3. Differential Equations

3.1. Linear Differential Equations of First Order

A first order differential equation is an equation of the form

$$\frac{dx(t)}{dt} \equiv \dot{x}(t) = F(x(t),t) \tag{3.1}$$

As noted above, there will in general be a whole class of functions $x(t;c)$ (parameterized by c) that satisfies (3.1). We need more information, like an initial condition for x_{t0}, pin down the solution exactly.

Result 10 Given that F is continuous and has continuous first derivatives, there is going to be a *one* function $x(t)$ that satisfies (3.1) and the initial condition.

3.1.1. The Simplest Case

If F is independent of x, the solution is trivial. Then since

$$\dot{x}(t) = f(t) \tag{3.2}$$

the class of functions satisfying this is a primitive function of f plus any constant, i.e.,

$$x(t) = \int_{t_0}^{t} f(s)ds + \tilde{c} = F(t) - F(t_0) + \tilde{c}, \tag{3.3}$$

where $dF(t)/dt \equiv f(t)$. Note that $F(t_0)$ is a constant. We can thus merge the two constants and write

$$x(t) = F(t) + c, \tag{3.4}$$

Certainly, for all c, (3.4) satisfies (3.2) or, equivalently, for any $\tilde{c}$, (3.3) satisfies (3.2). For example,

$$\begin{aligned} f(t) &= e^{at} \\ F(t) &= \frac{e^{at}}{a} \end{aligned} \tag{3.5}$$

The arbitrary constant is pinned down with some other piece of information. So, if we want to find $x(t)$ and we know the value of $x(0)$, we get

$$\begin{aligned} x(t) &= F(t) + c = F(t) - F(0) + \tilde{c} \\ &= \frac{e^{at}}{a} + c = \frac{e^{at}}{a} - \frac{1}{a} + \tilde{c} \end{aligned} \tag{3.6}$$

Where $c = x(0) + 1/a$.This satisfies $\dot{x}(t) = e^{at}$ and $x(0)=x(0)$.

A note on notation

Above, we have used the proper notation of an integral, where both the lower and upper limits and the dummy variable to integrate over have separate names and are all written out. Often, a more sloppy notation is used,

$$\int_{t_0}^{t} f(s)ds \equiv \int f(t)dt \equiv F(t) \tag{3.7}$$

where it is understood that

$$\frac{d}{dt}\int f(t)dt = f(t)\,. \tag{3.8}$$

This notation, called an indefinite integral, saves on variables, but can be confusing. Nevertheless, I will use it sometimes. Using this notation in (3.4), we write

$$x(t) = \int f(t)dt + \tilde{c}, \tag{3.9}$$

3.1.2. Linear First Order Differential Equations

Very often, a solution to a more complicated differential equation is derived by transforming the original equation into something that has the form of (3.2). Linear first order differential with constant coefficients equations can be solved directly using such a transformation. Consider

$$\dot{y}(t) + Py(t) = Q \tag{3.10}$$

In this case, we multiply both sides by e^{Pt} (often called the integrating factor). After doing that, note that the LHS becomes

$$e^{Pt}\left(\dot{y}(t) + Py(t)\right) = e^{Pt}\dot{y}(t) + Pe^{Pt}y(t) = \frac{d\left(e^{Pt}y(t)\right)}{dt} \tag{3.11}$$

Thus, thinking of $e^{Pt}y(t)$ as simply a function of *t,* as *x*(*t*) in (3.2), we get a LHS that is the time derivative of a known function of *t* and the RHS is also a function only of *t.* Then the solution is found as in (3.3).

$$
\begin{aligned}
&\frac{d}{dt}\left(e^{Pt}y(t)\right)=e^{Pt}Q\\
&e^{Pt}y(t)=Q\int_{t_0}^{t}e^{Ps}ds=Q\left[\frac{e^{Ps}}{P}\right]_{t_0}^{t}+c\\
&y(t)=\left(Q\frac{e^{Pt}}{P}-Q\frac{e^{Pt_0}}{P}+c\right)e^{-Pt}\\
&=\frac{Q}{P}+\left(c-Q\frac{e^{Pt_0}}{P}\right)e^{-Pt}\\
&=\frac{Q}{P}+\tilde{c}e^{-Pt}
\end{aligned}
\tag{3.12}
$$

If we know $y(t)$ at some point in time, e.g. at t=0.

$$
y(0)=y\Rightarrow\frac{Q}{P}+\tilde{c}e^{-P0}=y\Rightarrow\tilde{c}=y-\frac{Q}{P}
\tag{3.13}
$$

Note that there is only one degree of freedom in the constants c and t_0. Choosing another t_0 simply means that the constant c has to chosen in another way. Thus, one piece of information is sufficient to pin down the solution exactly.

