2 Some basics

2.1 Taylor series

We will often need to approximate a function around some point. Specifically, suppose we now $f(x_0)$ and some of its derivatives and want to approximate the value in some other point $x \neq x_0$. An efficient way of doing such an approximation is to use Taylors formula. This can be seen as an attempt to fit a polynomial (we will talk about such below) to a curve.

Result 1 *The Taylor approximation is*

$$\begin{aligned} f(x) &\approx f(x_0) + \frac{1}{1!} f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 \\ &+ \frac{1}{3!} f'''(x_0)(x - x_0)^3 + ... \end{aligned} \tag{27}$$

Usually we will just use the first order approximation $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ but sometimes a higher order approximation can be useful. It can furthermore be shown, that the approximation error of an n'th order Taylor approximation is given by

$$\frac{1}{(n+1)!} \frac{d^{(n+1)} f(c)}{dx^{(n+1)}} (x - x_0)^{n+1} \tag{28}$$

where $c \in [x_0, x]$. As we see, when $x - x_0$ is small, two forces imply that the approximation error tends to be small when n is large. In the denominator, $(n+1)!$ is large and $(x - x_0)^{n+1}$ is small. In general, we cannot say that for any $(x - x_0)$ a higher order approximation is better. But we can say that the approximation error goes faster to zero when we let $(x - x_0)$ go to zero if we use a higher order approximation.

2.2 Integration

If $b > a$, the expression

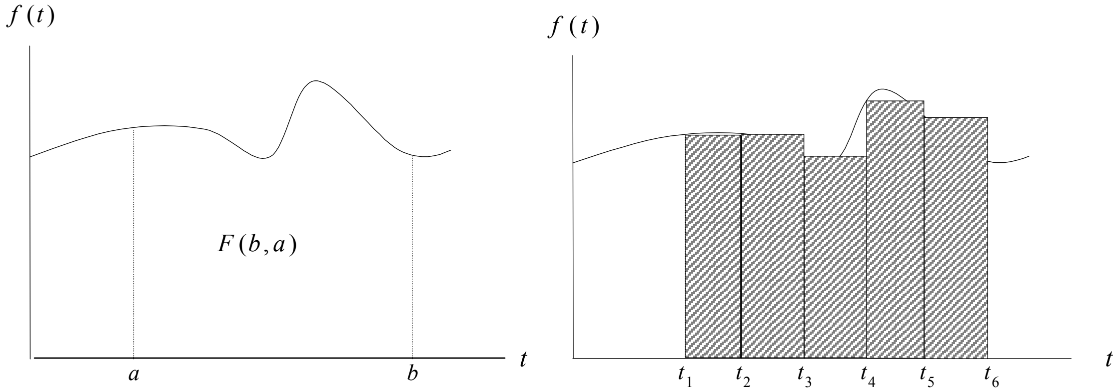
$$\int_a^b f(t)\, dt, \tag{1}$$

can be interpreted as the area under the graph $y = f(x)$ for $x \in [a, b]$. How should one compute such an area? The most natural way would be to divide the interval $[a, b]$ into (many) sub-intervals by choosing numbers $a = t_1 <$

$t_2 < t_3 ... t_n = b$. Suppose we choose uniform intervals, $t_{i+1} - t_i \equiv \Delta t = \frac{b-a}{n-1}$. Then we can approximate the area by

$$\int_a^b f(t)\, dt \approx \sum_{i=1}^{n-1} f(t_i)\, \Delta t, \tag{2}$$

i.e., by summing rectangles of base Δt and height $f(t_i)$.



If the function $f(t)$ is bounded and differentiable and we let the number of sub-intervals (n) increase and therefore size of each one decrease, the approximation (2) becomes perfect as $n \to \infty$. To see this, note that we can approximate the error in the approximation by the triangle $\frac{f(t_{i+1}) - f(t_i)}{2} \Delta t$. Since $f(t_{i+1}) \approx f(t_i) + f'(t_i)\, \Delta t$, by a first order Taylor approximation, each triangle can be approximated by

$$\frac{f'(t_i)\, \Delta t}{2} \Delta t = \frac{f'(t_i)}{2} \Delta t^2. \tag{3}$$

Furthermore, the sum of errors,

$$\left| \sum_{i=1}^{n-1} \frac{f'(t_i)}{2} \Delta t^2 \right| \leq \max_i |f'(t_i)| \frac{1}{2} \sum_{i=1}^{n-1} \Delta t^2 = \tag{4}$$

$$\max_i |f'(t_i)| \frac{1}{2} (n-1) \left(\frac{b-a}{n-1} \right)^2 = \max_i |f'(t_i)| \frac{1}{2} \frac{(b-a)^2}{n-1}. \tag{5}$$

Clearly, this goes to zero as n goes to infinity. If the function $f(t)$ has discontinuities or is non-differentiable somewhere, we can do the summation for each interval where $f(t)$ is continuous.

