5 Dynamic Optimization in Discrete Time

5.1 Non-Stochastic Dynamic Programming

Consider the dynamic problem

$$
\begin{aligned}
& \max_{\{c\}_{t=1}^{T}} \sum_{t=1}^{T} u\left(k_t, c_t, t\right) && (1)\\
\text{s.t. } k_1 &= \underline{k} && (2)\\
k_{t+1} &= f\left(k_t, c_t, t\right), t = 1..., T, && (3)\\
k_{T+1} &= \bar{k} && (4)
\end{aligned}
$$

Before trying to solve this, notice

1. Per period payoff is additive over time.
2. k_t cannot be changed in period t, but its future values, its law-of-motion can be changed by c_t. We will call k a state variable (to be more properly defined later) and c a control variable. A sequence $k_1, k_2, ..., k_{T+1}$ is said to be *admissible* if and only if it satisfies the constraints (2-4) for some sequence $c_1, c_2, ...c_T$.

A direct way to solve this would be to form the Lagrangian

$$
\begin{aligned}
L &= \sum_{t=1}^{T} u\left(k_t, c_t, t\right) + \lambda_t \sum_{t=1}^{T} \left(f\left(k_t, c_t, t\right) - k_{t+1}\right) && (5)\\
&\quad + \mu_1 \left(\underline{k} - k_1\right) + \mu_{T+1} \left(k_{T+1} - \bar{k}\right) && (6)
\end{aligned}
$$

with first order conditions

$$
\begin{aligned}
u_k\left(k_t, c_t, t\right) + \lambda_t f_k\left(k_t, c_t, t\right) - \lambda_{t-1} &= 0, \forall t = 2, .., T, && (7)\\
u_c\left(k_t, c_t, t\right) + \lambda_t f_c\left(k_t, c_t, t\right) &= 0, \forall t = 1, .., T, && (8)\\
u_k\left(k_1, c_1, 1\right) + \lambda_1 f_k\left(k_1, c_1, 1\right) - \mu_1 &= 0, && (9)\\
-\lambda_T + \mu_{T+1} &= 0, && (10)
\end{aligned}
$$

and (2-4).

This works, at least in principle, if T is finite. An alternative way is to recognize that in a problem like this, each sub-section of the path must be optimal in itself. This means that the problem has a recursive formulation i.e., it can be set up sequentially. We can thus solve the problem backwards starting from the last period. In any period, the remaining problem only depends on earlier actions through the "inherited" value of k.

For example, it the problem is over three periods ($T = 3$) we can rewrite (1)

$$\max_{c_1,k_2|k_1}\left(u\left(k_1,c_1,1\right)+\max_{c_2,k_3|k_2}\left(u\left(k_2,c_2,2\right)+\max_{c_3,k_4|k_3}u\left(k_3,c_3,3\right)\right)\right) \tag{11}$$

$$\text{s.t.} k_{t+1} = f\left(k_t,c_t,t\right), t = 1...,3 \tag{12}$$

$$k_4 = \bar{k} \tag{13}$$

In the final period ($T = 3$), the problem is trivial; simply set c_3 so that $k_4 = \bar{k}$. The value of c_3 that solves $\bar{k} = f\left(k_3,c_3,3\right)$ is a function of k_3.[3] Denote that function $c_3\left(k_3\right)$. We can then define

$$u\left(k_3,c_3\left(k_3\right),3\right) \equiv W\left(k_3,3\right). \tag{14}$$

The interpretation of $W\left(k_3,3\right)$, is the *maximum remaining pay-off in period 3,* being a function of the state variable k_3.

In period 2, we then want to solve

$$\max_{c_2,k_3}\left(u\left(k_2,c_2,2\right)+u\left(k_3,c_3\left(k_3\right),3\right)\right) \tag{15}$$

$$\text{s.t. } k_3 = f\left(k_2,c_2,2\right). \tag{16}$$

Using $W\left(k_3,3\right)$,we can write this

$$\max_{c_2}\left(u\left(k_2,c_2,2\right)+W\left(f\left(k_2,c_2,2\right),3\right)\right) \tag{17}$$

The solution and the maximized value depends on k_2 only and we called the latter the value function and k_2 the state variable. We can then define

$$W\left(k_2,2\right) \equiv \max_{c_2}\left(u\left(k_2,c_2,2\right)+W\left(f\left(k_2,c_2,2\right),3\right)\right). \tag{18}$$

The interpretation of this Bellman equation is straightforward. It says that the maximum remaining pay-off in period 2, being a function of k_2, is identically (i.e., for all k_2) equal to the maximum over the control in period 2, c_2, over period 2 pay-off and the maximum remaining pay-off in period 3 with period 3 state variable given by $f\left(k_2,c_2,2\right)$.

