6. Dynamic Optimization in Continuous Time

6.1 Calculus of variation

Look at the following simple dynamic problem

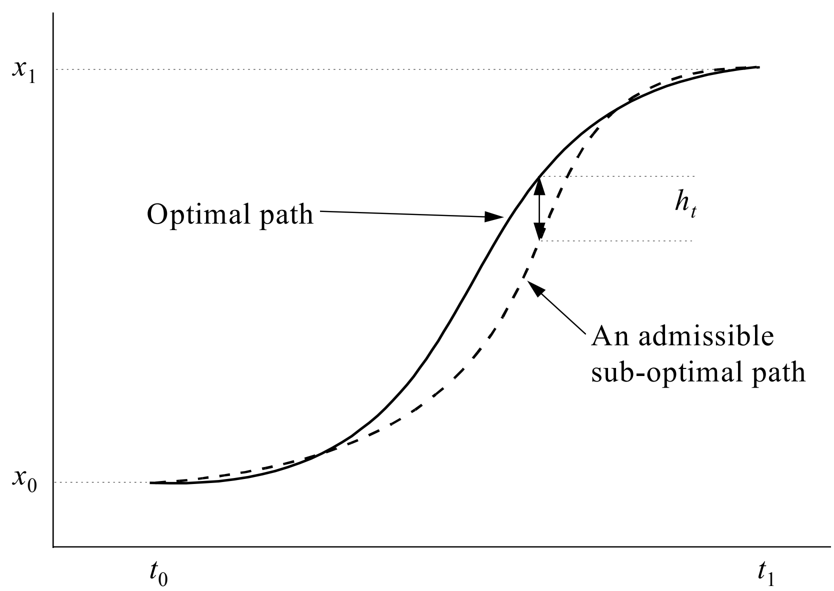
$$\max_{\{x_t\}_{t_0}^{t_1}} \int_{t_0}^{t_1} F(t, x_t, \dot{x}_t)\, dt \qquad (2.1)$$
$$s.t. x_{t_0} = x_0, x_{t_1} = x_1.$$

where F is continuous in its arguments. The problem is dynamic since $\dot{x}$ is included. Otherwise, we could maximize point by point in time.

An economic example could be that F represents profits from a firm that make output by employing labor (x). Time enters the profit function since the firm discounts future profits. If changes in the number of persons employed is costly $\dot{x}$ also enters the profit function. The firm can then not just in each moment hire the number of persons that maximize current profits.

A solution to the problem is a smooth path $x*$ (x being a function of time with a continuous derivative). To find it we try to find some characteristics of it that can help us to search. We will in particular now derive some *necessary* conditions that the solution must satisfy. From them we *may* find the solution.

The trick is to define *admissible* deviations h. These are the differences between the optimal path and an admissible but sub-optimal path, i.e. $h_t \equiv x_t^* - x_t$ The constraints in (2.1) imply that $h_{t_0} \equiv h_{t_1} \equiv 0$, which in this case are the only admissibility constraints (together with differentiability).



Now look at a linear combination of the optimal path and an admissible deviation. For any constant a let

$$y(a) = x^* + ah \tag{2.2}$$

which is clearly admissible. Note that $y(a)$ is a *one parameter family of admissible paths,* i.e., the parameter a pins down a particular path of the variable y over the whole interval t_0 to t_1..

Define the value of the program if we use $y(a)$

$$J(a) \equiv \int_{t_0}^{t_1} F\left(t, \overbrace{x* + ah}^{y(a)}, \overbrace{\dot{x}* + a\dot{h}}^{\dot{y}(a)}\right) dt \tag{2.3}$$

This value must by assumption be maximized when a=0. We also find a standard necessary first order condition for a maximum

$$J'(0) = \frac{\partial}{\partial a}\left(\int_{t_0}^{t_1} F\left(t, x* + ah, \dot{x}* + a\dot{h}\right) dt\right)\Bigg|_{a=0} = 0 \tag{2.4}$$

$$\begin{aligned}\Rightarrow 0 &= \left(\int_{t_0}^{t_1} \frac{\partial}{\partial a} F\left(t, x* + ah, \dot{x}* + a\dot{h}\right) dt\right)\Bigg|_{a=0} \\ &= \int_{t_0}^{t_1} \left(F_x h + F_{\dot{x}} \dot{h}\right) dt.\end{aligned} \tag{2.5}$$

We can rewrite this by integrating $F_{\dot{x}}\dot{h}$ by parts

$$\begin{aligned}\int_{t_0}^{t_1} F_{\dot{x}} \dot{h} dt &= \left[F_{\dot{x}} h\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} h \dot{F}_{\dot{x}} dt \\ &= F_{\dot{x}_{t_1}} \overbrace{h_{t_1}}^{=0} - F_{\dot{x}_{t_0}} \overbrace{h_{t_0}}^{=0} - \int_{t_0}^{t_1} h \dot{F}_{\dot{x}} dt.\end{aligned} \tag{2.6}$$

Substituting this into (2.5) we find that the necessary condition is that along the optimal path

$$\int_{t_0}^{t_1} \left(F_x - \dot{F}_{\dot{x}}\right) h dt = 0 \tag{2.7}$$

But this must be true for all the infinitely many different admissible deviations h. This requires that the value within parenthesis in (2.7) is zero for all t within the planning horizon.

