorem).

Typically the value function is of a similar for to the objective function. This is intuitive in the light of (6.14). For example if the utility function in (6.1) is logarithmic we guess that the value function is of the form $A \ln k + B$ for some constants A,B. For HARA utility functions (e.g., CRRA, CARA and quadratic) the value functions are generally of the same type as the utility function (Merton, 1971).

Iteration

An alternative way is try to find the limit of finite horizon Bellman equation as the horizon goes to infinity. Under for economical purposes quite general conditions this limit exists and is equal to the value function for the infinite horizon problem

$$\lim_{s\to\infty} V_s(k_{T-s}) \equiv V(k) \tag{6.17}$$

Using the notation in (6.16) we apply the operator T until the sequence converges.

$$V(k) = \lim_n T^n V(k) \tag{6.18}$$

if the limit exists. In this case we can be sure that this satisfies the Bellman equation which when we use the formulation in (6.16) and the definition in (6.18) becomes

$$
\begin{aligned}
&V(k) = T(V(k)) \\
&\lim_n T^n V(k) = T \lim_n T^n V(k) \\
&\lim_n T^n V(k) = \lim_n T^{n+1} V(k)
\end{aligned}
\tag{6.19}
$$

With discounting it is typically unimportant what we plug in for $V_0(k)$ in (6.17). We can then start with any function and iterate until we get convergence. This can easily be done numerically, either by specifying a functional form, if we know that, or by just choosing a grid. In the latter case we just a set of values for the state variable $\{k_0, k_1, \ldots, k_n\}$. $V_0(k)$ is then a set of preliminary values (numbers) for each of the state variables in the grid.

An Iteration Example

In (6.1) let $U(c)$=ln(c) and $f(k)$= k^{α} with $0<\alpha<1$.We then have

$$
\begin{aligned}
&V_1(k_{T-1}) = \max_{k_T} \ln\left(k_{T-1}^{\alpha} - k_T\right) + \beta \ln k_T^{\alpha} \\
&FOC \quad \frac{1}{\left(k_{T-1}^{\alpha} - k_T\right)} = \frac{\beta\alpha}{k_T} \Rightarrow k_T = \frac{\beta\alpha k_{T-1}^{\alpha}}{1+\alpha\beta}
\end{aligned}
\tag{6.20}
$$

Substitute into the value function

$$
\begin{aligned}
V_1(k_{T-1}) &= \ln\left(\frac{1}{1+\alpha\beta} k_{T-1}^{\alpha}\right) + \beta \ln\left(\frac{k_{T-1}^{\alpha}}{1+\alpha\beta}\right)^{\alpha} \\
&= -(1+\alpha\beta)\ln(1+\alpha\beta) + \alpha(1+\alpha\beta)\ln k_{T-1}
\end{aligned}
\tag{6.21}
$$

Then

$$
\begin{aligned}
&V_2(k_{T-2}) = \max_{k_{T-1}} \ln\left(k_{T-2}^{\alpha} - k_{T-1}\right) + \beta V_1(k_{T-1}) \\
&FOC \quad \frac{1}{k_{T-2}^{\alpha} - k_{T-1}} = \beta V_1'(k_{T-1}) = \beta\alpha(1+\alpha\beta)\frac{1}{k_{T-1}} \\
&\Rightarrow k_{T-1} = \frac{\beta\alpha(1+\alpha\beta)}{1+\beta\alpha(1+\alpha\beta)} k_{T-2}^{\alpha}
\end{aligned}
\tag{6.22}
$$

So

$$
\begin{aligned}
V_2(k_{T-2}) &= \ln\left(\frac{k_{T-2}^{\alpha}}{1+\beta\alpha(1+\alpha\beta)}\right) + \beta V_1\left(\frac{\beta\alpha(1+\alpha\beta)}{1+\beta\alpha(1+\alpha\beta)} k_{T-2}^{\alpha}\right) \\
&= \alpha\left(1+\alpha\beta+\alpha^2\beta^2\right)\ln k_{T-2} \\
&\quad +\left(\beta+\alpha\beta+\alpha^2\beta^2\right)\ln\alpha\beta \\
&\quad +(\alpha\beta+\alpha^2\beta^2-\beta-\beta^2)\ln(1+\alpha\beta) \\
&\quad -(1+\alpha\beta+\alpha^2\beta^2)\ln(1+\alpha\beta+\alpha^2\beta^2).
\end{aligned}
\tag{6.23}
$$

