2. Some Basics*

2.1. Integration

If $b>a$, the expression

$$\int_a^b f(t)dt \tag{2.1}$$

can be interpreted as the area under the graph $y=f(x)$ for $x \in [a,b]$. How should one compute such an area? The most natural way would be to divide the interval $[a,b]$ into (many) sub-intervals by choosing numbers $a = t_1 < t_2 < t_3 \ldots t_n = b$. Then we can approximate the area by

$$\sum_{i=1}^{n-1} f(t_i)(t_{i+1} - t_i) \tag{2.2}$$

Loosely speaking, if we let n go to infinity and each interval $[t_{i+1}, t_i]$ go to zero, we can hope that this approximation become a perfect approximation. If this is true, the function $f(t)$ is (Riemann) integrable. (See, Klein for a more formal treatment).

Now we can state the result:

Result 1 If $f(t)$ is bounded and continuous, except possibly on a countable number of points, on the compact interval $[a,b]$, the $f(t)$ is (Riemann) integrable. (See Klein.)

If we can handle integration over compact intervals, we can also define integrals over unbounded intervals by taking the limit value as integration limits (a, and/or b) approach infinity. Of course, this limit does not always exist.

2.1.1. The fundamental theorem of calculus.

An integral is a generalization of a sum, and a derivative is a generalization of a difference. The following theorem links these concepts. The first part of the *Fundamental Theorem* says that if

$$F(b) = \int_a^b f(t)dt \tag{2.3}$$

then

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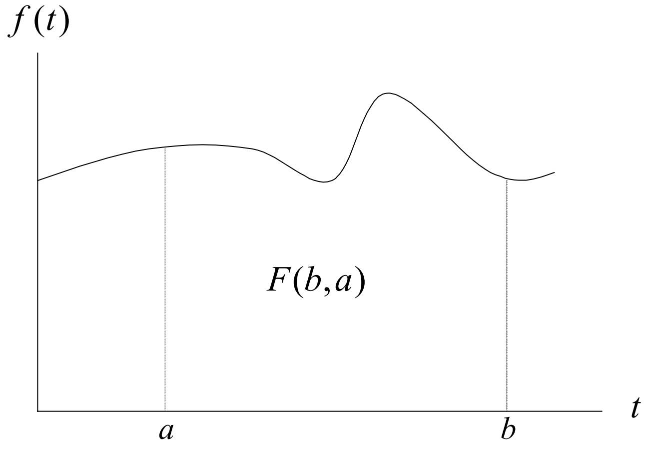
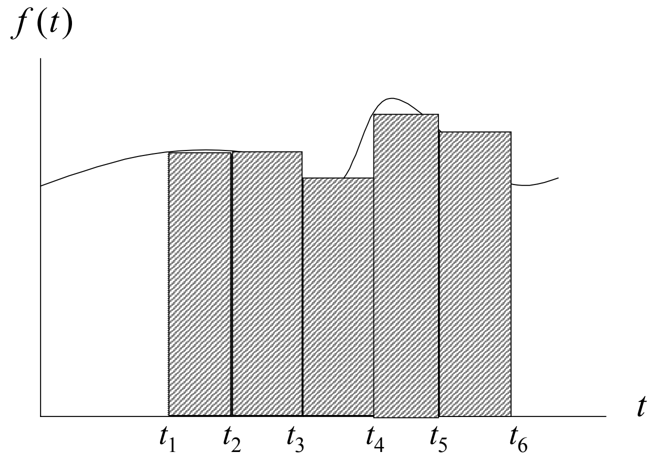
* Last changed on 3-Nov-98.

Result 2

$$F'(b) = \frac{d}{db}\int_a^b f(t)dt = f(b) \tag{2.4}$$

So in order to find a function that gives the area under $f(t)$, we must find a function that has $f(t)$ as first derivative.





Convince yourself that this is reasonable by making a drawing. A function $F(b)$ that has the property that $F'(b) = f(b)$ is called a primitive for f. Clearly, if $F(b)$ is a primitive for $f(b)$, then also $F(b)$ plus any constant, is a primitive.