3.1.3. Variable R.H.S.

Assume the R.H.S. is a function of t. Use the same method as above.

$$
\begin{aligned}
&\dot{y}(t)+Py(t)=Q(t)\\
&\frac{d}{dt}\left(e^{Pt}y(t)\right)=e^{Pt}Q(t)\\
&e^{Pt}y(t)=\int e^{Pt}Q(t)dt+c
\end{aligned}
\tag{3.14}
$$

E.g., if $Q(t)$=t

$$
\begin{aligned}
&e^{Pt}y(t)=\int_{t_0}^{t}se^{Ps}ds+\tilde{c}=\left[s\frac{e^{Ps}}{P}\right]_{t_0}^{t}-\int_{t_0}^{t}\frac{e^{Ps}}{P}ds+\tilde{c}=\left[s\frac{e^{Ps}}{P}\right]_{t_0}^{t}-\left[\frac{e^{Ps}}{P^2}\right]_{t_0}^{t}+c\\
&=t\frac{e^{Pt}}{P}-\frac{e^{Pt}}{P^2}+c,\Rightarrow y(t)=\frac{t}{P}-\frac{1}{P^2}+ce^{-Pt}
\end{aligned}
\tag{3.15}
$$

where we had to use integration by parts.

3.1.4. Variable Coefficients

If also the coefficient on $y(t)$ is variable we have to use a more general integrating factor to make the L.H.S. into the time differential of a known function. Here the integrating factor is

$$
e^{\int_{t_o}^{t}P(s)ds}
\tag{3.16}
$$

with

$$\frac{d}{dt}e^{\int_{t_o}^{t} P(s)ds} = P(t)e^{\int_{t_o}^{t} P(s)ds} \tag{3.17}$$

Using the sloppy notation:

$$\begin{aligned} &\dot{y}(t) + P(t)y(t) = Q(t) \\ &\frac{d}{dt}\left(e^{\int P(t)dt} y(t)\right) = e^{\int P(t)dt} Q(t) \\ &e^{\int P(t)dt} y(t) = \int e^{\int P(t)dt} Q(t)dt + c \end{aligned} \tag{3.18}$$

Check that you can verify that

$$\frac{d}{dt}\left(e^{\int_{t_0}^{t} P(s)ds} y(t)\right) = e^{\int_{t_0}^{t} P(s)ds} P(t)y(t) + e^{\int_{t_0}^{t} P(s)ds} \dot{y}(t) \tag{3.19}$$

Note that this is a generalization of the constant coefficient case, since $e^{\int_{t_0}^{t} Psds} = e^{Pt+c}$.

Example; money on a bank account with variable interest rate.

$$\begin{aligned} &\dot{y}(t) = r(t)y(t) \\ &\frac{d}{dt}\left(e^{-\int r(t)dt} y(t)\right) = 0 \\ &\Rightarrow e^{-\int r(t)dt} y(t) = c \\ &y(t) = y(0)e^{\int_0^t r(t)dt} \neq y(0)e^{rt} \end{aligned} \tag{3.20}$$

Separating Variables

Sometimes, we can write a differential equation such that the LHS only contains functions of x and $\dot{x}$ and the RHS only a function of t.

$$\dot{x}g(x) = h(t) \tag{3.21}$$

Letting $G(x)$ be defined as a primitive of $g(x)$, i.e., from $G'(x) = g(x)$, we have

$$\frac{dG(x(t))}{dt} = G'(x(t))\dot{x} = \dot{x}g(x) = h(t)$$
$$\Rightarrow G(x(t)) = \int h(t)dt + c \tag{3.22}$$

We can recover *x* by inverting *G.* An example;

$$\begin{aligned} &\dot{x}(t)x^{-2} = t^2 \\ &g(x) = x^{-2} \\ &G(x) = -x^{-1} = \int t^2 dt + c = t^3/3 + c \\ &x = G^{-1}\left(t^3/3 + c\right) = -\left(t^3/3 + c\right)^{-1}. \end{aligned} \tag{3.23}$$

3.2. Linear Differential Equations of Higher Order

3.2.1. Linear Second Order Differential Equations

A linear second order differential equation has the form.

$$\ddot{y}(t) + P(t)\dot{y}(t) + Q(t)y(t) = R(t) \tag{3.24}$$

This cannot be solved directly by transformations in the simple way we did with first order equations. Instead we use a more general method (that would have worked above also).

Result 11 The general solution (the complete set of solutions) to a differential equation is the general solution to the homogeneous part plus any particular solution to the complete equation.

Result 12 The general solution of the homogeneous part of a second order linear differential equation can be expressed as $c_1 y_1(t) + c_2 y_2(t)$ where $y_1(t)$ and $y_2(t)$ are two linearly independent particular solutions to the homogeneous equations.

Two functions $y_1(t)$ and $y_2(t)$ are linearly independent in a region Ω if

$$\text{there is no } c_1, c_2 \neq \{0,0\}, \text{ s.t. } c_1 y_1(t) = c_2 y_2(t), \\ \forall t \in \Omega. \tag{3.25}$$

3.2.2. Homogeneous Equations with Constant Coefficients

A homogeneous (part of a) differential equation has a zero LHS when expressed as in (3.24);

$$\ddot{y}(t) + P\dot{y}(t) + Qy(t) = 0 \tag{3.26}$$

To solve this we start by solving the *characteristic equation*, which always is a polynomial, in this case

$$r^2 + Pr + Q = 0 \tag{3.27}$$

As we know this polynomial has exactly two roots.

$$r_{1,2} = -\frac{P}{2} \pm \frac{\sqrt{P^2 - 4Q}}{2} \tag{3.28}$$

Now we can check that both $e^{r_i t}$ are solutions to (3.26) by noting that.

$$y_{h_1}(t) = e^{r_i t}, \dot{y}_{h_1}(t) = r_i e^{r_i t}, \ddot{y}_{h_1}(t) = r_i^2 e^{r_i t} \tag{3.29}$$

so using (3.29) in (3.26) yields

$$r_i^2 e^{r_i t} + P r_i e^{r_i t} + Q e^{r_i t} = e^{r_i t}\left(r_i^2 + P r_i + Q\right) = 0 \tag{3.30}$$

So then, the general solution to the homogenous equation is

$$y_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{3.31}$$

provided that the two parts are linearly independent, which they are unless $r_1 = r_2$.

