4. Difference equations

Difference equations can be solved in ways very similar to how we solve differential equations. Before looking for solution, we should define the analogy to integrals.

4.1. Sums, forward and backward solutions

4.1.1. Sums vs. Integrals

If

$$A_t - A_{t-1} \equiv \Delta A_t = q_t \tag{4.1}$$

where ΔA_t is the change in A per unit (interval) of time. We sum both sides to get

$$\sum_{s=1}^{t} \Delta A_s = A_t - A_0 = \sum_{s=1}^{t} q_s \ . \tag{4.2}$$

and we can then write the solution as

$$A_t = A_0 + \sum_{s=1}^{t} q_s \tag{4.3}$$

As we see, this is very much like integrals

$$\frac{dx(t)}{dt} = q(t) \Rightarrow \int_0^t \frac{dx(s)}{ds} ds = x(t) - x(0) = \int_0^t q(s) ds \tag{4.4}$$

and the notation is even closer if we use the Reimann-Stieltjes integral

$$\int_0^t dx(s) = x(t) - x(0) = \int_0^t q(s) ds \ . \tag{4.5}$$

4.1.2. Forward and backward bolutions

Sometimes the R.H.S. of (4.2) converges in one or both the directions. Then we can write the solutions in another way.

$$\lim_{T \to \infty} \sum_{s=-T}^{t} q_s \equiv \sum_{s=-\infty}^{t} q_s = \lim_{s \to -\infty} \left(A_t - A_s \right) \tag{4.6}$$

Then, since the left hand side converges, also the right must. So define

$$\lim_{s \to -\infty} A_s \equiv \underline{A}$$
$$\Rightarrow A_t = \underline{A} + \sum_{s=-\infty}^{t} q_s, \tag{4.7}$$

Similarily,

$$\lim_{T \to \infty} \sum_{s=t+1}^{T} q_s \equiv \sum_{s=t+1}^{\infty} q_s = \lim_{T \to \infty} \left(A_T - A_t \right) = \overline{A} - A_t$$
$$\Rightarrow A_t = \overline{A} - \sum_{s=t+1}^{\infty} q_s. \tag{4.8}$$

If they exist, (4.7) is called the "Backward solution" and (4.8) the "Forward solution". Note that the both solutions satisfy (4.1) regardless of the choice of the constant *A*. Here, the assumption of a forward or backward limit pins down the constant, just like any other condition on a particular A_t.

Example;

$$A_t - A_{t-1} = q_t$$
$$q_t = 0.9 q_{t-1}. \tag{4.9}$$

Here we can use the forward solution

$$\Rightarrow A_t = A - \sum_{s=t+1}^{\infty} q_s = A - \sum_{s=t+1}^{\infty} q_{t+1} 0.9^{s-(t+1)}$$
$$= A - q_{t+1} \sum_{s=0}^{\infty} 0.9^s = A - \frac{q_{t+1}}{1 - 0.9}. \tag{4.10}$$

4.1.3. First order difference equations with constant coefficients

A first order difference equation with constant coefficients has the following form.

$$x_t - a x_{t-1} = c \tag{4.11}$$

As we see, the L.H.S. is not a pure difference, as in (4.1), so we cannot simply sum over *t*. Instead we rely on the following result.

Result 13 The general solution (the complete set of solutions) to a difference equation is the general solution to the homogeneous part plus any particular solution to the complete equation.

Result 14 The general solution to the homogeneous first order difference equation with coefficient *a* can be written

$$A a^t \tag{4.12}$$

where *A* is an arbitrary constant.

A particular solution is, for example, the steady state which exists if $a \neq 1$;

$$
\begin{aligned}
&x_t = x_{t-1} = x^{ss} \\
&\Rightarrow x^{ss} - ax^{ss} = c \Rightarrow x^{ss} = \frac{c}{1-a}
\end{aligned} \tag{4.13}
$$

Non-constant R.H.S.

Now look at

$$
x_t - ax_{t-1} = q_t + c \tag{4.14}
$$

Here we use a little trick; We transform the equation into one with unitary coefficients and then simply solve by summing.