Now let us turn to he second part of the theorem, which provides a way of calculating the exact value of an integral like (2.1). Let F be any primitive of f, then

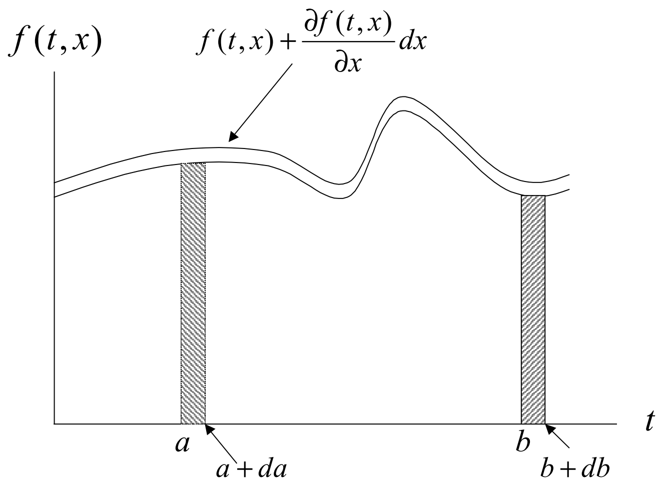
Result 3
$$\int_a^b f(t)dt = F(b) - F(a) \equiv \left[F(t)\right]_a^b \tag{2.5}$$

Sometimes it is easy to find the primitive, like in the cases $f(t) =$, t^{α}, $1/(at+b)$ or e^{at}, in which case $F(t)$ is $(t^{\alpha+1})/(\alpha+1)$, $\ln(at+b)/a$, $(e^{at})/a$. In other cases like $\pi^{0.5}e^{-x^2}$, the primitive cannot be expressed by using standard algebraic functions. This does *not* mean that the primitive does not exist. In the case $\pi^{0.5}e^{-x^2}$, a primitive is the cumulative normal distribution, which certainly exists.

Since the integral is a (kind of) sum, it is straightforward to understand *Liebniz'* rule.

Result 4 If f is continuous and u and v differentiable, then

$$\frac{\partial}{\partial x}\int_a^b f(t,x)dt = \int_a^b \frac{\partial f(t,x)}{\partial x}dt$$
$$\frac{\partial}{\partial b}\int_a^b f(t)dt = f(b) \tag{2.6}$$
$$\frac{\partial}{\partial a}\int_a^b f(t)dt = -f(a)$$



2.1.2. Change of variables

Suppose that $y=g(x)$, then the rules of differentiation gives $dy=g'(x)dx$. Now let us calculate the area under some function $f(y)$ but integrating over $x= g^{-1}(y)$ In a sense, this is like changing the scale of the x-axis. To do this, we simply substitute, $y=g(x)$, $dy=g'(x)dx$, $y=g(x)=$a $\Rightarrow x=g^{-1}(a)$ and , $y=g(x)=b \Rightarrow x=g^{-1}(b)$

Result 5
$$\int_a^b f(y)dy = \int_{g^{-1}(a)}^{g^{-1}(b)} f\left(g(x)\right)g'(x)dx \tag{2.7}$$

2.1.3. Integration by parts

Another important result, which we will use a lot, is the following rule for integration by parts. Assume we want to integrate a product of two functions of *x, i.e.,* $u(x)v(x)$. Then, let $U(x)$ be a primitive of *u.* The rule for differentiation of products says

$$\begin{aligned} &\frac{d}{dx}\left[U(x)v(x)\right] = U'(x)v(x) + U(x)v'(x) = u(x)v(x) + U(x)v'(x) \\ &\Rightarrow u(x)v(x) = \frac{d}{dx}\left[U(x)v(x)\right] - U(x)v'(x) \end{aligned} \tag{2.8}$$

Then we can integrate over *x* on both sides which yields

Result 6

$$\int_a^b u(x)v(x)dx = \left[U(x)v(x)\right]_a^b - \int_a^b U(x)v'(x)dx\,. \tag{2.9}$$

2.1.4. Double Integration

As we know, the integral $\int_a^b f(t)dt$ is an area under a curve $f(t)$ over an interval $[a,b]$ in the *t*-dimension. Similarly, we can compute the *volume* under a plane with a height $f(t,s)$ over an area in the *t,s* dimension. For example, let $f(t,s)$ be $t{\cdot}s$ and integrate over a rectangle with sides $[a_t,b_t]$ and $[a_s,b_s]$

$$\begin{aligned} &\int_{a_s}^{b_s}\int_{a_t}^{b_t} f(t,s)dtds = \int_{a_s}^{b_s}\left(\int_{a_t}^{b_t} t\cdot s\, dt\right)ds \\ &= \int_{a_s}^{b_s}\left[\frac{st^2}{2}\right]_{a_t}^{b_t} ds = \int_{a_s}^{b_s} s\left(\frac{b_t^{\,2}}{2} - \frac{a_t^{\,2}}{2}\right)ds \\ &= \left(\frac{b_t^{\,2}}{2} - \frac{a_t^{\,2}}{2}\right)\left[\frac{s^2}{2}\right]_{a_s}^{b_s} = \left(\frac{b_t^{\,2}}{2} - \frac{a_t^{\,2}}{2}\right)\left(\frac{b_s^{\,2}}{2} - \frac{a_s^{\,2}}{2}\right) \end{aligned} \tag{2.10}$$