3.2.3. Complex Roots

The general solution to the homogenous equation is here found the same way.

$$\begin{aligned}
r_{1,2} &= -\frac{P}{2} \pm \frac{\sqrt{P^2 - 4Q}}{2} \\
r_{1,2} &= -\frac{P}{2} \pm \frac{\sqrt{4Q - P^2}\sqrt{-1}}{2} \equiv a \pm bi \\
y(t) &= c_1 e^{(a+bi)t} + c_2 e^{(a-bi)t} \\
&= c_1 e^{at}\left(\cos bt + i \sin bt\right) + c_2 e^{at}\left(\cos bt - i \sin bt\right) \\
&= e^{at}\left((c_1 + c_2)\cos bt + i(c_1 - c_2)\sin bt\right).
\end{aligned} \tag{3.32}$$

where we used Result 8 to get the fourth equality. If we only care about real solutions, we restrict the constants in a way to make sure the solution is only on the real line

$$y(t) = e^{at}\left(\bar{c}_1 \cos bt + \bar{c}_2 \sin bt\right) \tag{3.33}$$

3.2.4. Repeated Roots

The general solution to the homogenous equation is in this case

$$y(t) = c_1 e^{rt} + c_2 t e^{rt} \tag{3.34}$$

Check that they are both solutions and convince yourself that they are linearly independent.

3.2.5. Non-Homogeneous Equations with Constant Coefficients

$$\ddot{y}(t) + P\dot{y}(t) + Qy(t) = R(t) \tag{3.35}$$

Relying on **Result 11,** the only added problem is that we have to find *one particular solution* to the complete equation. Typically we guess a form of this solution and then use the *method of undetermined coefficients.* Often it works if we guess on a form similar to $R(t)$, e.g., if it is a polynomial of degree n we guess on another n degree polynomial with unknown coefficients. Example:

$$\ddot{y}(t) - 2\dot{y}(t) + y(t) = 3t^2 + t \tag{3.36}$$

Guess that a particular solution is

$$At^2 + Bt + C \tag{3.37}$$

for some constants A,B,C. We then have to solve for these constants by substituting into the differential equation.

$$\begin{aligned} &\ddot{y}(t) - 2\dot{y}(t) + y(t) \\ &= 2A - 2(2At + B) + At^2 + Bt + C \\ &= At^2 - (4A - B)t + 2A - 2B + C = 3t^2 + t \end{aligned} \tag{3.38}$$

If this is going to hold for each t it is necessary that

$$\begin{aligned} &A = 3 \\ &-(4A - B) = 1 \Rightarrow B = 13 \\ &2A - 2B + C = 0 \Rightarrow C = 20 \end{aligned} \tag{3.39}$$

So a particular solution is

$$y(t) = 3t^2 + 13t + 20 \tag{3.40}$$

The characteristic equation is

$$\begin{aligned} &r^2 - 2r + 1 = (r-1)^2 = 0 \\ &\Rightarrow r_{1,2} = 1 \end{aligned} \tag{3.41}$$

So the general solution is

$$y(t) = c_1 e^t + c_2 t e^t + 3t^2 + 13t + 20 \tag{3.42}$$

3.2.6. Linear *n*th Order Differential Equations

$$y^n(t) + P_1(t)y^{n-1}(t) + \ldots + P_n y(t) = R(t) \tag{3.43}$$

The solution technique is here exactly analogous to the second order case. First we find the roots of the characteristic equation

$$r^n + P_1 r^{n-1} + \ldots + P_n = 0 \tag{3.44}$$

The general solution to the homogenous part is then the sum of the n solutions corresponding to each of n roots. The only thing to note is that if one root, r, is repeated $k \geq 2$ times, the solutions corresponding to this root is given by

$$c_1 e^{rt} + c_2 t e^{rt} + \ldots + c_n t^{k-1} e^{rt} \tag{3.45}$$

Repeated complex roots are handled the same way. Say the root $a \pm bi$ is repeated in k pairs. Their corresponding solution is given by

$$e^{at}\left(c_1 \cos bt + c_2 \sin bt + \ldots + t^{k-1} c_{2k-1} \cos bt + t^{k-1} c_{2k} \sin bt\right). \tag{3.46}$$

A particular solution to the complete equations can often be solved by the method of undetermined coefficients.

3.3. Stability

From the solutions to the differential equations we have seen we find that the terms corresponding to roots that have positive real parts tend to explode as t goes to infinity. This means that also the solution explodes unless the corresponding integration constant, c_i , is zero. Terms with roots that have negative real parts, on the other hand, always converge to zero. Global stability, i.e., regardless of initial conditions, is thus equivalent to all roots being negative.

