4 Difference equations

4.1 Sums, forward and backward solutions

4.1.1 Sums vs. Integrals

Difference equations can be solved in ways very similar to how we solve differential equations. First, we will look at the analogy to integrals.

Consider the difference equation

$$A_t - A_{t-1} \equiv \Delta A_t = q_t \tag{1}$$

where ΔA_t is the change in A per unit (interval) of time. We sum both sides from sum date t_0 until t to get

$$\sum_{s=t_0}^{t} \Delta A_s = A_t - A_{t_0-1} = \sum_{s=t_0}^{t} q_s. \tag{2}$$

We can then write the solution as

$$\Delta A_t = q_t \tag{3}$$

$$\rightarrow \quad A_t = \sum_{s=t_0}^{t} q_s + A_{t_0-1}. \tag{4}$$

As we see, this is very much like integrals

$$\frac{dA(t)}{dt} = q(t) \tag{5}$$

$$\rightarrow \quad A(t) = \int_{t_0}^{t} q(s)\, ds + A_{t_0}, \tag{6}$$

and the relation between q_t and $\sum_{s=t_0}^{t} q_s$ is the same as between $q(t)$ and its primitive, since

$$\Delta \sum_{s=t_0}^{t} q_s = q_t. \tag{7}$$

4.1.2 Forward and backward solutions

The part $\sum_{s=t_0}^{t} q_t$ of the RHS of (4) is exogenous, i.e., independent of A_t. Sometimes, it converges in one or both the directions. Then we can write the solutions in another way. Suppose the following limit exists

$$\lim_{T\to\infty}\sum_{s=-T}^{t} q_s \equiv \sum_{s=-\infty}^{t} q_s. \tag{8}$$

Then, the other part of RHS of (4) should also have a well-defined limit

$$\lim_{T\to\infty} A_{-T} \equiv \underline{A}, \tag{9}$$

so that the solution is

$$A_t = \sum_{s=-\infty}^{t} q_s + \underline{A}. \tag{10}$$

Clearly, (10) solves (1),

$$A_t - A_{t-1} = \sum_{s=-\infty}^{t} q_s + \underline{A} - \sum_{s=-\infty}^{t-1} q_s - \underline{A} = q_t. \tag{11}$$

If the she solution in (10) exists, it is called the *backward solution.*

Analogously, the limit

$$\lim_{T\to\infty}\sum_{s=t}^{T} q_s \equiv \sum_{s=t}^{\infty} q_s, \tag{12}$$

might exists, in which case

$$\lim_{T\to\infty} A_T \equiv \bar{A} \tag{13}$$

also exists. Then, we have

$$\begin{aligned} \lim_{T\to\infty}(A_T - A_t) &= \sum_{s=t+1}^{\infty} q_s, && (14)\\ &\to \quad A_t = \bar{A} - \sum_{s=t+1}^{\infty} q_s, && (15) \end{aligned}$$

which is called the *forward solution.*

Example. Suppose q_t follows a simple $AR(1)$ process

$$\begin{aligned} \Delta A_t &= q_t, && (16)\\ q_t &= r q_{t-1.} && (17) \end{aligned}$$

If, $r < 1$, we can use the forward solution and if $r > 1$, the backward solution works. In the former case,

$$\sum_{s=t+1}^{\infty} q_s = \sum_{s=0}^{\infty} q_{t+1} r^s = \frac{q_{t+1}}{1-r}, \tag{18}$$

$$\rightarrow \quad A_t = \bar{A} - \frac{q_{t+1}}{1-r}, \tag{19}$$

where $\bar{A}$ is determined from, for example, an initial condition. In the latter case,

$$\sum_{s=-\infty}^{t} q_s = \sum_{s=0}^{\infty} q_t r^{-s} = q_t \frac{r}{r-1} \tag{20}$$

$$A_t = q_t \frac{r}{r-1} + \underline{A}. \tag{21}$$

4.1.3 First order difference equations with constant coefficients

A first order difference equation with constant coefficients has the following form.

$$x_t - a x_{t-1} = c \tag{22}$$

As we see, the LHS is not a pure difference, as in (1), so we cannot simply sum over t. Instead we rely on the following result.

Result 12 *The general solution (the complete set of solutions) to a difference equation is the general solution to the homogeneous part plus any particular solution to the complete equation.*

The general solution to the homogeneous first order difference equation with coefficient a can be written

$$x_t^h = A a^t \tag{23}$$

where A is an arbitrary constant.