Result 1 A necessary condition for $x*$ to be an optimal path for the problem (2.1) is

$$\begin{aligned}&F_x\left(t, x*, \dot{x}*\right) = \frac{dF_{\dot{x}}\left(t, x*, \dot{x}*\right)}{dt}, \\ &\forall t \subseteq \left[t_0, t_1\right].\end{aligned} \tag{2.8}$$

This is the *Euler Equation* for the problem. We will see that this can be interpreted as a an arbitrage condition between different points in time. Sometimes we may be able to solve for the function $x*(t)$. At least we can derive some properties of it.

6.1.1 A simple consumption example

$$\max_{c_t} \int_0^T e^{-rt} U(c_t) dt$$

$$\begin{aligned} s.t. \quad & \dot{K}_t = iK_t + v_t - c_t \\ & K_0 = k_0, \qquad K_T = k_T. \end{aligned} \tag{2.9}$$

$$F(t, K, \dot{K}) = e^{-rt} U\left(\underbrace{iK_t + v_t - \dot{K}_t}_{c_t}\right),$$

$$\begin{aligned} F_K &= e^{-rt} i U'(c_t), \\ F_{\dot{K}} &= -e^{-rt} U'(c_t). \end{aligned} \tag{2.10}$$

The Euler equation is

$$\begin{aligned} & \dot{F}_{\dot{K}} = F_K \\ & \Rightarrow -d\left(e^{-rt} U'(c_t)\right)/dt = ie^{-rt} U'(c_t). \end{aligned} \tag{2.11}$$

Note that we express the Euler equation in terms of c rather than in K. This will make it easier to interpret and gives us a first order differential equation in c instead of a second order in k.

Now we use (2.11) to try to find a solution

$$\begin{aligned} & -de^{-rt}(U'(c_t))/dt = re^{-rt}(U'(c_t)) - e^{-rt}(U''(c_t))\dot{c}_t \\ & = ie^{-rt} U'(c_t) \\ & \Rightarrow \dot{c}_t = \frac{U'(c_t)}{-U''(c_t)}(i - r) \end{aligned} \tag{2.12}$$

(2.12) tells us about the shape of the optimal path. For a complete characterization, we need more, for example an initial and a final condition for the state variable.

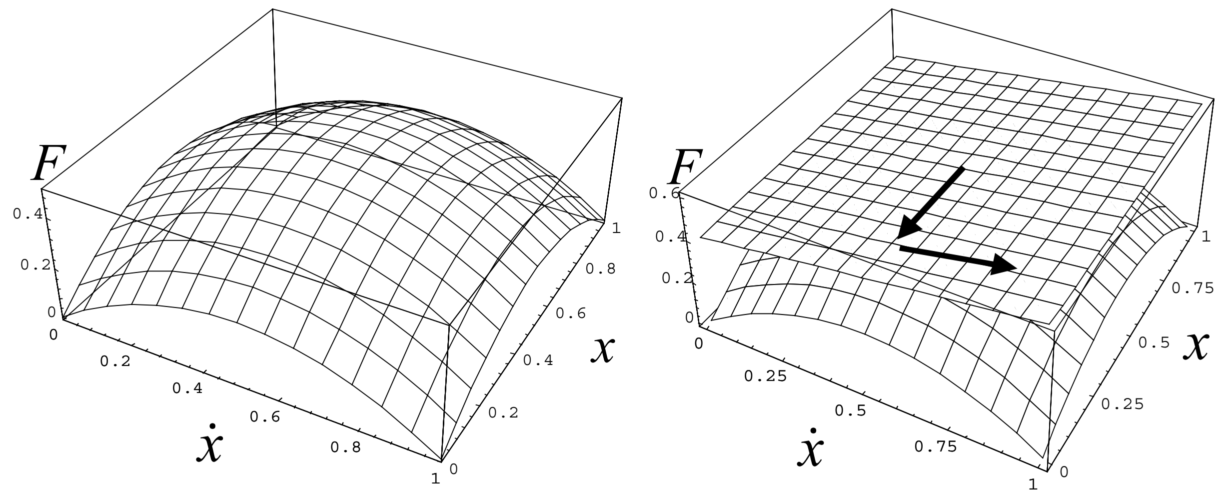
6.1.2 A sufficient condition*

The Euler condition is necessary but not sufficient. It is however also sufficient for a maximum if $F(t, x, \dot{x})$ is concave in $x, \dot{x}$.

Recall that if $F(t, x, \dot{x})$ is concave in $x, \dot{x}$ then

$$\begin{aligned} & F(t, x, \dot{x}) \\ & \leq F(t, x*, \dot{x}*) + (x - x*) F_x(t, x*, \dot{x}*) + (\dot{x} - \dot{x}*) F_{\dot{x}}(t, x*, \dot{x}*). \end{aligned} \tag{2.13}$$

To see this



Assume that $F\left(t, x_t^*, \dot{x}_t^*\right) \equiv F*$ satisfies the Euler equation and F is concave in $x, \dot{x}$. We then want to show that $F\left(t, x_t^*, \dot{x}_t^*\right)$ is optimal, i.e., that

$$\int_0^T F\left(t, x_t, \dot{x}_t\right) dt \leq \int_0^T F\left(t, x_t^*, \dot{x}_t^*\right) dt \tag{2.14}$$

for all admissible paths. Admissible deviations are defined $h(t) \equiv x(t) - x*(t)$ with $\dot{h}(t) \equiv \dot{x}(t) - \dot{x}*(t)$.
Now using (2.13) we have that

$$\begin{aligned} & F \leq F* + \overbrace{(x - x*)}^{h} F_x^* + \overbrace{(\dot{x} - \dot{x}*)}^{\dot{h}} F_{\dot{x}}^* \\ & \Rightarrow \int_0^T F dt \leq \int_0^T F* dt + \int_0^T \left(h F_x^* + \dot{h} F_{\dot{x}}^*\right) dt \end{aligned} \tag{2.15}$$

By integrating by parts we find that

$$\begin{aligned} \int_0^T \left(h F_x^* + \dot{h} F_{\dot{x}}^*\right) dt &= \int_0^T \left(h F_x^*\right) dt + \left[F_{\dot{x}}^* h\right]_0^T - \int_0^T h \dot{F}_{\dot{x}}^* dt \\ & F_{\dot{x}_T} \overbrace{h_T}^{=0} - F_{x_0} \overbrace{h_0}^{=0} + \int_0^T h\left(F_x^* - \dot{F}_{\dot{x}}^*\right) dt \\ &= 0. \end{aligned} \tag{2.16}$$

This shows that concavity, which gave us the inequality in (2.15), makes the Euler equation sufficient for optimality.

