MathII at SU and SSE

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1 Introduction

This course is about dynamic systems, i.e., systems that evolve over time. The analysis of the dynamic evolvement of economic systems is of core importance in many areas of economics. Growth, business cycles, asset pricing, and dynamic game theory are just a few examples.

1.1 Solving a simple dynamic system

Very often, our economic models provide a difference or differential equation for the endogenous variables. Take a very simple example; an arbitrage asset pricing model. Suppose there is a safe asset, a bond, that provides a return r every period. Suppose a share, giving rights to dividend flow d, is introduced. Now, arbitrage theory says that the share should also yield a return r in equilibrium. Defining the price on the share as p, arbitrage theory thus implies,

$$\frac{p_{t+1}+d}{p_t} = 1+r. \tag{1}$$

This is a simple difference equation,

$$p_{t+1} = (1+r)\, p_t - d. \tag{2}$$

One straightforward way of solving it is to substitute forward or backward, e.g., noting that

$$\begin{aligned} p_t &= (1+r)\, p_{t-1} - d & (3)\\ &= (1+r)\left((1+r)\, p_{t-2} - d\right) - d & (4)\\ &= (1+r)^2\, p_{t-2} - d\left(1+(1+r)\right) & (5) \end{aligned}$$

and so on. A more general approach is to first characterize all possible paths consistent with the law-of-motion. Here, this is quite simple. You will learn that set of possible paths is

$$p_t = c\,(1+r)^t + \frac{d}{r} \tag{6}$$

for any constant c. As we see, there is an infinite number of solutions, i.e., we need more information. If, for example, we know that the value of p_0, we can solve for the constant

$$\begin{aligned} c &= p_0 - \frac{d}{r} & (7)\\ &\rightarrow\ p_t = \left(p_0 - \frac{d}{r}\right)(1+r)^t + \frac{d}{r}. & (8) \end{aligned}$$

In finance, the solution $p = \frac{d}{r}$ is called the fundamental solution, and we see that if $r > 0$, the solution explodes as t goes to infinity if $p_0 \neq \frac{d}{r}$. This gives us another way of solving the difference equation. Suppose, we have reason to believe that the solution should remain bounded. Then, if $r > 0$, the only solution left is when $c = 0$. Note that $(1 + r)$ is called the root of the system. Note, the importance of whether the root is bigger or smaller than unity (in absolute values).

We will also work in continuous time. Then, I usually use put the time variable in parenthesis and use the dot symbol to indicate time derivatives. In continuous time, non-existence of arbitrage means that capital gains, i.e., the change in the price per unit of time plus dividends per unit of time should equal the opportunity cost, i.e., interest rate on the price of the assets. Thus, non-existence of arbitrage implies

$$\dot{p}(t) + d = rp(t). \tag{9}$$

This is a simple linear first-order differential equation. The set if solution is:

$$p(t) = ce^{rt} + \frac{d}{r}. \tag{10}$$

Also here, we have a term that is explosive if $r > 0$.

Later in the course, we will learn how to solve more complicated dynamic systems, involving, e.g., several endogenous variables and varying parameters.

1.2 Two approaches to Dynamic Optimization

The second part of the course, is to solve maximization problems in dynamic systems. Suppose there is a potentially infinite set of paths $x(t)_0^T$, each denoting a particular continuous function $x(t)$ for $t \in [0, T]$. Suppose also that we can evaluate them, i.e., they give different payoffs. Then, we will learn how to derive difference or differential equations, that are necessarily satisfied for optimal paths. If we can solve these equations, we can find the optimal path. We will use two approaches in this course.

1.2.1 Dynamic Programming (Bellman).

Suppose we want to find an optimal investment plan in discrete time and let x_t denote the stock of capital at time t. Also, let u_t denote investments and assume

$$x_{t+1} = g(x_t, u_t), \tag{11}$$

which we call the law-of-motion for x_t.