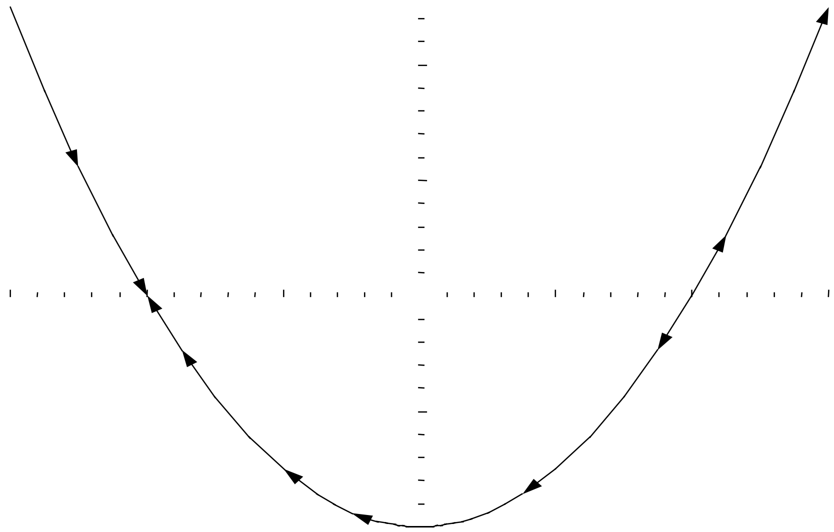
3.3.1. Local Stability

Look at the nonlinear differential equation

$$\dot{x}(t) = x(t)^2 - 4 \tag{3.47}$$

Although we have not learned how to solve such an equation, we can say something about it. Plot $x \rightarrow \dot{x}$

$$\dot{x}(t) = x(t)^2 - 4$$



We see that $x(t)$=2 and $x(t)$=-2 are stationary points. We also see that $x(t)$=-2 is locally stable. In the region [∞,2) x converge to x=-2. 2 is an unstable stationary point and in the region (2, ∞] x explodes over time.

In a plot $x \rightarrow \dot{x}$ local stability is equivalent to a negative slope at the stationary point.

3.4. Systems of Linear First Order Differential Equations

Consider the following system of two first order differential equations

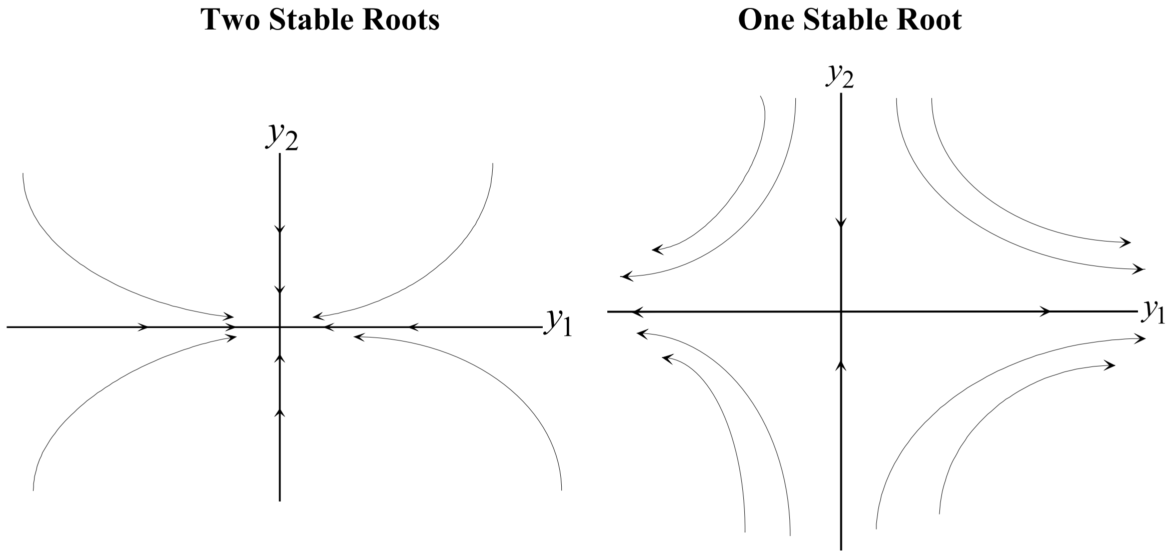
$$\begin{aligned} \begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} \\ &= \mathbf{A}\mathbf{y}_t + \mathbf{P}_t \end{aligned} \tag{3.48}$$

As in the one equations case we start by finding the general solutions to the homogeneous part. This plus some particular solution is the general solution to the complete system.

If the off diagonal terms are zero the solution to the homogeneous part is trivial, since there is no interdependency between the equations.

$$\begin{aligned} \begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} &= \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \\ \Rightarrow \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= \begin{bmatrix} e^{a_{11}t} & 0 \\ 0 & e^{a_{22}t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{aligned} \tag{3.49}$$

The solution can be showed in a graph, a phase diagram.



If the roots are stable, i.e., negative,. the homogeneous part always goes to zero. With only one root stable, there is just one stable path.