Let x_t denote the unknown solution to (4.14). Then defining the sequence A_t as time-varying coefficients on the general solution. Since we do not put any restrictions on A_t, this is clearly possible. Thus,

$$
x_t = A_t a^t + \frac{c}{1-a} \tag{4.15}
$$

By assumption this solves (4.14) so

$$
\begin{aligned}
&A_t a^t + \frac{c}{1-a} - a\left(A_{t-1}a^{t-1} + \frac{c}{1-a}\right) = q_t + c \\
&a^t\left(A_t - A_{t-1}\right) = q_t + c - (1-a)\frac{c}{1-a} \\
&A_t - A_{t-1} = a^{-t} q_t
\end{aligned} \tag{4.16}
$$

This may be solved using one of the methods in (4.2), (4.6) or (4.8). Assume for example the forward solution works, the we have

$$
\begin{aligned}
&A_t = A - \sum_{s=t+1}^{\infty} a^{-s} q_s \\
&x_t = A_t a^t + \frac{c}{1-a} = a^t\left(A - \sum_{s=t+1}^{\infty} a^{-s} q_s\right) + \frac{c}{1-a} \\
&= Aa^t - \sum_{s=t+1}^{\infty} a^{t-s} q_s + \frac{c}{1-a}.
\end{aligned} \tag{4.17}
$$

The first term of the solution is recognized as the general solution to the homogeneous equation.

4.2. Linear difference equations of higher order

4.2.1. Higher order homogeneous difference equations with constant coefficients

Consider the difference equation

$$x_{t+n}+a_1x_{t+n-1}+...+a_nx_t=0 \tag{4.18}$$

Definition The forward operator E is defined by

$$E^s x_t \equiv x_{t+s} \tag{4.19}$$

where s is any integer, positive or negative.

We can then write (4.18) in a condensed polynomial form.

$$\begin{aligned} P(\lambda) &= \lambda^n + a_1\lambda^{n-1} + \ldots + a_n, \forall\lambda \\ P(E)x_t &= 0. \end{aligned} \tag{4.20}$$

We then have to find the roots of the equation

$$P(r)=0 \tag{4.21}$$

Each root then contributes to the general solution with one term that is independent of the others, exactly as with differential equations.

Result 15 Let r_s denote the roots to the polynomial $P(E)$, i.e., all solutions to $P(r)=0$. Let the first $k\geq 0$ roots be distinct and the next $l=n-k$ roots repeated. Then, the general solution to $P(E)x_t=0$ is

$$x_t = \overbrace{c_1r_1^t+\ldots+c_kr_k^t}^{\text{distinct}} + \overbrace{c_{k+1}r_{k+1}^t + c_{k+2}tr_{k+2}^t + \ldots + c_{k+l}t^{l-1}r_{k+l}^t}^{\text{repeated roots}}. \tag{4.22}$$

In the case of complex roots, express the complex number in polar form

$$\begin{aligned} r &= a+bi = |r|(\cos\theta + i\sin\theta) \\ |r| &= \sqrt{a^2+b^2}, \theta = \arctan\frac{b}{a}. \end{aligned} \tag{4.23}$$

We then use the fact that

$$\begin{aligned} &e^{\theta i} = \cos\theta + i\sin\theta \\ &\Rightarrow r = |r|e^{\theta i} \Rightarrow r^t = |r|^t e^{\theta t i} = |r|^t(\cos\theta t + i\sin\theta t) \end{aligned} \tag{4.24}$$

to get

$$c_1(a+bi)^t + c_2(a-bi)^t = |r|^t(\tilde{c}_1\cos\theta t + \tilde{c}_2\sin\theta t) \tag{4.25}$$

which we can see is a generalization of (4.22). Complex roots thus give us oscillating solutions.

Stability

From (4.22) and (4.25) is clear that roots such that $|r|<1$ give converging terms.

4.2.2. Higher order non-homogeneous difference equations with constant coefficients

Result 16 The general solution to a non-homogeneous difference equation with constant coefficients is given by the general solution to the homogeneous part plus any solution to the full equation.

