1 Durables and non-durables consumption

1. Consider the following consumption problem. A consumer derives instantaneous utility according to $U(c,k) = c - c^2/2 + k - k^2/2$, where we may interpret c as consumption of a non-durable and k as a stock of durables that also give utility. The only way of saving is by investing in the durable. She gets a fixed income flow of $w > 0$ and the durables depreciates at rate $\delta \in [0,1]$ and the discount rate $r > 0$. The consumer thus solves

$$\begin{aligned} & \max_{\{c\}} \int_0^\infty e^{-rt} U(c,k)dt && (1) \\ s.t.\, \dot{k} &= w - c - \delta k && (2) \\ k(0) &= 0. && (3) \end{aligned}$$

   (a) Write the current value Hamiltonian and the necessary conditions for an optimum.

   Answer:

$$\begin{aligned} \mathcal{H} &= U(c,k) + \lambda (w - c - \delta k) && (4) \\ U_c(c,k) &= 1 - c = \lambda && (5) \\ \mathcal{H}_k &= 1 - k - \lambda\delta = r\lambda - \dot{\lambda} && (6) \end{aligned}$$

   (b) Take time derivatives of the condition involving the derivative of the Hamiltonian with respect to the control variable. Use this to get rid of the shadow value in the condition involving the derivative of the Hamiltonian with respect to the state variable. State your result in a system of differential equations for the control and the state variable.
   Answer:

$$\begin{aligned} -\dot{c} &= \dot{\lambda} && (7) \\ 1 - k - (1-c)\,\delta &= r\,(1-c) + \dot{c} && (8) \\ \dot{c} &= c\,(r+\delta) - k + 1 - \delta - r && (9) \\ \dot{k} &= w - c - \delta k && (10) \end{aligned}$$

   (c) Does the system have saddle path characteristics?
   Answer: Coefficient matrix is

$$\begin{bmatrix} r+\delta & -1 \\ -1 & -\delta \end{bmatrix} \quad (11)$$

   Eigenvalues: $\frac{1}{2}r + \frac{1}{2}\sqrt{\left(r^2 + 4 + 4r\delta + 4\delta^2\right)} > 0$, $\frac{1}{2}r - \frac{1}{2}\sqrt{\left(r^2 + 4 + 4r\delta + 4\delta^2\right)} < 0$.

(d) What are the steady states?

$$c^{ss} = \frac{w - (1 - (r+\delta))\,\delta}{(r+\delta)\,\delta + 1} \tag{12}$$

$$k^{ss} = \frac{1 + (w-1)\,(r+\delta)}{(r+\delta)\,\delta + 1} \tag{13}$$

(e) Draw a phase diagram of the system and mark the solution to the problem. Be careful when you draw the arrows. If you make further assumptions, state them clearly.
Answer:

$$\dot{c} = 0 \to c = \frac{r + k - 1 + \delta}{r+\delta} = 1 + \frac{k-1}{r+\delta} \tag{14}$$

$$\dot{k} = 0 \to c = w - \delta k \tag{15}$$

(f) Let the system be in a steady state. Assume that there is an unexpected increase in the wage w. Show how k and c responds to shock, immediately and in the long run. Indicate potential jumps as well as gradual adjustments.

(g) Provide an expression for the slope of the path towards the new steady state.
Answer: Stable eigenvector is:

$$\begin{bmatrix} -\frac{1}{2}r + \frac{1}{2}\sqrt{\left(r^2 + 4 + 4r\delta + 4\delta^2\right)} - \delta \\ 1 \end{bmatrix} \tag{16}$$

$$\frac{dc}{dk} = \left(-\frac{1}{2}r + \frac{1}{2}\sqrt{\left(r^2 + 4 + 4r\delta + 4\delta^2\right)} - \delta\right) > 0 \tag{17}$$

(h) Is the Hamiltonain concave? What are the transversality conditions?

$$D^2\mathcal{H} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} < 0. \tag{18}$$

$$\lim_{t\to\infty} e^{-rt}\lambda(t) = 0 \tag{19}$$

$$\lim_{t\to\infty} e^{-rt}\lambda(t)\,k(t) = 0 \tag{20}$$

2 Unemployment without savings

Consider an economy where individuals are either employed earning w or unemployed earning b. The probability per unit of time to get fired is q and the probability per unit of time to get hired is h. There is no access to savings and utility is given by the function $U(c)$ where $c = w$ or $c = b$. As we see, individuals have no choices here. Nevertheless, the standard recursive formulation of the value function holds.

Here the state variable is the employment status $s \in \{e, u\}$, the problem is autonmous and the value functions defined as

$$V(s_t) = E_t \int_0^\infty e^{-rs} U(c(s_{t+s}))\, ds \tag{21}$$

satisfies the continuous time Bellman equation

$$rV(s) = U(c(s)) + E\frac{dV(s)}{dt}. \tag{22}$$

Now, over a small interval dt,

$$\begin{aligned} EdV(e) &= (1 - qdt)0 + qdt(V(u) - V(e)) \qquad (23) \\ &\to E\frac{dV(e)}{dt} = q(V(u) - V(e)) \qquad (24) \end{aligned}$$

and

$$E\frac{dV(u)}{dt} = h(V(e) - V(u)) \tag{25}$$

We then write the equation for the value function as two equations, one for each of the two values the state variable can take.

$$\begin{aligned} rV(e) &= U(w) + q(V(u) - V(e)) \qquad (26) \\ rV(u) &= U(b) + h(V(e) - V(u)) \to \qquad (27) \\ V(e) &= \frac{(r+h)U(w) + qU(b)}{r(r+q+h)} \qquad (28) \\ V(u) &= \frac{(r+q)U(b) + hU(w)}{r(r+q+h)} \qquad (29) \end{aligned}$$