(3.49) suggests a way of finding the solution to the homogeneous part of (3.48). We make a transformation of the variables so that the transformed system is diagonal. Start by defining the new set of variables;

$$\begin{aligned}&\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \equiv \mathbf{B} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \\ \Rightarrow &\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} \equiv \mathbf{B} \begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} = \mathbf{BA} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \mathbf{BAB}^{-1} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.\end{aligned} \tag{3.50}$$

If we can find a **B** such that $\mathbf{BAB}^{-1}$ is diagonal we are half way. The solutions for x is then

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{r_1 t} & 0 \\ 0 & e^{r_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{3.51}$$

where r_i are the diagonal terms of the matrix $\mathbf{BAB}^{-1}$.The solution for y then follows from the definition of x

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \mathbf{B}^{-1} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathbf{B}^{-1} \begin{bmatrix} e^{r_1 t} & 0 \\ 0 & e^{r_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{3.52}$$

From linear algebra we know that $\mathbf{B}^{-1}$ is the eigenvectors of **A** and that the diagonal terms of $\mathbf{BAB}^{-1}$ are the corresponding eigenvalues. The eigenvalues are given by the characteristic equation of **A**

$$
\begin{vmatrix} a_{11}-r & a_{12} \\ a_{21} & a_{22}-r \end{vmatrix} = 0
$$

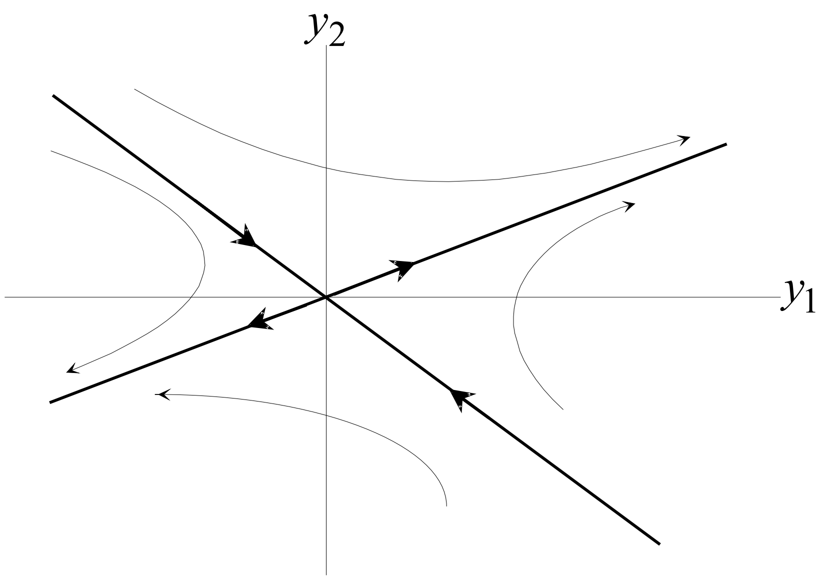
$$
\Rightarrow (a_{11}-r)(a_{22}-r)-a_{12}a_{21} = \tag{3.53}
$$

$$
r^2 - r\underbrace{(a_{11}+a_{22})}_{\text{tr}(\mathbf{A})} + \underbrace{a_{11}a_{22}-a_{12}a_{21}}_{\det(\mathbf{A})} = 0.
$$

The only crux is that we need the roots to be distinct, otherwise $\mathbf{B}^{-1}$ is not always invertible. Distinct root implies that $\mathbf{B}^{-1}$ is invertible. (If **A** is symmetric $\mathbf{B}^{-1}$ is also invertible.)

Let us draw a phase diagram with the eigenvectors of **A.** The dynamic system behaves as the diagonal one in (3.49) but the eigenvectors have replaced the standard, orthogonal axes.

One Stable and One Unstable Root



What is remaining is to find a particular solution of the complete system. One way is here to use the method of undetermined coefficients. Another is to look for a steady state of the system, i.e., a point where the time derivatives are all zero. This is easy if the second term in (3.48) is constant. We then set the differential equal to zero so

$$
0 = \begin{bmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}
$$

$$
\Rightarrow \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = -\begin{bmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \equiv \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix}. \tag{3.54}
$$

Given that we know the value of $y(0)$ we can now give the general solution of (3.48). The formula is given in matrix form and is valid for any dimension of the system. First define

$$\mathbf{r}_t \equiv \begin{bmatrix} e^{r_1 t} & 0 & \dots & 0 \\ 0 & e^{r_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{r_n t} \end{bmatrix} \tag{3.55}$$

and let bold letters define matrices. Then we have

$$\begin{aligned} \mathbf{y}_t &= \mathbf{B}^{-1}\mathbf{x}_t + \mathbf{y}^{ss} = \mathbf{B}^{-1}\mathbf{r}_t\mathbf{C} + \mathbf{y}^{ss} \\ \mathbf{y}_0 &= \mathbf{B}^{-1}\mathbf{r}_0\mathbf{C} + \mathbf{y}^{ss} = \mathbf{B}^{-1}\mathbf{C} + \mathbf{y}^{ss} \\ &\Rightarrow \mathbf{C} = \mathbf{B}\left(\mathbf{y}_0 - \mathbf{y}^{ss}\right) \\ &\Rightarrow \mathbf{y}_t = \mathbf{B}^{-1}\mathbf{r}_t\mathbf{B}\left(\mathbf{y}_0 - \mathbf{y}^{ss}\right) + \mathbf{y}^{ss} \end{aligned} \tag{3.56}$$