To solve non-homogeneous equations we thus have to find particular solution to the complete equation to add to the general solution of the homogeneous part. The simplest non-homogeneous difference equation is

$$P(E)x_t = c \tag{4.26}$$

Here we try with a steady state

$$\begin{aligned} &P(E)x^{ss} = P(1)x^{ss} = c \\ &\Rightarrow x^{ss} = \frac{c}{P(1)} \quad \text{if } P(1) \neq 0. \end{aligned} \tag{4.27}$$

Example

$$\begin{aligned} &P(E)x_t = \left(E^2 + 3E - 5\right)x_t = x_{t+2} + 3x_{t+1} - 5x_t = 3 \\ &\Rightarrow P(1)x^{ss} = (1+3-5)x^{ss} = 3 \\ &\Rightarrow x^{ss} = \frac{3}{-1} = -3. \end{aligned} \tag{4.28}$$

In the more general case we often have to guess a particular solution. We can also use the method of allowing time dependent coefficients. In Lang's lecture notes, it is shown how this is done for the second order case.

$$x_t + ax_{t-1} + bx_{t-2} = h_t + c \tag{4.29}$$

The solution is then

$$x_t = c_{1,t}u_t + c_{2,t}v_t + \frac{c}{1+a+b} \tag{4.30}$$

where u_t and v_t are solutions to the homogeneous equation with the property that

$$w_t \equiv u_t v_{t-1} - u_{t-1}v_t \neq 0 \forall t \tag{4.31}$$

$c_{1,t}$ and $c_{2,t}$ are the general solutions to

$$\begin{aligned} &c_{1,t} - c_{1,t-1} = \frac{v_{t-1}}{w_t}h_t \qquad \text{and} \\ &c_{2,t} - c_{2,t-1} = \frac{u_{t-1}}{w_t}h_t \end{aligned} \tag{4.32}$$

The general solutions to (4.32) are

$$c_{1,t} = c_1 + \sum_{i=1}^{t} \frac{v_{i-1}}{w_i} h_i,$$
$$c_{2,t} = c_2 + \sum_{i=1}^{t} \frac{u_{i-1}}{w_i} h_i \tag{4.33}$$

where c_1 and c_2 are arbitrary constants.

4.3. Systems of linear first order difference equations

Systems of first order difference equations are solved with the diagonalization method that we also used for the differential equation.

$$\begin{bmatrix} x_{1,t+1} \\ \vdots \\ x_{n,t+1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_{1,t} \\ \vdots \\ x_{n,t} \end{bmatrix} + \mathbf{P}$$
$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{P} \tag{4.34}$$

First we find the diagonalizing matrix of eigenvectors $\mathbf{B}^{-1}$. Then we solve the homogeneous equation by defining

$$\begin{aligned} &\mathbf{y}_t = \mathbf{B}\mathbf{x}_t \\ &\Rightarrow \mathbf{y}_{t+1} = \mathbf{B}\mathbf{x}_{t+1} = \mathbf{B}\mathbf{A}\mathbf{x}_t \\ &= \mathbf{B}\mathbf{A}\mathbf{B}^{-1}\mathbf{y}_t = \begin{bmatrix} r_1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & r_n \end{bmatrix} \mathbf{y}_t \end{aligned} \tag{4.35}$$

so that

$$\mathbf{y}_t = \begin{bmatrix} r_1^t & 0 & \dots & 0 \\ 0 & r_2^t & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r_n^t \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{r}_t \mathbf{c} \tag{4.36}$$

Then we transform back

$$\mathbf{x}_t = \mathbf{B}^{-1}\mathbf{y}_t = \mathbf{B}^{-1}\mathbf{r}_t\mathbf{c} \tag{4.37}$$

A particular solution to the complete equation then has to be added. Here we generally try to find a steady state as a particular solution.

$$\begin{aligned} &\mathbf{x}^{ss} = \mathbf{A}\mathbf{x}^{ss} + \mathbf{P} \\ &\Rightarrow (\mathbf{I} - \mathbf{A})\mathbf{x}^{ss} = \mathbf{P} \\ &(\mathbf{x}^{ss}) = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{P} \end{aligned} \tag{4.38}$$

If we have the initial conditions we find that

$$\begin{aligned}
&\mathbf{x}_t = \mathbf{B}^{-1}\mathbf{y}_t + \mathbf{x}^{ss} = \mathbf{B}^{-1}\mathbf{r}_t\mathbf{C} + \mathbf{x}^{ss} \\
&\mathbf{x}_0 = \mathbf{B}^{-1}\mathbf{r}_0\mathbf{C} + \mathbf{x}^{ss} = \mathbf{B}^{-1}\mathbf{C} + \mathbf{x}^{ss} \\
&\Rightarrow \mathbf{C} = \mathbf{B}\left(\mathbf{x}_0 - \mathbf{x}^{ss}\right) \\
&\Rightarrow \mathbf{x}_t = \mathbf{B}^{-1}\mathbf{r}_t\mathbf{B}\left(\mathbf{x}_0 - \mathbf{x}^{ss}\right) + \mathbf{x}^{ss}
\end{aligned} \tag{4.39}$$