[3] For now, we just assume it is unique.

The trade-off between generation of current and future pay-off is optimized by *one* simple FOC

$$u_c(k_2, c_2, 2) + W_k(f(k_2, c_2, 2), 3) f_c(k_2, c_2, 2) = 0. \tag{19}$$

Finally, in the first period,

$$W(k_1, 1) \equiv \max_{c_1} (u(k_1, c_1, 1) + W(f(k_1, c_1, 1), 2)). \tag{20}$$

If we know the value functions, the multidimensional problem has become much simpler. The Bellman equation provides a way of verifying that the value function we use is correct. It is of course straightforward to extend the analysis to any finite horizon problem, yielding

$$\begin{aligned} W(k_t, t) &\equiv \max_{c_t} (u(k_t, c_t, t) + W(k_{t+1}, t+1)) && (21) \\ s.t..k_{t+1} &= f(k_t, c_t, t) && (22) \end{aligned}$$

5.1.1 Discounting and the Current Value Bellman equation

Very often macroeconomics, the objective function is a discounted sum of pay-offs, i.e., (1) can be written

$$\max_{\{c\}_{t=1}^T} \sum_{t=1}^{T} \beta^{t-1} u(k_t, c_t, t). \tag{23}$$

In this case, it is convenient to work with *curren value functions,* $V(k, t)$ which should be interpreted as the maximum remaining value that can be achived from time t and onward, given k_t and *seen from period* t. In other words, given a problem

$$\begin{aligned} &\max_{\{c\}_{t=1}^T} \sum_{t=1}^{T} \beta^{t-1} u(k_t, c_t, t) && (24) \\ \text{s.t. } k_{t+1} &= f(k_t, c_t, t), t = 1..., T && (25) \\ k_1 &= \underline{k} && (26) \\ k_{T+1} &= \bar{k} \end{aligned}$$

we define for any $t \in \{1, ..., T\}$

$$\begin{aligned} V(k_t, t) &\equiv \max_{\{c,k\}_{s=t}^T} \sum_{s=t}^{T} \beta^{s-t} u(k_s, c_s, s) && (27) \\ \text{s.t. } k_{t+1} &= f(k_t, c_t, t), t = s..., T. && (28) \\ k_{T+1} &= \bar{k}. \end{aligned}$$

From this follows that

$$V(k,t) \equiv \beta^{-(t-1)} W(k,t).$$

Using this in the Bellman equation, (where I substitute for k_{t+1} from the law-of-motion) we get the *current value Bellman equation*

$$\begin{aligned}
W(k_t,t) &\equiv \max_{c_t} \left(\beta^{t-1} u(k_t,c_t,t) + W(f(k_t,c_t,t), t+1)\right) &(29)\\
\beta^{t-1} V(k,t) &\equiv \max_{c_t} \left(\beta^{t-1} u(k_t,c_t,t) + \beta^t V(f(k_t,c_t,t), t+1)\right) &(30)\\
V(k,t) &\equiv \max_{c_t} \left(u(k_t,c_t,t) + \beta V(f(k_t,c_t,t), t+1)\right) &(31)
\end{aligned}$$

In practice, the current value Bellman equation, is the most used variant in macroeconomics and, therefore, you will often see the word *current* dropped and (31) is simple referred to as the Bellman equation and $V(k,t)$ is referred to as the value function.

5.1.2 Infinite Horizon and Autonomous Problems

In an infinite horizon problem we cannot use the method of starting from the last period. Still, if the problem has a well-defined value function, *it satisfies the Bellman equation.* Furthermore, under conditions, which we will talk about later, there is *only one* function that solves the Bellman equation, so if we find one function that solves the Bellman equation, we have a solution to the dynamic optimization problem. Since geometric discounting will prove to be important for showing uniqueness, we will use that from now on.

To find a solution, we will use two different approaches.

1. Guess on a value function and make sure it satisfies the Bellman equation.

2. Iterate on the Bellman equation until it converges.

Guessing is often feasible when the problem is autonomous (stationary). Then, the problem is independent of time in the sense that given in initial condition on the state variable(s), the solution and the maximized objective is independent of the starting date. A problem is autonomous if

1. Time is infinite,

2. the law of motion for the state (including constraints on the control) is independent of time, and

3. the per-period return function is the same over time, except for possibly a geometric discount factor, i.e.,$u(k, c, t) = \beta^t u(k, c)$.