A particular solution is sometimes the steady state which exists for (22) if $a \neq 1$.

$$x^{ss} - a x^{ss} = c \tag{24}$$

$$\rightarrow \quad x^{ss} = \frac{c}{1-a}. \tag{25}$$

4.1.4 Non-constant RHS

Now look at

$$x_t - ax_{t-1} = q_t. \tag{26}$$

A way to solve (26) is to use a^{-t} as the analogue to the integrating factor. Multiplying both sides by a^{-t} yields

$$a^{-t}(x_t - ax_{t-1}) = a^{-t}q_t. \tag{27}$$

Now, we see that the LHS can be written $\Delta(a^{-t}x_t)$, implying

$$\begin{aligned}
\Delta\left(a^{-t}x_t\right) &= a^{-t}q_t, && (28)\\
&\Rightarrow a^{-t}x_t = \sum_{s=t_0}^{t} a^{-s}q_s + A, && (29)\\
a^{-t}x_t &= \sum_{s=t_0}^{t} a^{-s}q_s + A, && (30)\\
x_t &= Aa^t + \sum_{s=t_0}^{t} a^{t-s}q_s. && (31)
\end{aligned}$$

Again, we should verify that this satisfies our original difference equation.

4.1.5 Stable growth – the Solow growth model

Often in macro, the variables in the model grow in a way that precludes the existence of a steady state. However, some transformation of the variables might possess a steady state. The simplest example of this is the Solow growth model Here, savings is exogenous and denoted S and the labor supply N_t follows

$$N_t = e^g N_{t-1} \approx (1+g)\,N_{t-1}. \tag{32}$$

There is one good, used for consumption and as capital, which follows the law-of-motion

$$K_{t+1} = Sf(K_t, N_t), \tag{33}$$

where $f(K_t, N_t)$ is a concave production function. Let us specify production as the CRS Cobb-Douglas function,

$$\begin{aligned}
f(K_t, N_t) &= N_t^{1-\alpha}K_t^{\alpha}, && (34)\\
1 &> \alpha > 0. && (35)
\end{aligned}$$

Letting lower case letters denote natural logarithms, we have

$$k_{t+1} = s + (1-\alpha) n_t + \alpha k_t, \tag{36}$$
$$k_{t+1} - \alpha k_t = s + (1-\alpha) n_t, \tag{37}$$

and

$$\Delta n_t = g \tag{38}$$
$$\rightarrow \quad n_t = \sum_0^t g + n = tg + n, \tag{39}$$

for some constant n.

Here, we can guess on a particular solution of the same form as the RHS, i.e.,

$$k_t = k + tg_k \tag{40}$$
$$k + (t+1) g_k - \alpha (k + tg_k) = s + (1-\alpha) n_t \tag{41}$$
$$k (1-\alpha) + t (1-\alpha) g_k + g_k = s + (1-\alpha) n + t (1-\alpha) g \tag{42}$$

For this to hold for all t, we need

$$g_k = g, \tag{43}$$
$$k = \frac{s-g}{1-\alpha} + n. \tag{44}$$

giving a particular solution

$$k_{p,t} = \frac{s-g}{1-\alpha} + n + tg. \tag{45}$$

This solution is in economics often called a *balanced growth path*, in which all variables grow at the same rate. The complete solution is

$$k_t = Aa^t + \frac{s-g}{1-\alpha} + n + tg, \tag{46}$$
$$k_t = \left(k_0 - \left(\frac{s-g}{1-\alpha} + n\right)\right) a^t + \frac{s-g}{1-\alpha} + n + tg. \tag{47}$$

In this case, a convenient alternative is to define a new variable, capital per capita, $C_t \equiv K_t/N_t \rightarrow c_t = k_t - n_t$. Using this, and dividing (34) by N_t, we get

$$
\begin{aligned}
\frac{K_{t+1}}{N_t} &= \frac{Sf\left(K_t, N_t\right)}{N_t} && (48)\\
\frac{K_{t+1}N_{t+1}}{N_{t+1}N_t} &= \frac{SN_t^{1-\alpha}K_t^{\alpha}}{N_t} = && (49)\\
C_{t+1}e^{g} &= S\left(\frac{K_t}{N_t}\right)^{\alpha} && (50)\\
c_{t+1} + g &= s + \alpha c_t && (51)\\
c_{t+1} - \alpha c_t &= s - g && (52)\\
&\rightarrow c^{ss} = \frac{s-g}{1-\alpha} && (53)\\
c_t &= \left(c_0 - \frac{s}{1-\alpha}\right)\alpha^t + \frac{s}{1-\alpha}, && (54)
\end{aligned}
$$

coinciding with the solution in (46) but now expressed as steady state in capital per capita, rather than a balanced growth path for capital.