Result 2 If F is concave in $x, \dot{x}$, the Euler equation, given in (2.8) is both necessary and sufficient for an optimum.

6.1.3 Transversality conditions

Assume now that k_{t_1} is free. Before we used the terminal condition for k_{t1} to find one integration constant. Now we need some other condition to do this – the *transversality condition.*

An admissible deviation h is now *not* required to satisfy $h(t_1)$=0. The necessary condition (2.4), (2.5) are still valid but (2.6) is changed slightly.

$$\int_{t_0}^{t_1} F_{\dot{x}}\dot{h}dt = \left[F_{\dot{x}}h\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} h\dot{F}_{\dot{x}}dt$$
$$= F_{\dot{x}_T}\overbrace{h_T}^{\neq 0} - F_{x_0}\overbrace{h_0}^{=0} - \int_{t_0}^{t_1} h\dot{F}_{\dot{x}}dt. \quad (2.17)$$

So the necessary condition becomes that along the optimal path

$$\int_{t_0}^{t_1}\left(F_x - \dot{F}_{\dot{x}}\right)hdt + F_{\dot{x}}(t_1, x_{t_1}^*, \dot{x}_{t_1}^*)h(t_1) = 0. \quad (2.18)$$

So we see that the Euler equation is still valid. In addition to the Euler equation we have the added condition

$$F_{\dot{x}}(t_1, x_{t_1}^*, \dot{x}_{t_1}^*) = 0 \quad (2.19)$$

This is the transversality condition.

6.2 Optimal control

Optimal control can be seen as an extension of calculus of variation and is often more convenient when there is restrictions on the way the system can controlled. To facilitate this, we make a distinction between control variables (e.g., consumption or investments) and state variables (e.g., capital stocks or debt) that are governed by a differential equation (*transition* equation) and thus given in each point in time.

$$\max_{\{u_t\}_{t_0}^{t_1}} \int_{t_0}^{t_1} f(t, x_t, u_t) dt$$
$$s.t. \qquad \dot{x}_t = g(t, x_t, u_t), \qquad x_{t_0} = x_0. \tag{2.20}$$

Here, x is the state variable. The way it changes over time, can be affected by the control variable u. As a means to finding a solution we define a multiplier function λ_t for the transition equation.

For a combination of x,u to be admissible it must be that $\forall\ t \subseteq [t_0, t_1]\ \ g(t, x_t, u_t) - \dot{x}_t = 0$. Adding this zero to the maximand yields,

$$\int_{t_0}^{t_1} f(t, x_t, u_t) + \lambda_t (g(t, x_t, u_t) - \dot{x}_t) dt \tag{2.21}$$

Now we want to get rid $\dot{x}_t$. So integrate by parts

$$\int_{t_0}^{t_1} \lambda_t \dot{x}_t dt = [\lambda_t x_t]_{t_0}^{t_1} - \int_{t_0}^{t_1} \dot{\lambda}_t x_t dt \tag{2.22}$$

giving

$$\int_{t_0}^{t_1} \left(f(t, x_t, u_t) + \lambda_t g(t, x_t, u_t) + \dot{\lambda}_t x_t \right) dt - \lambda_{t_1} x_{t_1} + \lambda_{t_0} x_{t_0} \tag{2.23}$$

Now we use the same procedure as when deriving necessary conditions for the Calculus of Variation problem. Instead of looking at admissible variations of x we look at admissible variations of the control variable u. Let u^* represent the optimal control and u some other admissible control and define $h= u^*-u$ Let $y(a)$ denote the state variable generated by using the control u^*+ah. Let $J(a)$ denote the value of the program (2.23) when using the control u^*+ah. Clearly $J(0)$ is the maximum of J by definition and $J'(0)=0$. As before we will use this necessary condition to find necessary properties of the solution.

$$J(a) \equiv \int_{t_0}^{t_1} \left(f\left(t, y_t(a), u_t^* + ah_t\right) + \lambda g\left(t, y_t(a), u_t^* + ah_t\right) + \dot{\lambda}_t y_t(a) \right) dt$$
$$-\lambda_{t_1} y_{t_1}(a) \ + \lambda_{t_0} y_{t_0}(a)\ , \tag{2.24}$$

$$
\begin{aligned}
J'(0) &= \int_{t_0}^{t_1} \left(f_x y_t'(0) + f_u h_t + \lambda_t g_x y_t'(0) + \lambda_t g_u h_t + \dot{\lambda}_t y_t'(0) \right) dt \\
&\quad -\lambda_{t_1} y_{t_1}'(0) + \lambda_{t_0} y_{t_0}'(0) \\
&= \int_{t_0}^{t_1} \left(\left(f_x + \lambda_t g_x + \dot{\lambda}_t \right) y_t'(0) + \left(f_u + \lambda g_u \right) h_t \right) dt \\
&\quad -\lambda_{t_1} y_{t_1}'(0) + \lambda_{t_0} \overbrace{y_{t_0}'(0)}^{=0}, \\
&= 0.
\end{aligned} \tag{2.25}
$$