(Think about what would happen if any of these conditions is not satisfied). In this case, the *current* value function turns out to be independent of time.[4] We can then write the current Bellman equation in terms of the current value function

$$V(k_t) \equiv \max_{c_t} (u(k_t, c_t) + \beta V(k_{t+1})) \tag{32}$$

$$\text{s.t. } k_{t+1} = f(k_t, c_t). \tag{33}$$

Suppose we find a solution to the maximization problem in the RHS of (32). This will be a time invariant function $c(k)$, since u, V and f are time-invariant. Plugging $c(k)$ into (32), we get rid of the max-operator:

$$V(k) = u(k, c(k)) + \beta V(f(k, c(k))) \tag{34}$$

If (34) is satisfied for *all values of admissible k,* we have a solution to the value function, otherwise our guess was incorrect.

Note that (34) is a functional equation, i.e., the LHS and RHS have to be the same functions. It is convenient to define the RHS as

$$T(V(k)) \equiv \max_{c} u(k, c) + \beta V(f(k, c)) \tag{35}$$

where T operates on *functions* rather than on values. In the autonomous case, when the value function is unchanged over time, the Bellman equation then defines a *fixed point* for T in the space of functions V;

$$V(k) = T(V(k)). \tag{36}$$

(36) means that if we plug in some function of k in the RHS of (36) we must get out the same function on the LHS. The Bellman equation is a necessary condition for $V(k)$ being a correctly specified value function, we will later discuss conditions under which it is also sufficient.

Typically the value function is of a similar for to the objective function. This is intuitive in the light of (34). For example if the u function in is logarithmic we guess that the value function is of the form. For HARA utility functions (e.g., CRRA, CARA and quadratic) the value functions are generally of the same type as the utility function (Merton, 1971).

[4] The present value functions $W(k, t)$ is not independent ot time, but is separable so that we can write $W(k, t) = W(k)\beta^{t-1}$.

5.1.3 An example of guessing

In the problem (1) let time be infinite and

$$u(k,c,t) = \beta^t \ln(c), \tag{37}$$
$$f(k,c,t) = k^\alpha - c, 0 < \alpha < 1, \tag{38}$$

and let the end-condition $k_{T+1} = \bar{k}$ be replaced by $k_t > 0 \; \forall t$.

This is an autonomous problem, so we have

$$V(k_t) = \max_{c,k} \left(u(c_t) + \beta V(k_{t+1}) \right) \tag{39}$$
$$\text{s.t. } k_{t+1} = f(k_t, c_t), \tag{40}$$
$$\Rightarrow V(k_t) = \max_c \left(\ln c_t + \beta V\left(k_t^\alpha - c_t\right) \right) \tag{41}$$

Now, guess that V is of the same form as u, here $X \ln k_t + Y$, for some unknown constants X and Y, giving first order conditions

$$u'(c_t) = \beta V'\left(k_t^\alpha - c_t\right) \tag{42}$$
$$\frac{1}{c_t} = \beta \frac{X}{k_t^\alpha - c_t} \tag{43}$$
$$\Rightarrow c_t = \frac{k_t^\alpha}{1+\beta X} \equiv c(k_t). \tag{44}$$
$$k_{t+1} = k_t^\alpha - c_t = k_t^\alpha - \frac{k_t^\alpha}{1+\beta X} = \frac{\beta X}{1+\beta X} k_t^\alpha \tag{45}$$

Plugging $c(k_t)$ into the Bellman equation yields

$$X \ln k_t + Y = \ln \frac{k_t^\alpha}{1+\beta X} + \beta \left(X \ln \frac{\beta X}{1+\beta X} k_t^\alpha + Y \right) \tag{46}$$
$$= \ln \frac{k_t^\alpha}{1+\beta X} + \beta \left(X \ln \frac{k_t^\alpha}{1+\beta X} + X \ln \beta X + Y \right) \tag{47}$$
$$= (1+\beta X) \ln \frac{k_t^\alpha}{1+\beta X} + \beta \left(X \ln \beta X + Y \right) \tag{48}$$
$$= \alpha (1+\beta X) \ln k_t - (1+\beta X) \ln (1+\beta X) + \beta X \ln \beta X + \beta Y. \tag{49}$$

