MATFUII

Class Notes

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1. Introduction

This course is about dynamic systems, i.e., systems that evolve over time. The first task will be to simply describe how the system evolves, given some law of motion. Let us take a simple example.

1.1. A Simple Differential Equation

Consider a ball kicked up in the air. Define the height it has over the ground at time t as $h(t)$. When it is in the air the gravity affects it so it accelerates towards the ground. This gives us a very simple second order differential equation:

$$\begin{aligned} \ddot{h}(t) &= g, \\ g &\equiv -9.81\frac{m}{s^2} \end{aligned} \tag{1.1}$$

We could immediately see that the following function satisfies the differential equation

$$h(t) = \frac{gt^2}{2} + c_1 t + c_2 \tag{1.2}$$

for any constants c_1 and c_2.

It is clear that we need to know more to exactly characterize a particular ball's flight in the air. For example, if we know that it leaves the ground at time 0 and at an upward speed of 25 *m/s*, this should be enough, So we have

$$\begin{aligned} \dot{h}(0) &= g \cdot 0 + c_1 = 25 \Rightarrow c_1 = 25 \\ h(0) &= \frac{g \cdot 0^2}{2} + c_1 0 + c_2 = 0 \Rightarrow c_2 = 0 \end{aligned} \tag{1.3}$$

So the ball's height over the ground is given by

$$h(t) = \frac{gt^2}{2} + 25t \tag{1.4}$$

Note the units.

(1.4) is the solution to the problem. It can be used to, for example, calculating how the ball will reach.

$$\max_t h(t) = \frac{gt^2}{2} + 25t$$
$$FOC;\ gt + 25 = 0 \Rightarrow t = \frac{-25}{g} \approx 2.5 \tag{1.5}$$
$$\Rightarrow \max_t h(t) \approx \frac{g \cdot 2.5^2}{2} + 25 \cdot 2.5 \approx 31.25m$$

Solving problems of this type, in both continuous and discrete time, is the first topic of the course.

When we have learnt how to describe how a system evolves, we want to know how to optimize in some dimension given the restrictions given by the law of motion of the system. This is the topic of the second part of the course.

1.2. Approaches to Dynamic Optimization

The issue is to choose from a set of admissible controls of the path for $x(t)$ for $t \in [t_0,T]$ the one that maximizes a given objective *functional* which associates a particular value to each admissible path $V[x(t)]$.

Example: resource (oil) extraction. Choose an extraction plan $u(t)$ stating how extraction effort to use each point in time. This determines how much oil is pumped up, i.e., $-\dot{x}(t) = g(x(t),u(t))$ where $x(t)$ is the remaining oil left in the well. To be admissible $x(t)$ and has to satisfy:

$$\begin{aligned} &x(0) = K, \\ &-\dot{x}(t) = g(x(t),u(t)) \geq 0, \forall t, \\ &x(T) \geq 0. \end{aligned} \tag{1.6}$$

The problem is dynamic, since actions today affect profit opportunities in the future. Given the economic environment, $x(t)$ determines the remaining profit opportunities. Furthermore, $x(t)$ is given by the history and cannot jump since only its rate of change can be controlled. Such variables are called *state variables.*

The value (or objective) function associates a number to each path. For example;

$$\begin{aligned} V[x(t)] &= \int_0^T e^{-rt}\left(p(\dot{x}(t)) - c\left(x(t),u(t)\right)\right)dt \\ s.t. - \dot{x}(t) &= g(u(t),x(t)) \\ &= \int_0^T F\left(t, x(t), \dot{x}(t)\right)dt. \end{aligned} \tag{1.7}$$

Two approaches in this course:

1. Dynamic Programming (Bellman).

$$
\begin{aligned}
&V_1(x_1) = \max_{\{u_t\}_1^T} \sum_{t=1}^{T} F(t,x_t,u_t) \\
&s.t.\ x_{t+1} = g(t,x_t,u_t) \\
&x_1, x_T \text{ given.}
\end{aligned} \tag{1.8}
$$

Idea; splitting problem into sub-problems. Define;

$$
\begin{aligned}
&V_s(x_s) = \max_{\{u_t\}_s^T} \sum_{t=s}^{T} F(t,x_t,u_t) \\
&s.t.\ x_{t+1} = g(t,x_t,u_t), x_s, x_T \text{ given.}
\end{aligned} \tag{1.9}
$$

Bellman's principle of optimality says

$$
V_s(x_s) = \max_{u_s} F(s,x_s,u_{s+1}) + V_{s+1}(g(t,x_t,u_t)) \tag{1.10}
$$

If we know the functions V_s, it is easy to find the optimum policy from. If the soultion is interior, we use FOC

$$
F_u(s,x_s,u_s) + V'_{s+1}(g(t,x_t,u_t))g_u(t,x_t,u_t) = 0 \quad . \tag{1.11}
$$

2. Optimal Control (Pontryagin)

$$
\begin{aligned}
&\max_{u(t)} \int_0^T F(t,x(t),u(t))dt \\
&s.t.\ \dot{x}(t) = g(t,x(t),u(t)), x(0), x(T) \text{ given.}
\end{aligned} \tag{1.12}
$$

Idea; optimum is found by each point in time maximize

$$
\max_{u(t)} F(t,x(t),u(t)) + \lambda(t)g(t,x(t),u(t)), \tag{1.13}
$$

where $\lambda(t)$ is the shadow value of the state variable. If we know $\lambda(t)$ the optimum policy is easy to find from, for example, FOC,

$$
F_u(t,x(t),u(t)) + \lambda(t)g_u(t,x(t),u(t)) = 0, \tag{1.14}
$$

Note the similarity between (1.11) and (1.14).

Before starting, we need to establish some basic concepts.