We can also look at the log of output per capita,

$$
\begin{aligned}
y_t &= \alpha c_t, && (55)\\
y_t &= \left(y_0 - \alpha\frac{s}{1-\alpha}\right)\alpha^t + \alpha\frac{s}{1-\alpha}. && (56)
\end{aligned}
$$

This is the basis for the so-called growth regressions, pioneered by Barro,

$$
y_t - y_0 = -y_0\left(1-\alpha^t\right) + \frac{s}{1-\alpha}\alpha\left(1-\alpha^t\right), \tag{57}
$$

where growth over a sample period 0 through t is seen to depend negatively on initial output and positively on savings. Also,

$$
\begin{aligned}
y_t - y_{t-1} &= \left(y_0 - \alpha\frac{s}{1-\alpha}\right)\alpha^t + \alpha\frac{s}{1-\alpha} && (58)\\
&\quad - \left(y_0 - \alpha\frac{s}{1-\alpha}\right)\alpha^{t-1} - \alpha\frac{s}{1-\alpha} && (59)\\
&= -\left(y_0 - \alpha\frac{s}{1-\alpha}\right)\alpha^{t-1}\left(1-\alpha\right) && (60)\\
&= \left(\alpha\frac{s}{1-\alpha} - y_{t-1}\right)\left(1-\alpha\right). && (61)
\end{aligned}
$$

As we see, the growth rate (the log difference) of output per capita is a fraction $(1-\alpha)$ of the difference between the steady state and current output per capita. Note that convergence is *slower* the larger is the capital's share of output. In the limit when $\alpha \to 1$, the model shows no convergence and has become an *endogenous growth* model, with

$$\begin{aligned} \Delta k_t &= s, & (62) \\ \Delta c_t &= \Delta y_t = s - g. & (63) \end{aligned}$$

4.2 Linear difference equations of higher order

4.2.1 Higher order homogeneous difference equations with constant coefficients

Consider the homogeneous difference equation

$$x_{t+n} + a_1 x_{t+n-1} + ...a_n x_t = 0, \tag{64}$$

Definition 4 *The forward operator E is defined by*

$$E^s x_t \equiv E x_{t+s} \tag{65}$$

where s is any integer, positive or negative.

We can then write (64) in a condensed polynomial form

$$P(E) x_t = 0. \tag{66}$$

We then have to find the roots of the equation

$$P(r) = r^n + a_1 r^{n-1} + ...a_n = 0 \tag{67}$$

Each root contributes to the general solution with one term that is independent of the others, exactly as with differential equations.

Result 13 *Let r_s denote the roots to the polynomial $P(r)$, i.e., all solutions to $P(r) = 0$. Let the first $k \geq 0$ roots be distinct and the remaining $l = n - k$ roots repeated. Then, the general solution to (64) is*

$$x_t = c_1 r_1^t + ... + c_k r_k^t + r_{k+1}^t \left(c_{k+1} + t c_{k+2} + ... + t^{l-1} c_{k+l} \right) \tag{68}$$

If there is more than one set of repeated roots, each set of size $m \geq 2$ contributes with m linearly independent terms

$$r_{k+1}^t \left(c_{k+1} + t c_{k+2} + ... + t^{m-1} c_{k+m} \right). \tag{69}$$

In the case of complex roots, express the complex number in polar form

$$\begin{aligned} r &= a+bi = |r|\left(\cos\theta + i\sin\theta\right), & (70)\\ |r| &= \sqrt{a^2+b^2}, & (71)\\ \theta &= \tan^{-1}\frac{b}{a}. & (72) \end{aligned}$$

We then use the fact that

$$\begin{aligned} e^{i\theta} &= \left(\cos\theta + i\sin\theta\right) & (73)\\ r &= |r|\,e^{i\theta} \rightarrow r^t = |r|^t\,e^{i\theta t} = |r|^t\left(\cos t\theta + i\sin t\theta\right) & (74) \end{aligned}$$

to get for the complex conjugates $r_{1,2} = a \pm bi$,

$$c_1 r_1^t + c_2 r_2^t = |r|^t\left(\tilde{c}_1 \cos t\theta + \tilde{c}_2 \sin t\theta\right) \tag{75}$$

which we can see is a generalization of (68). Complex roots thus give us oscillating solutions.

4.2.2 Stability

From the solutions (68) and (75), it is clear that roots such that $|r| < 1$,give converging terms.