This is true for all values of k, if and only if

$$X = \alpha (1+\beta X) \tag{50}$$
$$Y = -(1+\beta X) \ln (1+\beta X) + \beta X \ln \beta X + \beta Y. \tag{51}$$

giving

$$X = \frac{\alpha}{1-\alpha\beta} \tag{52}$$

$$Y = \frac{\beta\alpha \ln(\beta\alpha) + (1-\beta\alpha)\ln(1-\alpha\beta)}{(1-\beta)(1-\alpha\beta)} \tag{53}$$

$$\Rightarrow V(k) = \frac{\alpha}{1-\alpha\beta}\ln k + \frac{\beta\alpha \ln(\beta\alpha) + (1-\beta\alpha)\ln(1-\alpha\beta)}{(1-\beta)(1-\alpha\beta)}. \tag{54}$$

Having $V(k)$, it is easy to find the *optimal control*, or the *policy rule*,

$$c(k_t) = \frac{k_t^{\alpha}}{1+\beta\frac{\alpha}{1-\alpha\beta}} = (1-\alpha\beta)k_t^{\alpha}, \tag{55}$$

$$\ln k_{t+1} = \ln \alpha\beta + \alpha \ln k_{t.} \tag{56}$$

5.1.4 Iteration

An alternative way is try to find the limit of finite horizon Bellman equation as the horizon goes to infinity. Under for economical purposes quite general conditions this limit exists and is equal to the value function for the infinite horizon problem. Let us change notation slightly, measuring time as the number or remaining periods s until the final period. We then denote the (current) value function with s periods left by

$$V(k,s). \tag{57}$$

and assume geometric discounting and that both pay-offs and the law-of-motion for the state variable are time-independent ($u(k,c,s) = u(k,c)$ and $f(k,c,s) = f(k,c)$ so that the infinite horizon problem is autonomous. If the following limit is well-defined, we denote

$$\lim{}_{s\to\infty} V(k,s) \equiv V(k). \tag{58}$$

The iteration method is usually done numerically, but it can (at some cost of messiness) be done also analytically. Using the T operator and the Bellman equation, we find the limit in the following way

$$V(k,s) = \max_{c_s}\left(u(k_s,c_s) + \beta V(f(k_s,c_s), s-1)\right) \tag{59}$$

$$\equiv T(V(k,s-1)) \tag{60}$$

$$V(k,s) = T^s V(k,0) \tag{61}$$

$$V(k) \equiv \lim_{s\to\infty} V(k,s) = \lim_{s\to\infty} T^s V(k,0). \tag{62}$$

If the limit exists, it clearly satisfies the Bellman equation

$$V(k) = T(V(k)) \tag{63}$$

$$\lim_{s\to\infty} T^s V(k,0) = T\left(\lim_{s\to\infty} T^s V(k,0)\right) = \lim_{s\to\infty} T^{s+1} V(k,0). \tag{64}$$

The remaining issue is what function $V(k,0)$ to start the iteration with. However, suppose that we can show that the limit $V(k)$ satisfies

$$\lim_{s\to\infty} \beta^s V(k_s) = 0 \tag{65}$$

for ALL PERMISSABLE (in other words FEASIBLE) values of k_{t+s} that can be reached, given relevant initial conditions and other constraints. Then, it can easily be shown that the Bellman equation is *sufficient.* Then, $\lim_{s\to\infty} T^s V(k,0)$ provides a unique solution to the Bellman equation. This means that the limit is independent of the choice of $V(k,0)$. As we see from (65), $\beta < 1$ and $V(k)$ bounded are sufficient for uniqueness. Intuitively, if $\beta < 1$ and pay-offs are bounded, the pay-off in the infinite horizon has no impact on the value function. Let us revert to measure time in the usual way. Then, if the Bellman equation is satisfied, we have

$$V(k_t) = \max_{c_t} \left(u(k_t, c_t) + \beta V(f(k_t, c_t))\right) = \tag{66}$$

$$\max_{c_t}\left(u(k_t, c_t) + \beta\left(\max_{c_{t+1}}\left(u(k_{t+1}, c_{t+1}) + \beta V(f(k_{t+1}, c_{t+1}))\right)\right)\right) \tag{67}$$

$$= \max_{\{c_{t+n}\}_0^1} \sum_{n=0}^{1} \beta^n u(k_{t+n}, c_{t+n}) + \beta^2 V(f(k_{t+1}, c_{t+1})). \tag{68}$$