We conclude that we should think of the integral as a sum of rectangles, each with height given by the function we are integrating and base dt.

If we can handle integration over compact intervals, we can also define integrals over unbounded intervals by taking the limit value as integration limits (a, and/or b) approach infinity. Of course, this limit does not always exist. If it does, we use the notation

$$\lim_{a\to-\infty}\int_a^b f(t)\,dt \equiv \int_{-\infty}^b f(t)\,dt \tag{6}$$

$$\lim_{b\to\infty}\int_a^b f(t)\,dt \equiv \int_a^\infty f(t)\,dt \tag{7}$$

2.2.1 Fundamental theorem of calculus

An integral is a generalization of a sum, and a derivative is a generalization of a difference. The following theorem links these concepts. The first part of the *Fundamental Theorem* says that if

$$F(b) = \int_a^b f(t)\,dt \tag{8}$$

then

$$F'(b) = \frac{\partial}{\partial b}\int_a^b f(t)\,dt = f(b). \tag{9}$$

(Convince yourself that this is reasonable by making a drawing.) A function $F(t)$ that has the property that $F'(t) \equiv f(t)$ is called a primitive for f. Clearly, if $F(t)$ is a primitive for $f(t)$, then also $F(t)$ plus any constant, is a primitive. Thus, the first part of the fundamental theorem provides a necessary, but not sufficient condition for finding the area.

Now let us turn to he second part of the theorem, which provides a way of calculating the exact value of an integral like in (1).

Result 2 *Let F be* any *primitive of f, then*

$$\int_a^b f(t)\,dt = F(b) - F(a) \equiv [F(t)]_a^b\,. \tag{10}$$

Sometimes it is easy to find the primitive, like in the cases $f(t) = t^a$, $1/(at+b)$ or e^{at}, in which case $F(t)$ is $t^{a+1}/(a+1)$, $\ln(at+b)/a$, or e^{at}/a respectively. In other cases like $\pi^{0.5}e^{-x^2}$, the primitive cannot be expressed by using standard algebraic functions. This does not mean that the primitive does not exist. In the case $\pi^{0.5}e^{-x^2}$, a primitive is the cumulative normal distribution, which certainly exists.

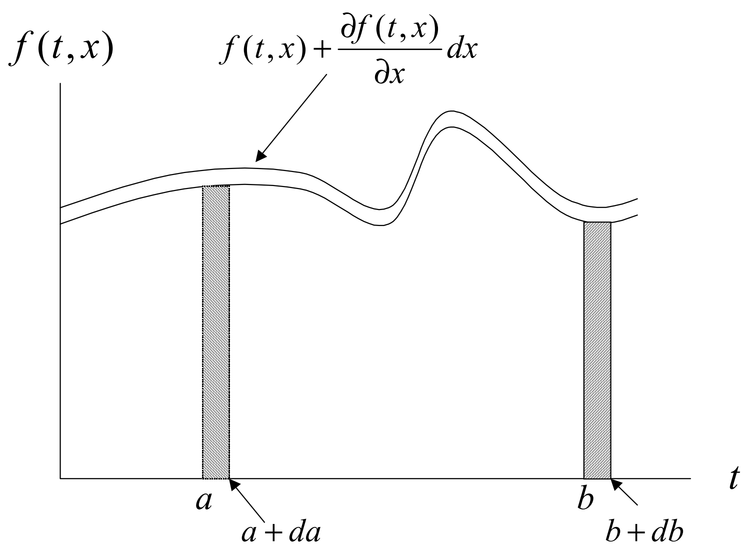
Since the integral is a (kind of) sum, it is straightforward to understand Liebniz' rule.

Result 3 *If f is differentiable, then*

$$\frac{\partial}{\partial x}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt, \tag{11}$$

$$\frac{\partial}{\partial a}\int_a^b f(x,t)\,dt = -f(x,a), \tag{12}$$

$$\frac{\partial}{\partial b}\int_a^b f(x,t)\,dt = f(x,b). \tag{13}$$



2.2.2 Change of variables

Suppose that $y = g(x)$, then the rules of differentiation gives $dy = g'(x)dx$. Now let us calculate the area under some function $f(y)$ but integrating over x. In a sense, this is like changing the scale of the horisontal axis. To do this, we simply substitute $y = g(x), dy = g'(x)dx$. To get the integration limits, we note that if $x = a$ then $y = g(a)$.