4.2.3 Higher order non-homogeneous difference equations with constant coefficients

Result 14 *The general solution to a non-homogeneous difference equation with constant coefficients is given by the general solution to the homogeneous part plus any solution to the full equation.*

To solve non-homogeneous equations we thus have to find particular solution to the complete equation to add to the general solution of the homogeneous part. The simplest non-homogeneous difference equation is

$$P\left(x_t\right) = c. \tag{76}$$

Here, we try a steady state

$$\begin{aligned} P\left(x^{ss}\right) &= P\left(1\right)x^{ss} = c & (77)\\ x^{ss} &= \frac{c}{P\left(1\right)} & (78) \end{aligned}$$

provided $P\left(1\right) \neq 0$.

In the more general case we often have to guess a particular solution.

4.3 Systems of linear first order difference equations

Systems of first order difference equations are solved with the diagonalization method that we also used for the differential equation.

$$\begin{aligned}
\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \\ . \\ x_{n,t+1} \end{bmatrix} &= \mathbf{A} \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ . \\ x_{n,t} \end{bmatrix} + \mathbf{P} && (79) \\
\mathbf{x}_{t+1} &= \mathbf{A}\mathbf{x}_t + \mathbf{P}. && (80)
\end{aligned}$$

First we find the diagonalizing matrix of eigenvectors $\mathbf{B}^{-1}$. Then we solve the homogeneous equation by defining

$$\begin{aligned}
\mathbf{y}_{t+1} &= \mathbf{B}\mathbf{x}_{t+1} && (81) \\
&= \mathbf{B}\mathbf{A}\mathbf{x}_{t+1} && (82) \\
&= \mathbf{B}\mathbf{A}\mathbf{B}^{-\mathbf{1}}\mathbf{y}_t && (83) \\
&= \begin{bmatrix} r_1^t & 0 & . & 0 \\ 0 & r_2^t & . & 0 \\ . & . & . & . \\ 0 & 0 & . & r_n^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ . \\ c_n \end{bmatrix} && (84) \\
&\equiv \mathbf{r}^t\mathbf{c}. && (85)
\end{aligned}$$

Then we transform back[2]

$$\mathbf{x}_{t+1} = \mathbf{B}^{-1}\mathbf{y}_{t+1} = \mathbf{B}^{-1}\mathbf{r}^t\mathbf{c}. \quad (86)$$

A particular solution to the complete equation then has to be added. Here we might try to find a steady state as a particular solution

$$\begin{aligned}
\mathbf{x}^{ss} &= \mathbf{A}\mathbf{x}^{ss} + \mathbf{P}, && (87) \\
\mathbf{x}^{ss} &= (\mathbf{I} - \mathbf{A})^{-1}\mathbf{P}. && (88)
\end{aligned}$$

If we have the initial conditions we find that

$$\begin{aligned}
\mathbf{x}_{t+1} &= \mathbf{B}^{-1}\mathbf{r}^t\mathbf{c} + \mathbf{x}^{ss} && (89) \\
\mathbf{x}_1 &= \mathbf{B}^{-1}\mathbf{r}^0\mathbf{c} + \mathbf{x}^{ss} && (90) \\
&= \mathbf{B}^{-1}\mathbf{c} + \mathbf{x}^{ss} && (91) \\
\mathbf{c} &= \mathbf{B}(\mathbf{x}_1 - \mathbf{x}^{ss}) && (92) \\
\mathbf{x}_t &= \mathbf{B}^{-1}\mathbf{r}^t\mathbf{B}(\mathbf{x}_1 - \mathbf{x}^{ss}) + \mathbf{x}^{ss} && (93)
\end{aligned}$$

[2] Since the vector of constants, $\mathbf{c}$, is arbitrary, we may equally well write $\mathbf{x}_t = \mathbf{B}^{-1}\mathbf{r}^t\mathbf{c}$.