Repeating this, and taking the limit yields

$$V(k_t) = \max_{\{c_{t+n}\}_0^s} \sum_{n=0}^{s} \beta^n u(k_{t+n}, c_{t+n}) + \beta^{s+1} V(f(k_{t+s}, c_{t+s})), \tag{69}$$

$$V(k_t) = \max_{\{c_{t+n}\}_0^\infty} \sum_{n=0}^{\infty} \beta^n u(k_{t+n}, c_{t+n}) + \lim_{s\to\infty} \beta^{s+1} V(k_{t+s+1}), \tag{70}$$

where we note that the constraint $k_{t+s+1} = f(k_{t+s}, c_{t+s})$ is satisfied by construction. Now, if $\lim_{s\to\infty} \beta^{s+1} V(k_{t+s+1}) = 0$, for all permissible paths of k we have showed that

$$V(k_t) = \max_{\{c_{t+s}\}_0^\infty} \sum_{n=0}^{\infty} \beta^s u(k_{t+s}, c_{t+s}) \tag{71}$$

$$\text{s.t. } k_{t+s+1} = f(k_{t+s}, c_{t+s})\, \forall s \geq 0, \text{ given} k_t, \tag{72}$$

i.e., that the Bellman equation implies optimality.

The iteration in can easily be done numerically, either by specifying a functional form if we know that, or by just choosing a grid. In the latter case we assume that the state variable must belong to a finite set of values, say for example that in every period k must be chosen from the set $\mathbf{K} \equiv \{k_1, k_2, ..., k_n\}$. Then, we can compute the corresponding set of possible controls, $c_{m,n} \in \mathbf{C}$ from the equation

$$k_m = f(k_n, c_{m,n}). \tag{73}$$

Then, in each iteration, we solve the Bellman equation for each $k \in \mathbf{K}$, giving for iteration s

$$V(k_n, s) = \max_{c_{m,n}\in\mathbf{C}} (u(k_n, c_{m,n}) + \beta V(c_m, s-1)). \tag{74}$$

This goes quickly on a computer and the iteration is repeated until $V(k, s)$ is sufficiently close to $V(k, s-1)$ over the set of $k \in \mathbf{K}$.

5.1.5 An envelope result

We will later have use for the following envelope result, which implies that we can we can evaluate a $dV(k)/dk$ as the partial derivative holding c constant. To see this, note that

$$\begin{aligned}\frac{dV(k_t)}{dk_t} &\equiv V'(k_t) = \frac{\partial u(k_t, c_t)}{\partial k_t} + \beta V'(k_{t+1}) \frac{\partial f(k_t, c_t)}{\partial k_t} &(75)\\ &+\frac{dc_t}{dk_t}\left(\frac{\partial u(k_t, c_t)}{\partial c_t} + \beta V'(k_{t+1}) \frac{\partial f(k_t, c_t)}{\partial c_t}\right). &(76)\end{aligned}$$

However, the Bellman equation implies that the last term above is zero, either because the first-order condition for and interior maximum is satisfied $\frac{\partial u(k,c)}{\partial c} + \beta V'(k_{t+1}) \frac{\partial f(k_t,c_t)}{\partial c_t} = 0$, of in the case of a corner because $\frac{dc_t}{dk_t} = 0$.

This envelope results is often very useful. Consider the example

$$\begin{aligned}u(k_t, c_t) &= v(c_t), &(77)\\ f(k_t.c_t) &= y(k_t) - c_t. &(78)\end{aligned}$$

The interior solution to the Bellman equation satisfies

$$\begin{aligned}0 &= v'(c_t) + \beta V'(k_{t+1}) \frac{\partial f(k_t, c_t)}{\partial c_t}, &(79)\\ &\rightarrow v'(c_t) = \beta V'(k_{t+1}). &(80)\end{aligned}$$

The envelope condition yields

$$\begin{aligned} V'(k_t) &= \frac{\partial u(k_t, c_t)}{\partial k_t} + \beta V'(k_{t+1}) \frac{\partial f(k_t, c_t)}{\partial k_t} && (81) \\ &= v'(c_t)\, y'(k_t). && (82) \\ V'(k_{t+1}) &\rightarrow v'(c_{t+1})\, y'(k_{t+1}). && (83) \end{aligned}$$

Using this in (79) yields the *Euler equation*

$$v'(c_t) = \beta v'(c_{t+1})\, y'(k_{t+1}). \qquad (84)$$

5.1.6 State Variables

We often solve the dynamics programming problem by guessing a form of the value function. The first thing to determine is then which variables should enter, i.e., which variables are the state variables. The state variables must satisfy both following conditions

1. To enter the value function at time they must be realized at t.

Note, however, that it sometimes may be convenient to use an conditional expectation $E_t(z_{t+s})$ as a state variable. The expectation as of t is certainly realized at t even if the stochastic variable z_{t+s} is not realized.