Result 4 *If, $y = g(x)$ then*

$$\int_{g(a)}^{g(b)} f(y)\, dy = \int_a^b f(g(x))\, g'(x) dx. \tag{14}$$

Sometimes a variable substitution makes integration simpler. Take the following example;

$$\int_1^2 \left(x^2 + 1\right)^{10} 2x dx. \tag{15}$$

Now define $y \equiv x^2 + 1 \equiv g(x)$ implying

$$dy = 2x dx, g(1) = 2, g(2) = 5 \tag{16}$$

$$\int_1^2 \left(x^2 + 1\right)^{10} 2x dx = \int_2^5 y^{10} dy \tag{17}$$

2.2.3 Integration by parts

Another important result, which we will use a lot, is the following rule for integration by parts. Assume we want to integrate a product of two functions of x, i.e., $u(x)v(x)$. Then, let $U(x)$ be a primitive of u. The rule for differentiation of products says

$$\frac{d(U(x)\, v(x))}{dx} = u(x)\, v(x) + U(x)\, v'(x) \tag{18}$$

$$\rightarrow u(x)\, v(x) = \frac{d(U(x)\, v(x))}{dx} - U(x)\, v'(x). \tag{19}$$

Then, we can integrate over x on both sides, giving

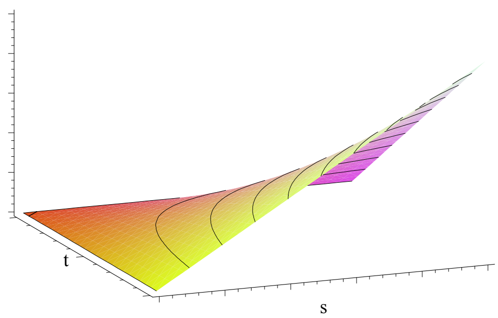
Result 5

$$\int_a^b u(x)\, v(x)\, dx = \int_a^b \frac{d\left(U(x)\, v(x)\right)}{dx} dx - \int_a^b U(x)\, v'(x)\, dx \tag{20}$$

$$= \left[U(x)\, v(x)\right]_a^b - \int_a^b U(x)\, v'(x)\, dx \tag{21}$$

2.2.4 Double integration

As we know, the integral is an area under a curve $f(t)$ over an interval $[a, b]$ in the t dimension. Similarly, we can compute the volume under a plane with a height $f(t, s)$ over an area in the t, s dimension. For example, let $f(t, s)$ be $t \cdot s$ and integrate over a rectangle with sides $[a_t, b_t]$ and $[a_s, b_s]$.



The function, $z = ts$

The volume under the plane $f(t, s)$ is then given by

$$\int_{a_s}^{b_s} \int_{a_t}^{b_t} f(t, s)\, dt ds = \int_{a_s}^{b_s} \left(\int_{a_t}^{b_t} ts dt \right) ds = \tag{22}$$

$$\int_{a_s}^{b_s} \left(\left[\frac{t^2}{2} s \right]_{a_t}^{b_t} \right) ds = \int_{a_s}^{b_s} s \left(\frac{b_t^2 - a_t^2}{2} \right) ds = \tag{23}$$

$$\left(\frac{b_t^2 - a_t^2}{2} \right) \left[\frac{s^2}{2} \right]_{a_s}^{b_s} = \left(\frac{b_t^2 - a_t^2}{2} \right) \left(\frac{b_s^2 - a_s^2}{2} \right) \tag{24}$$

Note, that we are first calculating the *area* under $f(t, s)$ over the interval $[a_t, b_t]$for each s. This area is a function of s which we then integrate over the interval $[a_s, b_s]$ in the s-dimension. If we integrate over other areas than rectangles, the limits of integration are not independent. For example, we may integrate over a triangle where $b_t = s$. Simplify and set $a_t = a_s = 0$. Then, we have

$$\int_0^{b_s} \int_0^s f(t,s)\, dtds = \int_0^{b_s} \left(\int_0^s tsdt \right) ds = \int_0^{b_s} \left(\left[\frac{t^2}{2} s \right]_0^s \right) ds = \tag{25}$$