So far we have not put any restrictions on λ. It will soon be clear that it is very convenient to let it follow the differential equation

$$
\begin{aligned}
\dot{\lambda} &= -f_x - \lambda g_x \\
\lambda_{t_1} &= 0.
\end{aligned} \tag{2.26}
$$

For (2.25) and (2.26)to hold we thus require that along the optimal path

$$
\int_{t_0}^{t_1} (f_u + \lambda g_u) h dt = 0, \tag{2.27}
$$

for all admissible deviations *h*. For this to hold for all such deviations we need that

$$
f_u + \lambda g_u = 0, \qquad \forall t \subseteq [t_0, t_1]. \tag{2.28}
$$

We have now found that if we define λ_t according to (2.26) the necessary condition for optimality can be written as (2.28). To remember this and the we construct the *Hamiltonian*.

$$
H(t, x_t, u_t, \lambda_t) = f(t, x_t, u_t) + \lambda_t g(t, x_t, u_t), \tag{2.29}
$$

from which we can derive the necessary conditions for optimality.

Result 3 Necessary conditions for a solution to (2.20), i.e., an optimal control, are

$$
\begin{aligned}
H_u &= f_u(t, x_t, u_t^*) + \lambda_t g_u(t, x_t, u_t^*) = 0 \\
-H_x &= -f_x(t, x_t, u_t^*) - \lambda_t g_x(t, x_t, u_t^*) = \dot{\lambda}_t \\
H_\lambda &= g(t, x_t, u_t^*) = \dot{x}_t
\end{aligned} \tag{2.30}
$$

and the initial condition $x_{t_0} = x_0$ and the terminal condition $\lambda_{t_1} = 0$.

6.2.1 Current value Hamiltonian

Often we have problems where *t* only enters as an exponential discounting. E.g.,

$$
\begin{aligned}
&\max_u \int_0^T e^{-rt} f(x_t, u_t) dt \\
&s.t. \qquad \dot{x}_t = g(x_t, u_t), \quad x_{t_0} = x_0.
\end{aligned} \tag{2.31}
$$

The Hamiltonian with necessary conditions is

$$H(t,x_t,u_t) = e^{-rt} f(x,u) + \lambda g(x_t,u_t)$$
$$H_u = e^{-rt} f_u + \lambda g_u = 0 \tag{2.32}$$
$$H_x = e^{-rt} f_x + \lambda g_x = -\dot{\lambda},\ \lambda_T = 0.$$

It is often convenient to use a *current* shadow value defined as

$$e^{-rt}\mu_t \equiv \lambda_t$$
$$\Rightarrow \dot{\lambda} = -re^{-rt}\mu + e^{-rt}\dot{\mu} = e^{-rt}(\dot{\mu} - r\mu) \tag{2.33}$$

Substitute into (2.32)

$$H(t,x_t,u_t) = e^{-rt} f(x,u) + e^{-rt}\mu g(x_t,u_t)$$
$$H_u = e^{-rt}(f_u + \mu g_u) = 0 \tag{2.34}$$
$$H_x = e^{-rt}(f_x + \mu g_x) = -e^{-rt}(\dot{\mu} - r\mu),\ e^{-rT}\mu_T = 0.$$

We get rid of all the discounting factors by defining the *current value Hamiltonian* with associated necessary conditions.

Result 4 Defining the current value Hamiltonian as $\mathcal{H}(t,x_t,u_t) \equiv e^{rt} H(t,x_t,u_t)$. Necessary conditions of an optimal control are

$$\mathcal{H}_u = f_u + \mu g_u = 0$$
$$\mathcal{H}_x = f_x + \mu g_x = -(\dot{\mu} - r\mu),\ e^{-rT}\mu_T = 0. \tag{2.35}$$

If T is infinity we may not divide the transversality condition by the discount factor since it is zero.

6.2.2 An alternative way of deriving the Hamiltonian

An alternative way of deriving the necessary conditions in (2.30) is to use the logic behind the Bellman equation, i.e., to separate the dynamic problem into two parts, current payoff and all future payoff, where the latter is calculated given that future decisions are taken optimally. This cannot be done directly in continuous time, but we can approximate by using a discrete time version and then let the length of the discrete time periods go to zero.

Now, consider the dynamic optimization problem

$$\max_{\{u_t\}_0^T} \int_0^T e^{-rt} f(x_t,u_t)dt$$
$$s.t. \qquad \dot{x}_t = g(x_t,u_t),\ \ x_0 = \bar{x}_0. \tag{2.36}$$

In discrete time, time intervals *dt,* we can rewrite this as

$$\max_{\{u_s\}_0^{T/dt}} \sum_{s=0}^{T/dt} e^{-rsdt} f(x_{sdt},u_{sdt})dt$$
$$s.t. \qquad x_{t+dt} = x_t + g(x_t,u_t)dt, \qquad x_0 = \bar{x}_0, \tag{2.37}$$

with an associated Bellman equation given by

$$V_t(x_t) = \max_{u_t} f(x_t, u_t)dt + (1-rdt)V_{t+dt}\left(x_t + g(x_t, u_t)dt\right), \tag{2.38}$$

where we used the approximation $e^{-rdt} \approx 1 - rdt$. Now, let us denoted the (shadow) value of the state variable by μ_t. Clearly, this means that $\mu_t \equiv V'(x_t)$ which means that we can write the first order condition of (2.38) as

$$f_u(x_t, u_t)dt + (1-rdt)\mu_{t+dt} g_u(x_t, u_t)dt = 0 . \tag{2.39}$$

Dividing by dt and letting dt go to zero yields

$$\begin{aligned} 0 &= \lim_{dt \to 0} f_u(x_t, u_t) + (1-rdt)\mu_{t+dt} g_u(x_t, u_t) \\ &= f_u(x_t, u_t) + \mu_t g_u(x_t, u_t) = 0, \end{aligned} \tag{2.40}$$

as in (2.35). Furthermore, let us use the Bellman equation to calculate an explicit formula for μ.