2. The set of variables chosen as state variables must together give sufficient information so that the value of the program from t and onwards when the optimal control is chosen can be calculated.[5]

Note, we should try to find the smallest such set. If, for example we have an investment problem with several assets to invest in and without any costs of adjusting the portfolio, total wealth may be a sufficient as a state variable.

5.2 Stochastic Dynamic Programming

As long as the recursive structure of the problem is intact adding a stochastic element to the transition equation does not change the Bellman equation. Consider the problem

$$\max_{\{c_t\}_0^\infty} E_0 \sum_{t=0}^{\infty} \beta^t u(k_t, c_t) \qquad (85)$$

$$\begin{aligned} \text{s.t. } k_{t+1} &= f(k_t, c_t, \varepsilon_{t+1})\, \forall t \geq 0, k_0 \text{ given, and} && (86) \\ k_t &> 0\ \forall t. && (87) \end{aligned}$$

[5] Can you figure out what do we need if the per period utility function in (1) were $u(c_t, c_{t-1})$?

where E_0 is the expectations operator, conditional on time 0 information and we assume that c_t can be chosen conditional on information about ε_s for all $s \leq t$. Furthermore, let us assume that the distribution of ε_t is *i.i.d.* over time. Then, the Bellman equation becomes

$$V(k_t) = \max_{c_t} \left(u(k_t, c_t) + \beta E_t V(f(k_t, c_t, \varepsilon_{t+1})) \right), \tag{88}$$

with a first-order condition

$$0 = u_c(k_t, c_t) + \beta E_t \left(V'(k_{t+1}) f_k(k_t, c_t, \varepsilon_{t+1}) \right). \tag{89}$$

Note that, in general $E_t \left(V'(k_{t+1}) f_k(k_t, c_t, \varepsilon_{t+1}) \right) \neq E_t V'(k_{t+1}) E_t f_k(k_t, c_t, \varepsilon_{t+1})$.

5.2.1 A Stochastic Consumption Example

Consider the following problem

$$\max_{\{c_t\}_0^\infty} E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t \tag{90}$$

$$\text{s.t. } A_{t+1} = (A_t - c_t)(1 + \tilde{m}_{t+1}) \, \forall t \geq 0, \tag{91}$$

$$A_t \geq 0, \, \forall t \geq 0, \, A_0 \text{ given.} \tag{92}$$

The consumer decides how much to consume each period. The savings is placed in a risky asset with gross return $(1 + \tilde{m}_{t+1})$, that is drawn from an i.i.d. distribution with $E(\ln(1 + \tilde{m}_{t+1})) = m > -\infty$. If, for example the gross return is log-normal, with mean $\bar{m}$ and variance σ^2 then, $E(1 + \tilde{m}_{t+1}) = e^{\bar{m} + \frac{\sigma^2}{2}}$.

The problem is autonomous so we write the current value Bellman equation with time independent value function V

$$V(A_t) = \max_{c_t} \left\{ \ln c_t + \beta E_t V((A_t - c_t)(1 + \tilde{m}_{t+1})) \right\}. \tag{93}$$

The necessary first order condition for c_t yield

$$\frac{1}{c_t} = \beta E_t V'(A_{t+1})(1 + \tilde{m}_{t+1}). \tag{94}$$

Now we use Merton's result and guess that the value function is

$$V(A_t) = Y + X \ln A_t, \tag{95}$$

for some constants Y and X. Substituting into (94), we get

$$\begin{aligned}
\frac{1}{c_t} &= \beta E_t \frac{X}{A_{t+1}} \left(1+\tilde{m}_{t+1}\right), && (96)\\
&= \beta E_t \frac{X\left(1+\tilde{m}_{t+1}\right)}{\left(A_t - c_t\right)\left(1+\tilde{m}_{t+1}\right)}, && (97)\\
&= \beta \frac{X}{A_t - c_t}, && (98)\\
&\rightarrow \; c_t = \frac{A_t}{1+\beta X}, && (99)\\
A_t - c_t &= \frac{\beta X A_t}{1+\beta X}. && (100)
\end{aligned}$$