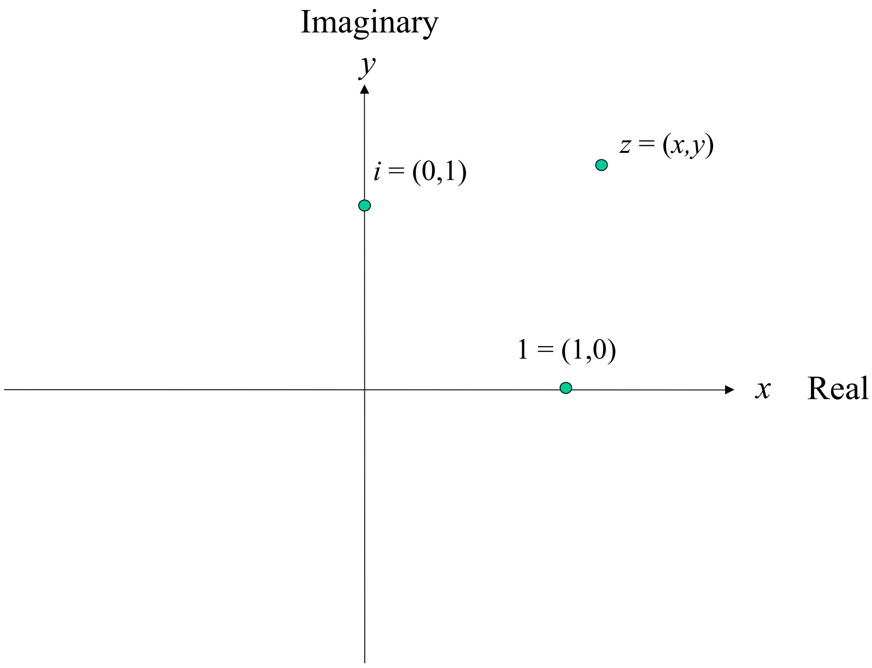
$$\int_0^{b_s} s \frac{s^2}{2} ds = \left[\frac{s^4}{8} \right]_0^{b_s} = \frac{b_s^4}{8}. \tag{26}$$

2.3 Complex numbers and trigonometric functions

The formula for the solution to a quadratic equation $ax^2 + bx + c = 0$, is

$$x = -\frac{b}{2a} \pm \frac{\sqrt{(b^2 - 4ac)}}{2a}. \tag{27}$$

If $b^2 - 4ac < 0$. The solution is not in the set of real numbers. The introduction of complex numbers intended to extend the space of solutions to accommodate such cases and it turns out that for all numbers in the extended space, we can always find solutions to such equations. We can think of complex numbers as two-dimension objects $z = (x, y)$. The first number, x, provides the value in the standard real dimension, while the second provides the value in the other dimension, called *imaginary*. Thus, real numbers are a special sub-set of complex number such that $y = 0$.



The rules for addition and multiplication with complex numbers are the following

$$
\begin{aligned}
z_1 + z_2 &= (x_1 + x_2, y_1 + y_2) && (28)\\
z_1 z_2 &= (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) && (29)
\end{aligned}
$$

Using these rules, we can compute the square of a complex number only consisting of a unitary imaginary part, i.e., $z = i = (0, 1)$.

$$(0,1)^2 = (0 - 1, 0 + 0) = (-1, 0) = -1 \tag{30}$$

We thus established the important result

Result 6 *$x = i$ is a solution to the equation $x^2 = -1$.*

Using the rules above, it is also straightforward to show that an alternative we of writing z is given by the following

$$z \equiv (x, y) = (x, 0) + (0, y) = x\,(1, 0) + y\,(0, 1) = x + yi. \tag{31}$$

We should also note that

$$(x, y)(x, -y) = \left(x^2 + y^2, -xy + xy\right) = \left(x^2 + y^2, 0\right) \equiv |(x, y)|^2 \tag{32}$$

The numbers (x, y) and $(x, -y)$ are called complex conjugates and the value $|x, y|$ is called the modulus of (x, y).