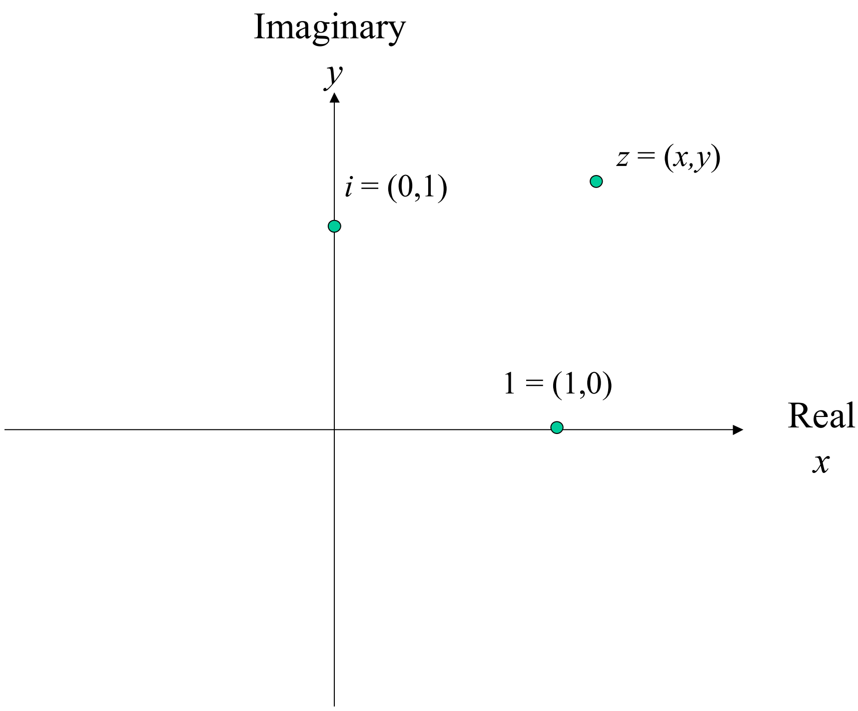
Note, that we are first calculating the area under $f(t,s)$ over the interval $[a_t,b_t]$ for each *s*. This area is a function of *s* which then is integrate over the interval $[a_s,b_s]$ in the *s*-dimension. If we integrate over other areas than rectangles, the limits of integration are not independent. For example, we may integrate over a triangle where $b_t = s$.If $a_t = a_s =$, we have

$$
\begin{aligned}
&\int_0^{b_s}\int_0^{s} f(t,s)\,dt\,ds = \int_0^{b_s}\left(\int_0^{s} t\cdot s\,dt\right)ds \\
&= \int_0^{b_s} s\left[\frac{t^2}{2}\right]_0^{s} ds = \int_0^{b_s} s\left(\frac{s^2}{2}\right)ds \\
&= \left[\frac{s^4}{8}\right]_0^{b_s} = \frac{b_s{}^4}{8}
\end{aligned}
\tag{2.11}
$$

2.2. Complex numbers and trigonometric functions

The formula for the solution to a quadratic equation $ax^2 + bx + c = 0$ is $x = \left(-b \pm \sqrt{b^2 - 4ac}\right)\Big/2a$.

If $(b^2 - 4ac) < 0$. The solution is not in the set of real numbers. The introduction of complex numbers intended to extend the space of solutions to accommodate such cases and it turns out that for all numbers in the extended space, we can always find solutions to such equations. We can think of complex numbers as two-dimension objects $z=(x,y)$. The first number, x, provides the value in the standard real dimension, while the second provides the value in the other dimension, called *imaginary*.



The rules for adding and multiplication with complex numbers are the following:

$$
\begin{aligned}
z_1 + z_2 &= (x_1 + x_2\,, y_1 + y_2) \\
z_1 z_2 &= (x_1 x_2 - y_1 y_2\,, x_1 y_2 + x_2 y_1)
\end{aligned}
\tag{2.12}
$$

Now let us compute the square of a complex number only consisting of a unitary imaginary part, i.e., $z=i$

$$(0,1)^2 = (0\cdot 0 - 1\cdot 1, 0\cdot 1 + 1\cdot 0) = (-1,0) = -1 \qquad (2.13)$$

We thus established the important

Result 7 A solution to the equation x^2=-1 is i.