Now we have to solve for the constant X. This is done by substituting the solutions to the first order conditions and the guess into the Bellman equations,

$$\begin{aligned}
Y + X \ln A_t &= \max_{c_t,\omega} \left\{ \ln c_t + \beta E_t V\left(\left(A_t - c_t\right)\left(1+\tilde{m}_{t+1}\right)\right)\right\}, && (101)\\
&= \ln \frac{A_t}{1+\beta X} + \beta E_t V\left(\frac{\beta X A_t}{1+\beta X}\left(1+\tilde{m}_{t+1}\right)\right), && (102)\\
&= \ln \frac{A_t}{1+\beta X} + \beta E_t \left(Y + X \ln\left(\frac{\beta X A_t}{1+\beta X}\left(1+\tilde{m}_{t+1}\right)\right)\right), && (103)\\
&= \left(1+\beta X\right)\ln A_t - \left(1+\beta X\right)\ln\left(1+\beta X\right) + \beta Y && (104)\\
&\quad + \beta X \ln \beta X + \beta X m. && (105)
\end{aligned}$$

This is satisfied for all A_t iff

$$\begin{aligned}
X &= \left(1+\beta X\right) && (106)\\
&= \frac{1}{1-\beta}, && (107)\\
Y &= -\frac{1}{1-\beta}\ln\left(\frac{1}{1-\beta}\right) + \beta Y + \beta\frac{1}{1-\beta}\ln\beta\frac{1}{1-\beta} + \beta\frac{m}{1-\beta} && (108)\\
&= -\frac{1}{1-\beta}\ln\frac{1}{1-\beta} + \frac{\beta}{\left(1-\beta\right)^2}\left(m+\ln\beta\right). && (109)
\end{aligned}$$

Thus,

$$
\begin{aligned}
c_t &= \frac{A_t}{1+\beta\frac{1}{1-\beta}} = (1-\beta)\,A_t, && (110)\\
A_t - c_t &= \beta A_t. && (111)\\
A_{t+1} &= A_t\beta\,(1+\tilde{m}_{t+1})\,, && (112)\\
\ln A_{t+1} &= \ln A_t + \ln\beta + \ln(1+\tilde{m}_{t+1}) && (113)\\
E_t \ln A_{t+1} &= \ln A_t + \ln\beta + m. && (114)
\end{aligned}
$$

Note that since $\ln(1+\tilde{m}_{t+1})$ is normally distributed, $(1+\tilde{m}_{t+1}) > 0$, for all t, implying $A_t > 0$ for all t. If, on the other hand, $(1+\tilde{m}_{t+1})$ can be negative with positive probability, $E_t \ln(1+\tilde{m}_{t+1})$ is minus infinity implying that the value functions is ill-defined.

5.3 Contraction mappings

In the previous section we discussed guessing on solutions to the Bellman equation. However, we would like to know whether there exists a solution and whether it is unique. If the latter is not the case, it is not in principle sufficient to guess and verify, since we might have other value functions that also satisfy the Bellman equation. To prove existence and uniqueness we will apply a contraction mapping argument.[6] For this purpose, we first have to define some concepts.

5.3.1 Complete Metric Spaces and Cauchy Sequences

Let $\mathbf{X}$ be a vector space, i.e., a set on which addition and scalar multiplication is defined. Also define an operator d: which we can think of as measuring the (generalized) distance between any two elements of $\mathbf{X}$. We call d a norm assumed to satisfy

1. Positivity $\forall x, y \in \mathbf{X}$, $d(x,y) \geq 0$ and $d(x,y) = 0 \Rightarrow x = y$.
2. Symmetry $\forall x, y \in \mathbf{X}$, $d(x,y) = d(y,x)$.
3. Triangle inequality $\forall x, y, z \in \mathbf{X}$, $d(x,z) \geq d(x,y) + d(y,z)$

Now, we call $(\mathbf{X}, d)$ a *normed vector space* or a *metric space*. An example of such a space would be $\mathbf{R}^n$ together with the Euclidean norm $d(x,y) \equiv$

[6] An alternative is sometimes to look for the limit $\lim_{s\to\infty} T^s(V(k,0))$, which typically is the solution we are interested in (at least in macroeconomics).