It is easy to see that the coefficient on k is a power series that converge to $\alpha/(1-\alpha\beta)$ when the horizon goes to infinity (provided $\alpha\beta<1$). Also the constants converge if also $0<\beta<1$ and the resulting function is

$$
\begin{aligned}
&\lim_{s\to\infty} V_s(k_{T-s}) \\
&\equiv V(k) = \frac{\alpha}{1-\alpha\beta}\ln k + \frac{\ln(1-\alpha\beta)}{1-\beta} + \frac{\alpha\beta\ln\alpha\beta}{(1-\beta)(1-\alpha\beta)} \\
&= A\ln k + B.
\end{aligned}
\tag{6.24}
$$

From this we can derive the optimal policy function by using the first order condition for the Bellman equation

$$\begin{aligned} &V(k_t) = \max_{k_{t+1}} \ln\left(k_t^{\alpha} - k_{t+1}\right) + \beta V(k_{t+1}) \\ &FOC \quad \frac{1}{k_t^{\alpha} - k_{t+1}} = \beta V'(k_{t+1}) = \frac{\beta A}{k_{t+1}} \\ &\Rightarrow k_{t+1} = \frac{\beta A}{1+\beta A} k_t^{\alpha} \end{aligned} \tag{6.25}$$

Note that the policy function is a stable difference equation under the assumptions about α, β.

Verification of Guess

If we had guessed the form $A \ln k + B$ the Bellman equation had become

$$V(k_t) = \max_{k_{t+1}} U\left(k_t^{\alpha} - k_{t+1}\right) + \beta V(k_{t+1}) \tag{6.26}$$

giving first order conditions

$$\begin{aligned} &U'\left(k_t^{\alpha} - k_{t+1}\right) = \beta V'(k_{t+1}) \\ &\frac{1}{k_t^{\alpha} - k_{t+1}} = \beta A \frac{1}{k_{t+1}} \Rightarrow k_{t+1} = \frac{A\beta}{1+A\beta} k_t^{\alpha} \end{aligned} \tag{6.27}$$

Plugging this into the Bellman equation yields

$$\begin{aligned} A \ln k_t + B &= \ln\left(k_t^{\alpha} - \frac{A\beta}{1+A\beta} k_t^{\alpha}\right) + \beta\left(A \ln \frac{A\beta}{1+A\beta} k_t^{\alpha} + B\right) \\ &= \alpha \ln k_t + \ln \frac{1}{1+A\beta} + \alpha\beta A \ln k_t + \beta A \ln \frac{A\beta}{1+A\beta} + \beta B \\ &= (\alpha + \alpha\beta A) \ln k_t + \ln \frac{1}{1+A\beta} + \beta A \ln \frac{A\beta}{1+A\beta} + \beta B \end{aligned} \tag{6.28}$$

which is satisfied if we set A and B to the values in (6.24).

State Variables

We often solve the dynamics programming problem by guessing a form of the value function. The first thing to determine is then which variables should enter, i.e., which variables are the state variables. The state variables must satisfy both following conditions

1. To enter the value function at time they must be realized at t.

   Note, however, that it sometimes may be convenient to use $E_t\ (z_{t+s})$ as a state variable. The expectation as of t is certainly realized at t even if the stochastic variable is not realized.

2. The set of variables chosen as state variables must together give sufficient information so that the value of the program from t and onwards when the optimal control is chosen can be calculated.

What do we need if the per period utility function in (6.1) was $U(c_t, c_{t-1})$?

Note, we should try to find the smallest such set. Look for example on the following problem.

$$
\begin{aligned}
&\max_{c,k} \sum_{t=0}^{T} \beta^t U(c_t) \\
&s.t. \quad k_0 = \bar{k}_0, \\
&\qquad\quad l_0 = \quad \bar{k}_0 \\
&\qquad\quad k_{t+1} + l_{t+1} = f(k_t, l_t) - c_t,\ t = 0, \ldots, T, \\
&\qquad\quad k_{T+1} = 0
\end{aligned}
\tag{6.29}
$$

In general we need both k, and l in the value function but if f is linear we may only need a linear combination. If $f(k_t, l_t) = a(k_t + l_t)$ we could define a new state variable $w = k+l$ and use $V(w)$ as our value function. The reason is that to compute the value of the program we only need to know the sum of k and l, their share are superfluous information.

6.2. Contraction mappings

In the previous section we discussed guessing on solutions to the Bellman equation. However, we would like to know whether there exists a solution and whether it is unique. If the latter is not the case, it is not in principle sufficient to guess and the verify the solution since we might have other value functions that also satisfy the Bellman equation. To prove existence and uniqueness we will apply a contraction mapping argument.