Each period, the payoff is given by $F(t, x_t, u_t)$ and the problem is to solve

$$\begin{aligned} & \max_{u_0^T} \sum_{t=0}^{T} F(t, x_t, u_t), && (12) \\ \text{s.t. } x_{t+1} &= g(x_t, u_t) \forall t, && (13) \\ x_0 &= x. && (14) \end{aligned}$$

Note that this is a dynamic problem; The choice of investment at time t, u_t may affect payoffs in many future period. First, the payoff at t, $F(t, x_t, u_t)$ is affected directly and also the next periods payoff since

$$F(t+1, x_{t+1}, u_{t+1}) = F(t+1, g(x_t, u_t), u_{t+1}).$$

Furthermore, also payoffs further away can be affected since, for example, $x_{t+2} = g(x_{t+1}, u_{t+1}) = g(g(x_t, u_t), u_{t+1})$, affecting the payoff in period $t+2$. The choice of u_t must thus take into account all future payoff relevant effects.

Sometimes the dynamic problem degenerates in the sense that this dynamic link breaks. To illustrate this, let us simplify and take the example where $F(t, x_t, u_t) = \left(\frac{1}{1+r}\right)^t (f_t(x_t) - u_t)$ and $g(x_t, u_t) = (1-\delta) x_t - u_t$. We can interpret δ as the rate of capital depreciation. If $\delta = 1$, we have $x_{t+1} = u_t$ and by substituting from the law-of-motion, we can write the problem as

$$\begin{aligned} & \max_{x_1^T} \sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^t (f_t(x_t) - x_{t+1}), && (15) \\ x_0 &= x. && (16) \end{aligned}$$

The first-order condition for choosing x_t for any $t > 0$ is $f_t'(x_s) = 1+r$, so to know how much to invest in period t, we only need to know the marginal productivity of capital next period, i.e., we maximize period-by-period. With $\delta < 1$, we cannot do this. Then dynamic programing is a handy way to attack the problem. We will use the Bellman's principle of optimality, saying that there exists a sequence of functions $V_t(x_t)$ such that

$$V_t(x_t) = \max_{u_t} F(t, x_t, u_t) + V_{t+1}(g(x_t, u_t)) \tag{17}$$

and where

$$\begin{aligned} V_t(x_t) &\equiv \max_{u_t^T} \sum_{s=t}^{T} F(s, x_s, u_s) && (18) \\ \text{s.t. } x_{t+1} &= g(x_t, u_t) && (19) \\ x_t &= x_t. && (20) \end{aligned}$$

$V_t(x_t)$ is called the value function and x_t a state variable. We find that through the Bellman principle, we have split up the problem into a sequence of one period problems. We can then solve the problem more easily since the first-order condition for maximizing the RHS of (17)

$$F_u(t, x_t, u_t) + V'_{t+1}(g(x_t, u_t))\, g_u(x_t, u_t) = 0. \tag{21}$$

implicitly yields a system of difference equations in x_t and u_t that we may be able to solve.

1.2.2 Optimal control (Pontryagin)

The other way of solving dynamic optimization problems that we will use is called Optimal Control. We will use it for continuous time problems. Suppose we want to solve

$$\begin{aligned} & \max_{u_0^T} \int_0^T F(t, x(t), u(t))\, dt && (22) \\ s.t.\ \dot{x}(t) &= g(x(t), u(t)) && (23) \\ x(0) &= x_0 && (24) \\ x(T) &= x_T. && (25) \end{aligned}$$

Then, Pontryagin's maximum principle says that for each point in time, the optimal control, call it $u^*(t)$, satisfies

$$u^*(t) = \arg\max_{u(t)} F(t, x(t), u(t)) + \lambda(t)\, g(x(t), u(t)), \tag{26}$$

where $\lambda(t)$ can be interpreted as the shadow value of the state variable (capital). The sum $F(t, x(t), u(t)) + \lambda(t)\, g(x(t), u(t))$, is called the Hamiltonian. Again, we have turned the dynamic problem into a sequence of static problems. The first order condition for (26) will implicitly define a system of differential equations. Note the similarity between (26) and (17).