The method outlined above works also in the case of complex roots of the characteristic equation. If the roots are $a \pm bi$ we have

$$\begin{aligned} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= \mathbf{B}^{-1} \begin{bmatrix} e^{(a+bi)t} & 0 \\ 0 & e^{(a-bi)t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \\ &= \mathbf{B}^{-1} \begin{bmatrix} c_1 e^{at}(\cos bt + i \sin bt) \\ c_2 e^{at}(\cos bt - i \sin bt) \end{bmatrix} + \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \end{aligned} \tag{3.57}$$

Example;

$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} \tag{3.58}$$

$$\begin{aligned} r_{1,2} &= -1 \pm i \\ \mathbf{B}^{-1} &= \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \end{aligned} \tag{3.59}$$

So from using (3.57) we get

$$\begin{aligned} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-1t}(\cos t + i \sin t) \\ c_2 e^{-1t}(\cos t - i \sin t) \end{bmatrix} + \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \\ &= e^{-t} \begin{bmatrix} i(c_1 - c_2)\cos t - (c_1 + c_2)\sin t \\ (c_1 + c_2)\cos t + i(c_1 - c_2)\sin t \end{bmatrix} + \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \\ &= e^{-t} \begin{bmatrix} \tilde{c}_1 \cos t - \tilde{c}_2 \sin t \\ \tilde{c}_2 \cos t + \tilde{c}_1 \sin t \end{bmatrix} + \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \end{aligned} \tag{3.60}$$

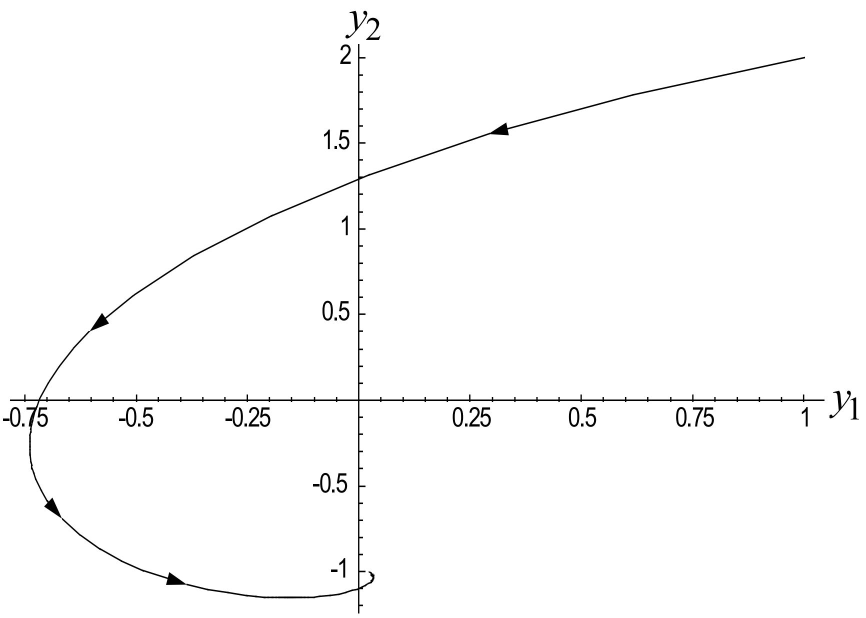
So if we know $y(0)$ we get

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = e^{-t} \begin{bmatrix} \left(y_1(0) - y_1^{ss}\right)\cos t - \left(y_2(0) - y_2^{ss}\right)\sin t \\ \left(y_2(0) - y_2^{ss}\right)\cos t + \left(y_1(0) - y_1^{ss}\right)\sin t \end{bmatrix} + \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \tag{3.61}$$

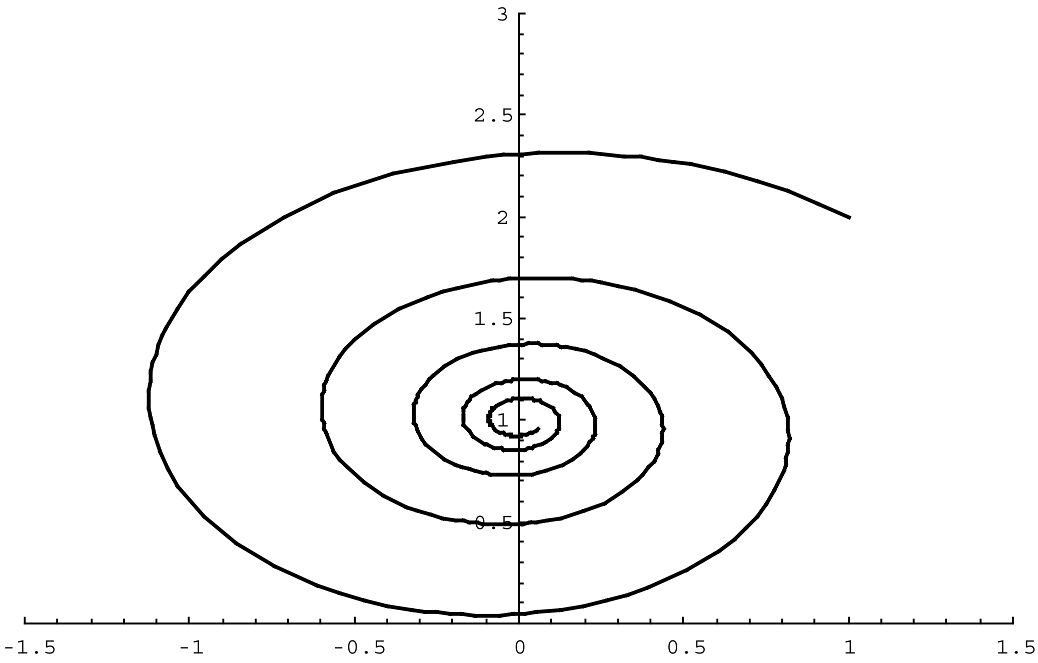
where

$$\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \tag{3.62}$$

A phase diagram of the system in (3.61) looks like



If the root has a real part closer to zero we get more pronounced circles. Here is a phase diagram with a real part of -0.1.