Complete Metric Spaces and Cauchy Sequences

Let X be a metric space, i.e., a set on which addition and scalar multiplication is defined. Also define an operator $d\colon\ X \times X \to \mathbf{R}$ which we can think of as measuring the (generalized) distance between any two elements of X. We call d a norm. It is assumed to satisfy

1. Positivity $\begin{array}{l} d(x,y) \geq 0 \\ d(x,y) = 0 \Leftrightarrow x = y \end{array}$

2. Symmetry $d(x,y) = d(y,x)$ (6.30)

3. Triangle inequality $d(x,z) \leq d(x,y) + d(y,z)$

Now, we call (X,d) a normed vector space or a *metric space*. An example of such a space would be R^n together with the Euclidian norm $d(x,y) = \|x,y\|$. Another example is the space $C(S)$ of continuous and bounded functions where each element is a function from $S \subset \mathbf{R}^n \to \mathbf{R}$ together with the "sup-norm" defined as follows. For any two elements in $C(S)$, i.e., any two functions f and g, the distance d between them is the maximal euclidian distance, i.e,.,

$$d(f,g) \equiv \sup_{y \in S} \|f(y), g(y)\| \tag{6.31}$$

Now let us define a Cauchy sequence. This is a sequence of elements $\{x_n\}$ in a space X that come closer and closer to each other, using some particular norm. More precisely, for all $\varepsilon>0$, there exist a number n, such that for all $m,p \geq n$, $d(x_m,x_p) < \varepsilon$. An example of this would be the sequence $\{1,1/2,1/3,\ldots\}$ which is a Cauchy sequence using the Euclidian norm. A Cauchy sequence converges if there is an element in X such that $d(x_n,x)$ approach zero as n goes to infinity. It may, of course, be the case that the Cauchy sequence does not converge to a point in X. An eaxmple would be if we let X be the open interval $(0,\infty)$ and look at the Cauchy sequence $\{1,1/2,1/3,\ldots\}$ which converges to zero which is not in X.

Complete metric spaces

Now we are ready to define the complete metric space. This is a metric space where all Cauchy sequences in it are convergent, i.e., they converge to a point in the space.

Contraction Mapping

Consider the metric space (X,d) and look at the function T that maps each element in X to some element in X, $T: X \to X$. T is a contraction mapping if there exists a non-negative number ρ which is *strictly* smaller than unity, $0 \leq \rho < 1$, such that for all elements x, y in X,

$$d\left(T(x), T(y)\right) \leq \rho d\left(x, y\right) \tag{6.32}$$

An example of such a mapping whould we a map in say scale 1:10 000 put on top of a map in scale 1:1000 covering the same geographical area. The norm can be the distance between the points on the map. Clearly, (6.32) is satisfied for ρ =0.1.

The Contraction Mapping Theorem

Now we can state the very important contraction mapping theorem.

Result 22 Consider a complete metric space, and let $T: X \to X$ be a contraction mapping. The T has one unique fixed point x, *i.e.*, $x=T(x)$.

Another very useful result is the following

Result 23 Let S be a subset of $\mathbf{R}^n$ and $B(S)$ the space of all bounded functions from S to $\mathbf{R}$. Let T be a map that maps all elements of $B(S)$ into itself. Then, T is a contraction mapping if

1. For any functions $w(s)$ and $v(s)$ $w(s) - v(s) \geq 0, \forall s \in S \Rightarrow Tw(s) - Tv(s) \geq 0, \forall s \in S$, and

2. There is a non-negative $0 \leq \beta\,\beta$ strictly smaller than unity such that for any number c in $\mathbf{R}$, and any function w in $B(S)$, $T(w(s)+c) = T(w(s)) + \beta c, \forall s.$

Usually it is straightforward to apply the previous result to show that if we have positive discounting the Bellman equation is a contraction mapping. The only problem is that it is confined to bounded functions.

6.3. Stochastic Dynamic Programming

As long as the recursive structure of the problem is intact adding a stochastic element to the transition equation does not change the Bellman equation. Consider the problem

$$\begin{aligned}
&\max_{\{u_t\}_0^\infty} E\sum_{t=0}^{\infty}\beta^t r(k_t,u_t)\\
&s.t. \qquad k_0=\bar{k}_0,\\
&\qquad\quad\ k_{t+1}=g(k_t,u_t,\varepsilon_{t+1}),\ \forall t\geq 0.\\
&\qquad\quad\ \varepsilon_t=\begin{cases}\bar{\varepsilon}, & \text{with probability } p\\ \underline{\varepsilon}, & \text{with probability (1- } p)\end{cases}
\end{aligned} \tag{6.33}$$

where E is the expectations operator. Note that we have to specify the set of information that u_t can be conditioned on. Clearly it will in general be optimal to condition for example consumption on observed realizations of ε_t. If the agent may condition on information available at t we get the Bellman equation with first order conditions