$$\begin{aligned} &V_t'(x_t) \equiv \mu_t = f_x(x_t, u_t)dt + (1-rdt)\mu_{t+dt}\left(1 + g_x(x_t, u_t)dt\right) \\ &\Rightarrow 0 = f_x(x_t, u_t)dt + \mu_{t+dt} - \mu_t - rdt\mu_{t+dt} + (1-rdt)\mu_{t+dt} g_x(x_t, u_t)dt \end{aligned}, \tag{2.41}$$

Dividing by dt and letting dt go to zero yields

$$\begin{aligned} &\lim_{dt \to 0}\left[f_x(x_t, u_t) + \frac{\mu_{t+dt} - \mu_t}{dt} - r\mu_{t+dt} + (1-rdt)\mu_{t+dt} g_x(x_t, u_t)\right] \\ &= f_x(x_t, u_t) + \dot{\mu}_t - r\mu_t + \mu_t g_x(x_t, u_t) = 0 \\ &\Rightarrow f_x(x_t, u_t) + \mu_t g_x(x_t, u_t) = -\dot{\mu}_t + r\mu_t \end{aligned}, \tag{2.42}$$

as in (2.35). Thus, if we want the variable μ to express the shadow value of capital, it has to satisfy the differential equation in (2.42).

We can now interpret the current value Hamiltonian,

$$\mathcal{H}(x_t, u_t, \mu_t) \equiv f(x_t, u_t) + \mu_t g(x_t, u_t) \tag{2.43}$$

in the following way; The first term is the flow of current payoff and $g(x_t, u_t)$ is the accumulation of the state variable. Multiplying the latter by the shadow value of the state variable gives the accumulation of *future* payoffs. The sum of these two terms should be maximized over the control variable. Exactly as in Dynamic Programming, we can thus interpret the optimality condition $\mathcal{H}_u=0$ as a necessary condition for maximizing the sum of *current* and *future* profits.

6.2.3 The simple consumption example

$$\begin{aligned} &\max_{\{c_t\}_0^T} \int_0^T e^{-rt} U(c_t)dt \\ &s.t. \ \dot{K} = f(K) - c, \\ &K_0 \text{ given.} \end{aligned} \tag{2.44}$$

The current value Hamiltonian is

$$\begin{aligned}
&\mathcal{H} = U(c_t) + \mu_t \left(f(K_t) - c_t \right) \\
&\mathcal{H}_c = U'(c_t) - \mu_t = 0 \\
&\mathcal{H}_K = \mu_t f'(k) = -\left(\dot{\mu} - r\mu \right), \\
&e^{-rT} \mu_T = 0.
\end{aligned} \tag{2.45}$$

Taking first time derivative of the first condition yields,

$$U''(c_t)\dot{c}_t = \dot{\mu}_t, \tag{2.46}$$

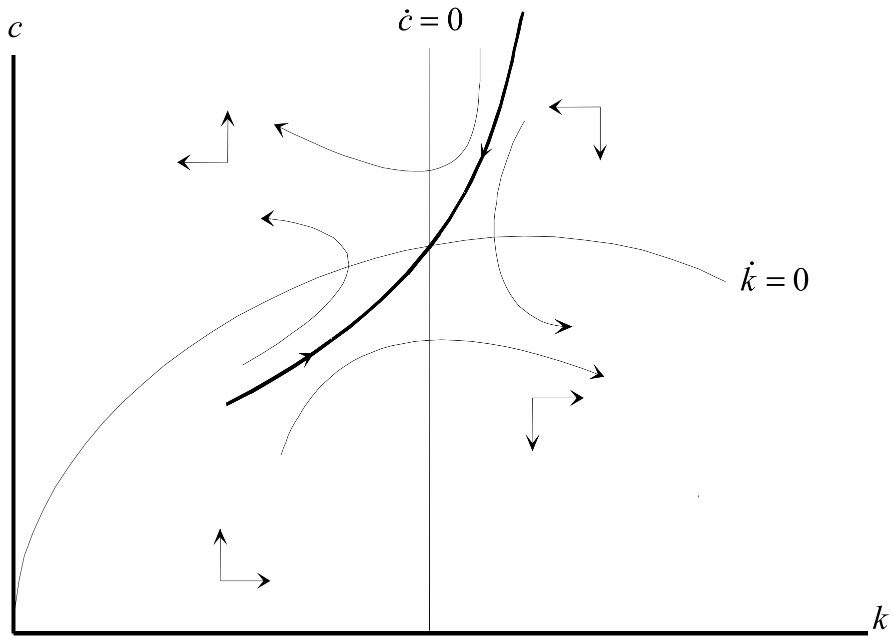
implying

$$U'(c_t) f'(k) = -\left(U''(c_t)\dot{c}_t - rU'(c_t) \right), \tag{2.47}$$

Solving this for $\dot{c}_t$ and adding the law-of-motion for capital yields

$$\begin{aligned}
&\dot{c}_t = \frac{U'(c_t)}{-U''(c_t)} \left(f'(k_t) - r \right) \\
&\dot{k}_t = f(k_t) - c_t
\end{aligned} \tag{2.48}$$

This system, we can solve qualitatively using the phase diagram.