In the *repeated root case* the matrix of the eigenvectors may be singular, so that we cannot find $\mathbf{B}^{-1}$. Then we use the method of *equivalent systems.*

3.4.1. Equivalent Systems

A linear *n*th order differential equation is equivalent to a system of *n* first order differential equations. So

$$\dddot{y}(t) + a_1\ddot{y}(t) + a_2\dot{y}(t) + a_3 y(t) = P(t) \tag{3.63}$$

can be transformed by the following substitution

$$\begin{aligned}
&\dot{y}(t) = x_1(t) \\
&\ddot{y}(t) = x_2(t) = \dot{x}_1(t) \\
&\dddot{y}(t) = \dot{x}_2(t) \\
&\Rightarrow \begin{bmatrix} \dot{y}(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} y(t) \\ x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ P(t) \end{bmatrix}
\end{aligned} \tag{3.64}$$

Since the equations are equivalent they consequently have the same solutions. Sometimes one of the transformations is more convenient to solve. Let us also transform a two dimensional system into a second order equation

$$\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + c_1 \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + c_2
\end{aligned} \tag{3.65}$$

First use the first equation to express x_2 and then take the time derivative of the first. Then we can eliminate x_2 and its timederivative.

$$
\begin{aligned}
&x_2 = \frac{\dot{x}_1}{a_{12}} - \frac{a_{11}x_1}{a_{12}} - \frac{c_1}{a_{12}} \\
&\ddot{x}_1 = a_{11}\dot{x}_1 + a_{12}\dot{x}_2 = a_{11}\dot{x}_1 + a_{12}a_{21}x_1 + a_{12}a_{22}x_2 + a_{12}c_2 \\
&= a_{11}\dot{x}_1 + a_{12}a_{21}x_1 + a_{22}\dot{x}_1 - a_{22}a_{11}x_1 - a_{22}c_1 + a_{12}c_2 \\
&\ddot{x} - (a_{11} + a_{22})\dot{x}_1 + (a_{11}a_{22} - a_{12}a_{21})x_{11} = \tilde{c}
\end{aligned}
$$

3.5. Phase Diagrams and Linearisation

Phase diagrams are convenient to analyze the behavior of a 2 dimensional system qualitatively. E.g.,

$$
\begin{aligned}
\dot{c}(t) &= g_1\big(c(t), k(t)\big) + p_1 \\
\dot{k}(t) &= g_2\big(c(t), k(t)\big) + p_2
\end{aligned} \tag{3.66}
$$

The first step here is to find the two curves in the *c,k*-space where *c* and *k,* respectively are constant. Setting (3.66) equal to zero defines two relations between *c* and *k,* which we denote by G_1 and G_2

$$
\begin{aligned}
\dot{c}(t) = 0 &\Rightarrow c = G_1(k) \\
\dot{k}(t) = 0 &\Rightarrow c = G_2(k)
\end{aligned} \tag{3.67}
$$

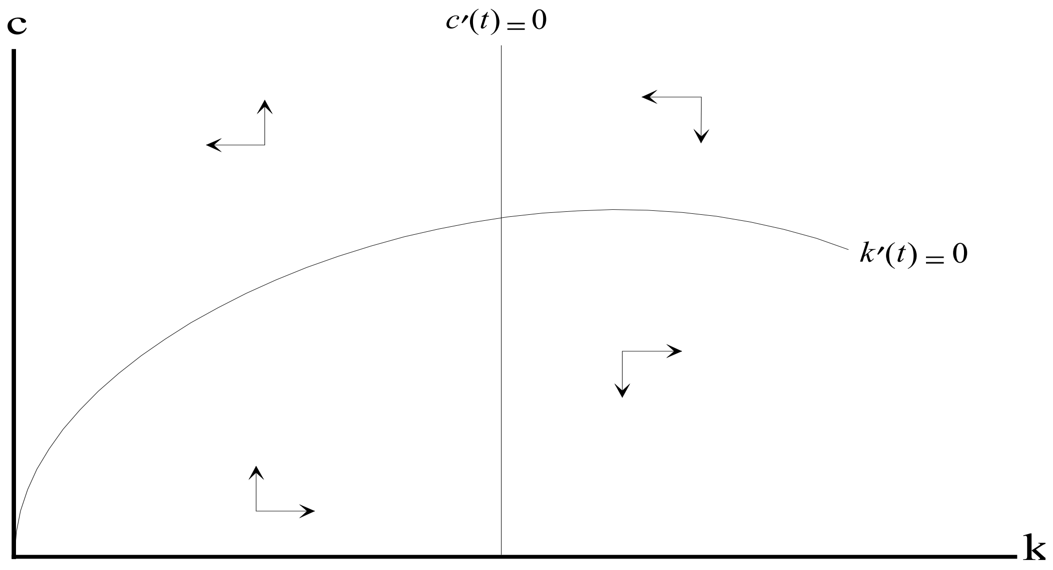
We then draw these curves in the *c,k*-space. For example, you will soon be able to show that the dynamic solution to the Ramsey consumption problem is given by the following system of differential equations