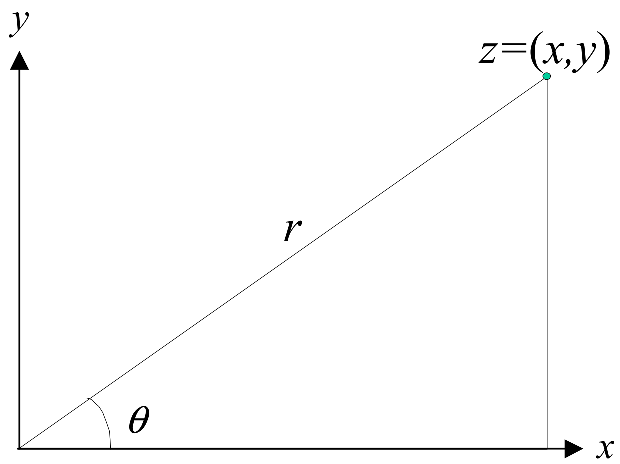
2.3.1 Polar representation

Recall that in the right-angled triangle in the figure, we have

$$\cos(\theta) = \frac{x}{r} \tag{33}$$

$$\sin(\theta) = \frac{y}{r} \tag{34}$$

$$r = \sqrt{x^2 + y^2}. \tag{35}$$



Using this, we can represent the complex number z either by its coordinates, (x, y) or alternatively as $r(\cos(\theta) + i\sin(\theta))$. The latter form is called the polar representation and r is the *modulus* and θ is the *argument* of z. Usually we measure θ in radians.

We will use the following result below.

Result 7 *Let z be the complex number (x, y) , then z satisfies*

$$z = re^{i\theta} \tag{36}$$

where r is the modulus of z and θ is the argument. In particular, for $r = 1$ we get,

$$\begin{aligned} e^{i\theta} &= \cos(\theta) + i\sin(\theta) && (37)\\ e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta) && (38)\\ e^{i\pi} &= \cos(\pi) + i\sin(\pi) = -1 && (39) \end{aligned}$$

Optional proof:

The Taylor formula around zero for any function $f(x)$ is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3... \quad (40)$$

Using this for $f(x) = e^x, \cos(x)$ and $\sin(x)$, and the rules

$$\begin{aligned} \frac{\partial}{\partial x}\cos(x) &= -\sin(x) && (41)\\ \frac{\partial}{\partial x}\sin(x) &= \cos(x) && (42) \end{aligned}$$

we have respectively

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}... && (43)\\ \cos(x) &= \cos(0) - \sin(0)x - \frac{\cos(0)}{2!}x^2 + \frac{\sin(0)}{3!}x^3 && (44)\\ &+ \frac{\cos(0)}{4!}x^4 - \sin\frac{\sin(0)}{5!}x^5 - \frac{\cos(0)}{6!}x^6... && (45)\\ &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}... && (46)\\ \sin(x) &= \sin(0) + \cos(0)x - \frac{\sin(0)}{2!}x^2 - \frac{\cos(0)}{3!}x^3 && (47)\\ &+ \frac{\sin(0)}{4!}x^4 + \frac{\cos(0)}{5!}x^5 - \frac{\sin(0)}{6!}x^6 - \frac{\cos(0)}{7!}x^7... && (48)\\ &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 && (49) \end{aligned}$$

Thus, using the Taylor formula around zero to evaluate $f(i\theta) = e^{i\theta}$, we have

$$
\begin{aligned}
e^{i\theta} &= 1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\frac{(i\theta)^5}{5!}+\frac{(i\theta)^6}{6!}... && (50)\\
&= 1+i\theta-\frac{1}{2}\theta^2-\frac{1}{3!}i\theta^3+\frac{1}{4!}\theta^4+\frac{1}{5!}i\theta^5-\frac{1}{6!}\theta^6 && (51)\\
&= 1-\frac{1}{2}\theta^2+\frac{1}{4!}\theta^4-\frac{1}{6!}\theta^6 && (52)\\
&\quad +i\theta-\frac{1}{3!}i\theta^3+\frac{1}{5!}i\theta^5 && (53)\\
&= \cos(\theta)+i\sin(\theta) && (54)
\end{aligned}
$$

2.3.2 Polynomials

A polynomial $P(z)$ of order n is defined as weighted sum of z^s for $s \in \{0, ..., n\}$, i.e.,

$$P(z) \equiv a_n z^n + a_{n-1} z^{n-1} +, ..., +a_1 z^1 + a_0, \qquad (55)$$

for some sequence of constants $\{a_s\}_0^n$. The following will be important for our analysis of difference and differential equations.

Result 8 ***Result 9*** *A polynomial $P(z)$ of order n has exactly n, not necessarily distinct, roots. I.e., it can be expressed as*

$$P(z) = a_n (z - r_1)(z - r_2) ... (z - r_n) \qquad (56)$$

From (56), we see that each root r_i is a solution to the equation $P(z) = 0$. Note that the roots may be repeated, i.e.,$r_i = r_j$ and that, of course, some root may be complex. It turns out, also, that complex roots always come in pairs, the complex conjugates.