Using the rules in (2.12), it is also straightforward to show that an alternative we of writing z is given by the following

$$z \equiv (x,y) = (x,0) + (0,y) = x(1,0) + y(0,1) = x + iy \qquad (2.14)$$

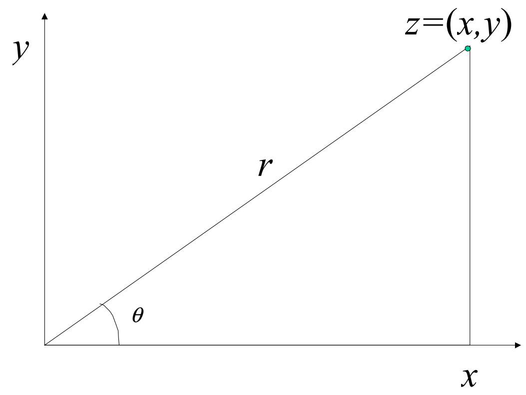
We should also note that

$$(x,y)(x,-y) = (x^2 + y^2, -xy + yx) = (x^2 + y^2, 0) = |(x,y)|^2 \qquad (2.15)$$

The numbers (x,y) and $(x,-y)$ are called complex conjugates and the value |x,y| is the modulus of (x,y).

2.2.1. Polar representation

Recall that in the right-angled triangle in the figure, we have.



$$\cos(\theta) = \frac{x}{r}$$
$$\sin(\theta) = \frac{y}{r} \qquad (2.16)$$
$$r = \sqrt{x^2 + y^2}$$

This means that we can represent the complex number z either by its coordinates, (x,y) or alternatively as $r(\cos(\theta)+i\sin(\theta))$. This is called the polar representation and r is the modulus and θ is the argument ($\theta \equiv \arg(z)$). Note that θ is measured in radians.

We will use the following result below.

Result 8 If z is the complex number (x,y) , then

$$z = re^{i\theta} \tag{2.17}$$

where r is $|z|$ and $\theta \equiv \arg(z)$. In particular, for $r = 1$ we get,

$$\begin{aligned} e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\ e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta) \\ e^{i\pi} &= \cos(\pi) + i\sin(\pi) = (-1,0) = -1. \end{aligned} \tag{2.18}$$

Optional proof:

The Taylor formula around zero is,

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}... \tag{2.19}$$

Using this for $f(x)=e^x$,$\cos(x)$ and $\sin(x)$, we have respectively

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}... \\ \cos(x) &= \cos(0) - \sin(0)x - \cos(0)\frac{x^2}{2!} + \sin(0)\frac{x^3}{3!} + \cos(0)\frac{x^4}{4!} - \sin(0)\frac{x^5}{5!} - \cos(0)\frac{x^6}{6!}... \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}... \\ \sin(x) &= \sin(0) + \cos(0)x - \sin(0)\frac{x^2}{2!} - \cos(0)\frac{x^3}{3!} + \sin(0)\frac{x^4}{4!} + \cos(0)\frac{x^5}{5!} - \sin(0)\frac{x^6}{6!} - \cos(0)\frac{x^7}{7!}... \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}... \end{aligned} \tag{2.20}$$

Thus, using the Taylor formula around zero to evaluate $f(i\theta)=e^{i\theta}$,

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!}.... \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!}... \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!}... \\ &\quad + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^6}{6!}...\right) \\ &= \cos(\theta) + i\sin(\theta) \end{aligned} \tag{2.21}$$

2.2.2. Polynomials

A polynomial P(z) of order n has the form

$$P(z) \equiv a_n z^n + a_{n-1} z^{n-1} + \ldots a_1 z + a_0 \tag{2.22}$$

where the *a's* are parameters of the polynomial. The following will be important for our analysis of difference and differential equations.

Result 9 A polynomial $P(z)$ of order n has exactly n roots. I.e., it can be expressed as

$$P(z) = a_n (z - r_1)(z - r_2)\ldots(z - r_n) \tag{2.23}$$

From(2.23), we see that each root r_i is a solution to the equation $P(z) = 0$. Note, however, that the roots may be repeated, i.e, $r_i = r_j$. Consider for example the polynomial

$$z^2 - 4z + 4 = (z - 2)(z - 2) \tag{2.24}$$