$$
\begin{aligned}
\dot{c}(t) &= -\frac{u'(c)}{u''(c)}\big(f'(k) - \theta\big) \\
\dot{k}(t) &= f(k) - c
\end{aligned} \tag{3.68}
$$

where *c* is consumption, *u* is some utility function *k* is a capital stock and *f* a net production function. Setting the time derivatives to zero we get

$$
\begin{aligned}
0 &= -\frac{u'(c)}{u''(c)}\big(f'(k) - \theta\big) \Rightarrow f'^{-1}(\theta) = k \\
0 &= f(k) - c \Rightarrow c = f(k)
\end{aligned} \tag{3.69}
$$

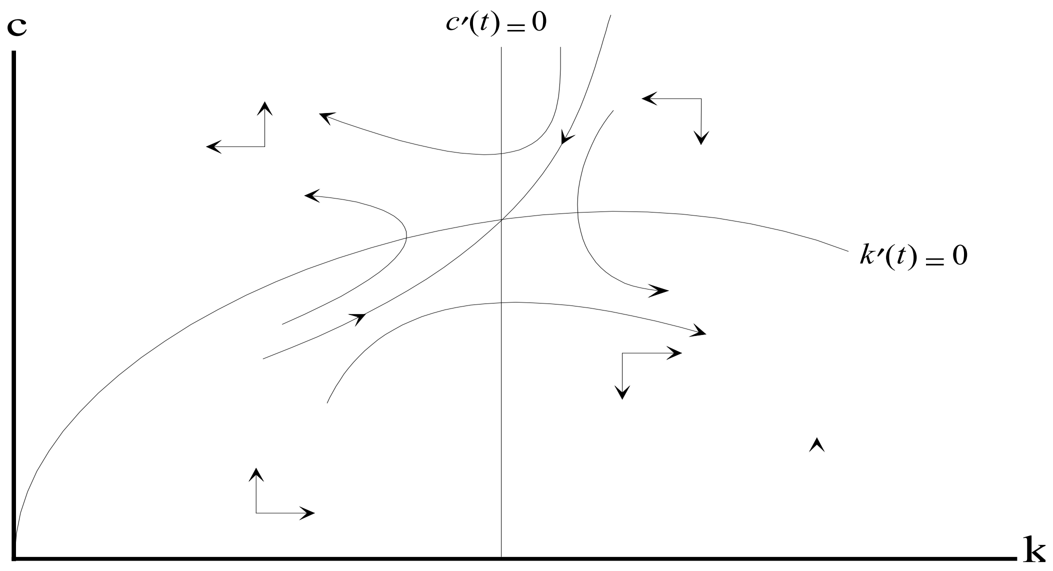
Draw these curves in the *c,k* space



We then have to find the signs of $\dot{k}$, and $\dot{c}$ above and below their respective zero motion curves. From (3.68) we see that

$$\frac{\partial \dot{c}(t)}{\partial k} = -\frac{u'(c)}{u''(c)} f''(k) < 0$$
$$\frac{\partial \dot{k}(t)}{\partial c} = -1 \tag{3.70}$$

This means that $\dot{c}$ is positive to the left of $\dot{c}(t)=0$ and negative to the right. For $\dot{k}'$ we find that it is positive below $\dot{k}(t)=0$ and negative above. Then draw these motions as arrows in the phase diagram. Note that no paths ever can cross.



We conclude that this system has saddle point characteristics and thus have only one stable trajectory towards the steady state.

The behavior close to the steady state should also be evaluated by means of linearization around the steady state. We do that by approximating in the following way

$$\begin{bmatrix} \dot{c}(t) \\ \dot{k}(t) \end{bmatrix} = \begin{bmatrix} \partial g_1(c^{ss}, k^{ss})/\partial c & \partial g_1(c^{ss}, k^{ss})/\partial k \\ \partial g_2(c^{ss}, k^{ss})/\partial c & \partial g_2(c^{ss}, k^{ss})/\partial k \end{bmatrix} \begin{bmatrix} c - c^{ss} \\ k - k^{ss} \end{bmatrix} \tag{3.71}$$

with an obvious generalization to higher dimensions.

We now evaluate the roots of the matrix of derivatives. In the example we find that the characteristic equation is

$$\left| \begin{bmatrix} -(f' - \theta)\partial(u'/u'')/\partial c - r & -f'u'/u'' \\ -1 & f' - r \end{bmatrix} \right| = 0. \tag{3.72}$$

Since $f' = \theta$ at the steady state, the 1,1 element of the coefficient matrix is zero and the roots of the system are given by

$$\begin{aligned} &-r(f' - r) - f'u'/u'' = 0 \\ &\Rightarrow r = \frac{f'}{2} \pm \frac{\sqrt{(f')^2 + 4f'u'/u''}}{2} \end{aligned} \tag{3.73}$$

so that one root is stable and the other explosive.