Another example

$$\begin{aligned}
&\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \begin{bmatrix}-1\\-1\end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} \\
&\Rightarrow \mathbf{x}^{\mathbf{ss}} = \left(\mathbf{I} - \mathbf{A}\right)^{-1}\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}.
\end{aligned} \tag{4.40}$$

Roots and eigenvectors are

$$\begin{aligned}
&r_i = 0 \pm i \Rightarrow r_i^t = 1^t\left(\cos\frac{\pi}{2}t \pm i\sin\frac{\pi}{2}t\right) \\
&\mathbf{B}^{-1} = \begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}
\end{aligned} \tag{4.41}$$

So the solution is

$$\begin{aligned}
&\mathbf{x}_t - \begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}\begin{bmatrix}i^t & 0\\ 0 & (-i)^t\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} \\
&= \begin{bmatrix}i^t & (-i)^t\\ i^{t+1} & (-i)^{t+1}\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}c_1 i^t + c_2(-i)^t\\ c_1 i^{t+1} + c_2(-i)^{t+1}\end{bmatrix}
\end{aligned} \tag{4.42}$$

which can be further evaluated as

$$\begin{aligned}
&\begin{bmatrix}
c_1\left(\cos\left(\frac{\pi}{2}t\right) + i\sin\left(\frac{\pi}{2}t\right)\right) + c_2\left(\cos\left(\frac{\pi}{2}t\right) - i\sin\left(\frac{\pi}{2}t\right)\right) \\
c_1\left(\cos\left(\frac{\pi}{2}(t+1)\right) + i\sin\left(\frac{\pi}{2}(t+1)\right)\right) + c_2\left(\cos\left(\frac{\pi}{2}(t+1)\right) - i\sin\left(\frac{\pi}{2}(t+1)\right)\right)
\end{bmatrix} \\
&= \begin{bmatrix}
\overbrace{(c_1 + c_2)}^{\tilde{c}_1}\cos\left(\frac{\pi}{2}t\right) + \overbrace{(c_1 - c_2)i}^{\tilde{c}_2}\sin\left(\frac{\pi}{2}t\right) \\
\overbrace{(c_1 + c_2)}^{\tilde{c}_1}\cos\left(\frac{\pi}{2}(t+1)\right) + \overbrace{(c_1 - c_2)i}^{\tilde{c}_2}\sin\left(\frac{\pi}{2}(t+1)\right)
\end{bmatrix}
\end{aligned} \tag{4.43}$$

4.3.1. Non-invertible eigenvectors

If some roots are repeated $\mathbf{B}^{-1}$ may be non-invertible. In this case we cannot use the diagonalisation method. Instead we can use the existence of a higher order single difference equation that is equivalent to the systems of first order difference equations we want to solve. This is exactly analogous to the case of differential equations.

Look at the following example

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} \tag{4.44}$$

The eigenvalues of the coefficient matrix are both 1 and the matrix of eigenvectors are non-invertible. Now we transform (4.44) to a second order single difference equation.

From the first row of (4.44) we have that

$$y_t = x_{t+1} + x_t \tag{4.45}$$

Mowing this one step ahead in time and substituting into the second row of (4.44) we get

$$\begin{aligned} & x_{t+2} + x_{t+1} = -4x_t + 3(x_{t+1} + x_t) \\ & \Rightarrow x_{t+2} - 2x_{t+1} + x_t = 0. \end{aligned} \tag{4.46}$$

Note that the polynomial $P(E)$ of (4.46) is identical to the characteristic equation of the coefficient matrix in (4.44). Consequently the have the same (repeated) roots 1. The solution to (4.46) is

$$x_t = c_1 1^t + c_2 t 1^t = c_1 + c_2 t \tag{4.47}$$

With knowledge of x_0 and x_1, (4.47) becomes

$$x_t = x_0 + (x_1 - x_0)t \tag{4.48}$$

By substituting the solution for x_t back into (4.45), we have the solution to (4.44).