Alternatively, we can simplify (or linearize) to make the system linear. Suppose the production function is linear, so

$$\dot{c}_t = \frac{U'(c_t)}{-U''(c_t)}(i-r)$$
$$\dot{k}_t = ik_t - c_t \tag{2.49}$$

Then, if we specify a utility function we can go further.

In the CARA utility (exponential) case

$$U = \frac{-e^{-\lambda c}}{\lambda}, \qquad U' = e^{-\lambda c}, \qquad U'' = -\lambda e^{-\lambda c} \tag{2.50}$$

so

$$\dot{c}_t = \frac{i-r}{\lambda}, \qquad c_t = c_0 + \frac{i-r}{\lambda}t. \tag{2.51}$$

This means that the absolute growth of consumption is constant. Note that this just defines the slope of the optimal path, the level is determined from the dynamic budget constraint.

$$\dot{k}_t - ik_t = -c_t \tag{2.52}$$

Multiplying by the integration factor and integrating we have

$$e^{-it}\left(\dot{k}_t - ik_t\right) = \frac{de^{-it}k_t}{dt} = -e^{-it}c_t$$
$$e^{-iT}k_T - k_0 = -\int_0^T e^{-it}c_t dt. \tag{2.53}$$

This is the *intertemporal* budget constraint. Solving this

$$\int_0^T e^{-it}\left(c_0 + \frac{i-r}{\lambda}t\right)dt = -\left[c_0\frac{e^{-it}}{i}\right]_0^T + \int_0^T \overbrace{\left(\frac{i-r}{\lambda}t\right)}^{u} \overbrace{e^{-it}dt}^{dv}$$
$$= -\left[c_0\frac{e^{-it}}{i}\right]_0^T - \left[\frac{i-r}{\lambda}t\frac{e^{-it}}{i}\right]_0^T - \int_0^T -\frac{i-r}{\lambda}\frac{e^{-it}}{i}dt$$
$$= -\left[c_0\frac{e^{-it}}{i}\right]_0^T - \left[\frac{i-r}{\lambda}t\frac{e^{-it}}{i}\right]_0^T - \left[\frac{i-r}{\lambda}\frac{e^{-it}}{i^2}\right]_0^T \tag{2.54}$$
$$= c_0\left(\frac{1-e^{-iT}}{i}\right) + \frac{i-r}{\lambda i^2} - e^{-iT}\left(\frac{i-r}{\lambda i}T + \frac{i-r}{\lambda i^2}\right) = W_0.$$

where $W_0 = k_0 - e^{-iT}k_T$, is the wealth of the individual.

Note that if i=r consumption is simply a fraction of wealth, that decreases with the length of the planning horizon.

$$c_0 = \frac{i}{(1-e^{-iT})}W_0 \tag{2.55}$$

So with an infinite horizon $c_t = iW_t$. Note that in the infinite horizon case, we usually require that

$$\lim_{T\to\infty} e^{-iT} k_T \geq 0, \tag{2.56}$$

since otherwise someone in the economy would be giving up resources for free, loosely speaking.

Similarly on the case of CRRA utility

$$U = \frac{c^{1-1/\sigma}}{1-1/\sigma}, \qquad U' = c^{-1/\sigma}, U'' = -\frac{c^{-1/\sigma-1}}{\sigma} \tag{2.57}$$

the first line of (2.49) becomes

$$\begin{aligned} &\dot{c}_t / c_t = \sigma(i-r) \\ &\Rightarrow c_t = c_0 e^{\sigma(i-r)t}. \end{aligned} \tag{2.58}$$

Here the growth *rate* of consumption is constant and proportional to the difference between market and subjective discount rates. Compare this to (2.51).

Using the intertemporal budget constraint

$$\begin{aligned} &\int_0^T e^{-it} c_0 e^{\sigma(i-r)t} dt = \int_0^T c_0 e^{((\sigma-1)i-\sigma r)t} dt = W_0 \\ &\Rightarrow c_0 = \left(\frac{(\sigma r - (\sigma - 1)i)}{1 - e^{((\sigma-1)i-\sigma r)T}} \right) W_0 \end{aligned} \tag{2.59}$$

Note the results when σ=1 and when $T\to\infty$.

6.2.4 Sufficiency

As in the Calculus of Variation we get a sufficiency condition by imposing the right concavity condition. Assume that f and g are concave in x,u and $\lambda \geq 0$. This implies that the Hamiltonian is concave in x,u. Then since f is concave

$$\begin{aligned} &f \leq f* + (x - x*) f_x^* + (u - u*) f_u^* \\ &\int_{t_0}^{t_1} f dt \leq \int_{t_0}^{t_1} f* dt + \int_{t_0}^{t_1} \left((x - x*) f_x^* + (u - u) f_u^* \right) dt. \end{aligned} \tag{2.60}$$

So we want to show that the last integral in (2.60) is ≤0. Substitute for f_x from (2.30) and then integrate by parts the term involving $\dot{\lambda}$ to get rid of that.

$$\int_{t_0}^{t_1}\left((x-x_*)f_x^* + (u-u_*)f_u^*\right)dt$$

$$=\int_{t_0}^{t_1}\left((x-x_*)\left(-\dot{\lambda}-\lambda g_x^*\right)+(u-u_*)(-\lambda g_u^*)\right)dt$$

$$=-\overbrace{\left[\lambda(x-x_*)\right]_{t_0}^{t_1}}^{0}+\int_{t_0}^{t_1}\overbrace{(\dot{x}-\dot{x}_*)}^{g-g_*}\lambda dt \tag{2.61}$$

$$+\int_{t_0}^{t_1}\left((x-x_*)\left(-\lambda g_x^*\right)+(u-u_*)(-\lambda g_u^*)\right)dt$$

$$=\int_{t_0}^{t_1}\lambda\left((g-g_*)-\left((x-x_*)g_x^*+(u-u_*)g_u^*\right)\right)dt\leq 0.$$

If $\lambda \leq 0$ we need that g is convex in x,u. Then H is still concave. If g is linear we see that its sufficient that f is concave in x,u.