An example;

$$\mathbf{x}_{t+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}_t + \begin{bmatrix} -1 \\ -1 \end{bmatrix}. \tag{94}$$

$$\mathbf{x}^{ss} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{95}$$

Eigenvalues are

$$r_{1,2} = i, -i \to r_{1,2}^t = 1^t \left(\cos\left(\frac{\pi}{2} t\right) \pm i \sin\left(\frac{\pi}{2} t\right) \right) \tag{96}$$

$$= \left(\cos\left(\frac{\pi}{2} t\right) \pm i \sin\left(\frac{\pi}{2} t\right) \right) \tag{97}$$

and

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}. \tag{98}$$

The solution is

$$\mathbf{x}_t = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} i^t & 0 \\ 0 & (-i)^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{99}$$

$$= \begin{bmatrix} i^t & (-i)^t \\ i^{t+1} & (-i)^{t+1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{100}$$

$$= \begin{bmatrix} c_1 i^t + c_2 (-i)^t \\ c_1 i^{t+1} + c_2 (-i)^{t+1} \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{101}$$

$$= \begin{bmatrix} c_1 \left(\left(\cos\left(\frac{\pi}{2} t\right) + i \sin\left(\frac{\pi}{2} t\right) \right) \right) \\ + c_2 \left(\cos\left(\frac{\pi}{2} t\right) - i \sin\left(\frac{\pi}{2} t\right) \right) \\ c_1 \left(\left(\cos\left(\frac{\pi}{2} (t+1)\right) + i \sin\left(\frac{\pi}{2} (t+1)\right) \right) \right) \\ + c_2 \left(\cos\left(\frac{\pi}{2} (t+1)\right) - i \sin\left(\frac{\pi}{2} (t+1)\right) \right) \end{bmatrix} \tag{102}$$

$$+ \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{103}$$

$$= \begin{bmatrix} (c_1 + c_2) \left(\cos\left(\frac{\pi}{2} t\right) + (c_1 - c_2) i \sin\left(\frac{\pi}{2} t\right) \right) \\ (c_1 + c_2) \left(\cos\left(\frac{\pi}{2} (t+1)\right) + (c_1 - c_2) i \sin\left(\frac{\pi}{2} (t+1)\right) \right) \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{104}$$

$$\begin{bmatrix} \tilde{c}_1 \left(\cos\left(\frac{\pi}{2} t\right) + \tilde{c}_2 \sin\left(\frac{\pi}{2} t\right) \right) \\ \tilde{c}_1 \left(\cos\left(\frac{\pi}{2} (t+1)\right) + \tilde{c}_2 \sin\left(\frac{\pi}{2} (t+1)\right) \right) \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \tag{105}$$

4.3.1 Non-invertible eigenvectors

If some roots are repeated $\mathbf{B}^{-1}$ may be non-invertible. In this case we cannot use the diagonalization method. Instead we can use the existence of a higher order single difference equation that is equivalent to the systems of first order difference equations we want to solve. This is exactly analogous to the case of differential equations.

Look at the following example

$$\mathbf{x}_{t+1} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} \mathbf{x}_t \tag{106}$$

The eigenvalues of the coefficient matrix are both 1 and the matrix of eigenvectors are non-invertible. Now we transform the system into a second order single difference equation.

From the first row we have that

$$\begin{aligned} x_{1,t+1} &= -x_{1,t} + x_{2,t} && (107) \\ &\rightarrow x_{2,t} = x_{1,t+1} + x_{1,t}, && (108) \\ x_{2,t+1} &= x_{1,t+2} + x_{1,t+1} && (109) \end{aligned}$$

Using the second row,

$$\begin{aligned} x_{2,t+1} &= -4x_{1,t} + 3x_{2,t} && (110) \\ &\rightarrow x_{1,t+2} + x_{1,t+1} = -4x_{1,t} + 3\left(x_{1,t+1} + x_{1,t}\right) && (111) \\ 0 &= x_{1,t+2} - 2x_{1,t+1} + x_{1,t} && (112) \end{aligned}$$

Note that the polynomial this second order difference equation is identical to the characteristic equation of the coefficient matrix in (106). Consequently the have the same (repeated) roots 1. The solution to (110) is

$$x_{1,t} = (c_1 + tc_2)\, 1^t = (c_1 + tc_2) \tag{113}$$

With knowledge of x_0 and x_1, we have

$$x_{1,t} = \left(x_{1,0} + t\left(x_{1,1} - x_{1,0}\right)\right) \tag{114}$$

By using, (107), we can express the solution in terms of $x_{1,t}$ and $x_{2,t}$

$$\begin{aligned} x_{1,t} &= x_{1,0} + t\left(x_{1,1} - x_{1,0}\right) && (115) \\ x_{2,t} &= x_{1,t+1} + x_{1,t} && (116) \\ &= 2x_{1,0} + (2t+1)\left(x_{1,1} - x_{1,0}\right) && (117) \\ &\rightarrow \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \begin{bmatrix} 1 & t \\ 2 & 2t+1 \end{bmatrix} \begin{bmatrix} x_{1,0} \\ x_{1,1} - x_{1,0} \end{bmatrix}. && (118) \end{aligned}$$