$$\begin{aligned}
&V(k_t)\equiv\max_{u_t}\left\{r(k_t,u_t)+\beta\left[pV(g(k_t,u_t,\bar{\varepsilon}))+(1-p)V(g(k_t,u_t,\underline{\varepsilon}))\right]\right\}\\
&FOC\\
&r_u(k_t,u_t)+\\
&\beta\left[pV'(g(k_t,u_t,\bar{\varepsilon}))g_u(k_t,u_t,\bar{\varepsilon})+(1-p)V'(g(k_t,u_t,\underline{\varepsilon}))g_u(k_t,u_t,\underline{\varepsilon})\right]\\
&=0
\end{aligned} \tag{6.34}$$

or for a general distribution F of ε

$$\begin{aligned}
&V(k_t)\equiv\max_{u_t}\left\{r(k_t,u_t)+\beta EV(g(k_t,u_t,\bar{\varepsilon}))\right\}\\
&FOC \quad r_u(k_t,u_t)+\beta E\left(V'(g(k_t,u_t,\varepsilon))g_u(k_t,u_t,\varepsilon)\right)=0
\end{aligned} \tag{6.35}$$

where E denotes the expectations operator. Note that $V(k_t)$ in (6.34) and (6.35) is a current value function.

A Stochastic Consumption Example

Consider the following program

$$\begin{aligned}
&\max_{c,\omega}\sum_{t=0}^{\infty}\beta^t\ln c_t\\
&s.t. \qquad A_{t+1}=(A_t-c_t)\left((1+r)\omega+(1+z_t)(1-\omega)\right).
\end{aligned} \tag{6.36}$$

The consumer decides how much to consume each period. The share ω of here assets is placed in a riskless asset yielding r in return and (1-ω) in a risky asset with return z_t, that is i.i.d.

The problem is autonomous so we write the current value Bellman equation with time independent value function V

$$V(A_t) = \max_{c_t,\omega}\left[\ln c_t + \beta E_t V\left((A_t - c_t)\left((1+r)\omega + (1+z_t)(1-\omega)\right)\right)\right] \tag{6.37}$$

Necessary first order conditions yield

$$\begin{aligned} &c_t; \qquad \frac{1}{c_t} - \beta E_t V'(A_{t+1})\left((1+r)\omega + (1+z_t)(1-\omega)\right) = 0, \\ &\omega_t; \qquad E_t V'(A_{t+1})(A_t - c_t)(r - z_t) = 0. \end{aligned} \tag{6.38}$$

Now we use Merton's result and guess that the value function is

$$V(A_t) = a \ln A_t + B \tag{6.39}$$

for some constants a and B. Substituting into (6.38) we get

$$\begin{aligned} &\frac{1}{c_t} = \beta E_t a \frac{1}{A_{t+1}}\left((1+r)\omega + (1+z_t)(1-\omega)\right) \\ &= \beta a \frac{1}{(A_t - c_t)} \Rightarrow c_t = \frac{A_t}{1 + a\beta}. \end{aligned} \tag{6.40}$$

and

$$\begin{aligned} &E_t V'(A_{t+1})(A_t - c_t)(r - z_t) \\ &= E_t \frac{\cancel{(A_t - c_t)}(r - z_t)}{\cancel{(A_t - c_t)}\left((1+r)\omega + (1+z_t)(1-\omega)\right)} \\ &= E_t \frac{(r - z_t)}{\left((1+r)\omega + (1+z_t)(1-\omega)\right)} = 0. \end{aligned} \tag{6.41}$$

Note that (6.41) implies that ω is constant since z_t is i.i.d.

Now we have to solve for the constant a. This is done by substituting the solutions to the first order conditions and the guess into the Bellman equations.

$$
\begin{aligned}
& a\ln A_t + B \\
&= \ln A_t - \ln(1+a\beta) + \beta E_t\left(a\ln A_{t+1} + B\right) \\
&= \ln A_t - \ln(1+a\beta) + \beta a \ln\left(A_t - c_t\right) + \beta B \\
&\quad + \beta a E_t \ln\left((1+r)\omega + (1+z_t)(1-\omega)\right) \\
&= \ln A_t - \ln(1+a\beta) + \beta a(\ln A_t + \ln a\beta - \ln(1+a\beta)) \\
&\quad + \beta B + \beta a E_t \ln\left((1+r)\omega + (1+z_t)(1-\omega)\right) \\
&= (1+a\beta)\ln A_t + k \\
&\Rightarrow a = \frac{1}{1-\beta}, \quad c_t = (1-\beta)A_t
\end{aligned}
\tag{6.42}
$$