6.2.5 Some infinite horizon results

In the infinite horizon case, we must, of course, first be sure that the integral in the objective function converges, so that the value of the program is bounded. If so,the optimality condition and the differential equation for the shadow value and the state variable

$$\begin{aligned} H_u &= f_u\left(t,x_t,u_t^*\right)+\lambda_t g_u\left(t,x_t,u_t^*\right)=0 \\ -H_x &= -f_x\left(t,x_t,u_t^*\right)-\lambda_t g_x\left(t,x_t,u_t^*\right)=\dot{\lambda}_t \\ H_\lambda &= g_x\left(t,x_t,u_t^*\right)=\dot{x}_t \end{aligned} \tag{2.62}$$

are necessary also in the infinite horizon case. The finite horizon transversality conditions can, however, not immediately be used in the infinite horizon case. In the following infinite horizon case, we have sufficient conditions.

Result 5 (Arrow) Consider the problem

$$\begin{aligned} &\max_{\{u_t\}_0^T}\int_0^\infty e^{-rt}f\left(x_t,u_t,t\right)dt \\ &s.t. \qquad \dot{x}_t = g\left(x_t,u_t\right), \;\; x_0=\bar{x}_0. \end{aligned} \tag{2.63}$$

Now, consider the *maximized* Hamiltonian, i.e.,

$$H^*(t,x,\lambda)\equiv \max_u H(t,x,u,\lambda)\,. \tag{2.64}$$

If H^* is concave in x for every t. Then, the conditions in (2.62) (or equivalently in (2.35)) together with

$$\begin{aligned} \lim_{t\to\infty}\lambda_t &= \lim_{t\to\infty}e^{-rt}\mu_t \geq 0, \\ \lim_{t\to\infty}\lambda_t x_t &= \lim_{t\to\infty}e^{-rt}\mu_t x_t = 0. \end{aligned} \tag{2.65}$$

are *sufficient* for the control being optimal. It may be more difficult to check the property of a derived function, but if H is concave in x,u H^* is necessarily concave in x. However, concavity of H^* in x does not require concavity of H in x,u. On the other hand, concavity in H is easier to check, then we just need that $\lambda \geq 0$ and f and g being concave in x,u. Note, that here, it does not matter whether we use the current value or the discounted value Hamiltonian since they only differ by multiplicative discount factor.

Sometimes, the sufficiency conditions allow us to find the optimal control as the stable arm (saddle-path) leading to a saddle-point stable steady state. For example, consider the following consumption-investment problem,

$$\begin{aligned} &\max_{s_t} \int_0^\infty e^{-rt} U(c_t) dt \\ &s.t. \dot{k}_t = s_t f(k_t) - \delta k_t, \\ &\quad c_t = (1-s_t) f(k_t), \end{aligned} \tag{2.66}$$

where U and f are increasing concave functions. The current value Hamiltonian is

$$\begin{aligned} &\mathcal{H} = U((1-s)f(k_t)) + \mu_t \left(sf(k_t) - \delta k_t \right) \\ &\mathcal{H}_s = f(k)\left(\mu - U'\right) = 0. \\ &\Rightarrow U'\left((1-s)f(k)\right) = \mu \\ &\Rightarrow U'^{-1}\left(\mu\right) = (1-s)f(k) \\ &\Rightarrow 1-s = \frac{U'^{-1}\left(\mu\right)}{f(k)} \end{aligned} \tag{2.67}$$

We could now check the concavity of the Hamiltonian. Alternatively, we check the *maximized Hamiltonian.* For this purpose, we substitute the optimal control into the Hamiltonian, yielding

$$H^*(t,x,\lambda) = U\left(U'^{-1}\left(\mu\right)\right) + \mu_t \left(sf(k_t) - \delta k_t \right) \tag{2.68}$$

which is concave in k if f is, since $\mu > 0$ if $U' > 0$.

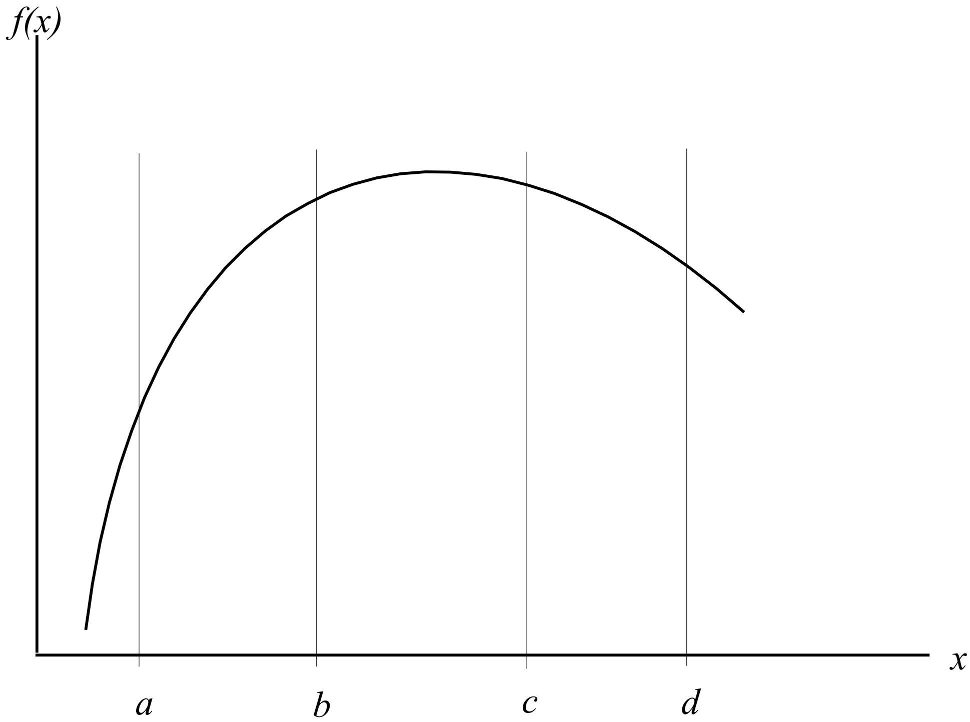
We have seen that this problem, has saddle-path properties. Thus, we can find a saddle path leading to a steady state x^{ss} and μ^{ss}. Then, it is clear that (2.65) is satisfied.

6.2.6 Bounded controls

For a control to be optimal it is necessary that it solves

$$\max_u H(t, x_t, u_t, \lambda_t) \tag{2.69}$$

If u is bounded $H_u = 0$ is not necessary for an optimum. As in standard maximization the first order conditions only holds for interior solutions.



If we maximize $f(x)$ over [a,b] b is optimal and $f'(x*)$ >0. If the range is [c,d] c is optimal with $f'(x*)$ <0. Also in optimal control we may use *Kuhn-Tucker* multipliers in this case. Assume we solve problem (2.20) but restrict u to the range [a,b]. We then form the appended Hamiltonian

$$\begin{aligned} H(t,x_t,u_t,\lambda_t) &= f(t,x_t,u_t)+\lambda_t g(t,x_t,u_t) \\ &+w_1(b-u)+w_2(u-a). \end{aligned} \tag{2.70}$$

The optimality condition now becomes

$$\begin{aligned} & f_u+\lambda g_u-w_1+w_2=0 \\ & w_1,w_2\geq 0, \\ & w_1(b-u*)=w_2(u*-a)=0. \end{aligned} \tag{2.71}$$

Except for the knife-edge case we have that if w_1>0, $(b-u*)$=0 so the H_u>0 as in the figure

6.2.7 The Pontryagin maximum principle

Now we come to a more general formulation of the necessary conditions for a solution to the optimal control problem. We allow for a any finite number of discontinuity points in the control, *n* control and state variables and that the controls are restricted to a constant weak subset of R^n.

$$
\begin{aligned}
&\max_{\mathbf{u}} \int_{0}^{T} f(t,\mathbf{x},\mathbf{u})dt \\
&s.t. \quad \dot{x}_i = g_i(t,\mathbf{x},\mathbf{u}), \quad i = 1,\ldots n, \\
&\qquad \mathbf{x}(0) = \overline{\mathbf{x}}, \\
&\qquad x_i(T) = x_{iT}, \quad i = 1,\ldots p \\
&\qquad x_i(T) \geq x_{iT}, \quad i = p+1,\ldots q \\
&\qquad x_i(T) \text{ free} \quad i = q+1,\ldots n. \\
&\qquad \mathbf{u} \in U \subseteq R^n
\end{aligned}
\tag{2.72}
$$

Result 6

For **u*** *and the resulting state vector* **x*** *to maximize* (2.72) *it is necessary that there exists a constant* λ_0 *and continuous functions* $\boldsymbol{\lambda}$(t) (λ_i(t) *i*=1,...,*n*) *such that* $\forall t \in [0,T]$

$$\lambda_0 = 0 \text{ or } 1, \ \{\lambda_0, \boldsymbol{\lambda}(t)\} \neq \{0, \mathbf{0}\}, \tag{2.73}$$

$$\mathbf{u}^* = \arg\max_{\mathbf{u}} H(t,\mathbf{x}^*,\mathbf{u},\boldsymbol{\lambda}), \tag{2.74}$$

where

$$H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda}) = \lambda_0 f(t,\mathbf{x},\mathbf{u}) + \sum_{i=1}^{n} \lambda_i g_i(t,\mathbf{x},\mathbf{u}) \tag{2.75}$$

except at points of discontinuity of **u**

$$\dot{\lambda}_i = -H_{x_i} \tag{2.76}$$

and the transversality conditions

$$
\begin{aligned}
&\lambda_i(T) \text{ free}, && i = 1,\ldots p, \\
&\lambda_i(T) \geq 0, && i = p+1,\ldots q, \\
&\lambda_i(T) = 0, \text{if } x_i(T) > x_{iT}, && i = p+1,\ldots q, \\
&\lambda_i(T) = 0 && i = q+1,\ldots n.
\end{aligned}
\tag{2.77}
$$

The strange shadow value on the objective function λ_0 may under some perverse circumstances be 0. I believe you can safely ignore this possibility for the coming courses in economics.

Note that by specifying the control region we may formulate *Kuhn-Tucker* first order conditions instead of (2.74).

At the points in time when the control jumps λ has a kink. It is, however, *always continuous.* Note also that *H* is always continuous. Kuhn-Tucker shadow values on the control constraints, μ_t, may be discontinuous.