$\|x, y\|$. Another example is the space $\mathbf{C(S)}$ of bounded functions where each element is a function from $\mathbf{S} \subseteq \mathbf{R}^n \to \mathbf{R}$ together with the "sup-norm" defined as follows. For any two elements in $\mathbf{C}(\mathbf{S})$, i.e., any two functions $w(s)$ and $v(s)$, the distance d between them is the maximal euclidean distance, i.e.,.,

$$d(w, v) \equiv \sup_{s \in \mathbf{S}} \|w(s), v(s)\| \tag{115}$$

Now let us define a Cauchy sequence. Intuitively, this is a sequence of elements $\{x_n\}$ in a space $\mathbf{X}$ that come closer and closer to each other, using some particular norm. More precisely, $\{x_n\}$ is defined as a sequence of elements in $\mathbf{X}$ such that for all $\varepsilon > 0$, there exist a number n, such that for all $m, p \geq n$, $d(x_m, x_p) < \varepsilon$.. An example of such a sequence would be the sequence $\{1, 1/2, 1/3, \}$ which is a Cauchy sequence using the Euclidean norm. A Cauchy sequence converges if there is an element $y \in \mathbf{X}$ such that $\lim_{n \to \infty} d(x_n, y) = 0$. It may, of course, be the case that the Cauchy sequence does not converge to a point in $\mathbf{X}$. An example would be if we let $\mathbf{X}$ be the open interval $(0, 1]$ and look at the Cauchy sequence $\{1, 1/2, 1/3, \}$ which is in $\mathbf{X}$ but converges to zero which is not in $\mathbf{X}$.

5.3.2 Complete metric spaces

Now we are ready to define the *complete metric space*. This is a metric space in which all Cauchy sequences converge to a point in the space.

5.3.3 Contraction Mapping

Consider a metric space $(\mathbf{X}, d)$ and look at an operator T that maps $\mathbf{X} \to \mathbf{X}$. T is a contraction mapping by definition if there exists a non-negative number $\rho \in [0, 1)$, such that for all elements $x, y \in \mathbf{X}$,

$$d(T(x), T(y)) \leq \rho d(x, y), \tag{116}$$

where we note that ρ must be *strictly* smaller than one.

An example of contraction mapping would we a map in say scale $1 : 10000$ put on top of a map in scale $1 : 1000$ covering the same geographical area. The norm can be the distance between the points on the map. Clearly, (116) is satisfied for $\rho = 0.1$.

5.3.4 The Contraction Mapping Theorem

Now we can state the very important *contraction mapping theorem.*

Result 15 *Consider a complete metric space* $(\mathbf{X}, d)$ *and let* $T : \mathbf{X} \rightarrow \mathbf{X}$ *be a contraction mapping. Then,* T *has one unique fixed point* $x^* \in \mathbf{X}$*, i.e., the solution to* $x = T(x)$ *always exists and is unique. Furthermore, the sequence* x_0*,* $T(x_0)$*,*$T^2(x_0)$*,...,*$T^n(x_0)$ *converges to* x^* *for all* $x_0 \in \mathbf{X}$*.*

There are theorems that can be used to show that T is a contraction mapping.

Result 16 *Let the state space* $\mathbf{S}$ *be a subset of* $\mathbf{R}^n$ *and* $\mathbf{B}(\mathbf{S})$ *the set of all* bounded *functions from* $\mathbf{S}$ *to* $\mathbf{R}$*. Let* T *be a map that maps all elements of* $\mathbf{B}(\mathbf{S}) \rightarrow \mathbf{B}(\mathbf{S})$*. Then,* T *is a contraction mapping if*

1. *for any functions* $w(s), v(s) \in \mathbf{B}(\mathbf{S})$*, the following holds; if* $w(s) \geq v(s) \forall s \in \mathbf{S}$ *then* $T(w(s)) \geq T(v(s)) \forall s \in \mathbf{S}$ *(*monotonicity*), and*
2. *there is a* $\beta \in [0, 1)$ *such that for any constant* $\kappa \in R$*, and any function* $w(s) \in \mathbf{B}(\mathbf{S}), T(w(s) + \kappa) = T(w(s)) + \beta\kappa$*. (*discounting*).*

Usually it is straightforward to apply the previous result to show that if we have strict discounting, the Bellman equation is a contraction mapping. There is one major limitation which we have to live with, however, result 16 and variants of it require bounded value functions.

Let us look at an example, where we apply result 16. Consider a simple growth model,

$$\max_{\{c_t\}_0^\infty} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{117}$$

$$\text{s.t. } k_{t+1} = f(k_t) - c_t, \forall t \geq 0, k_0 \text{ given, and} \tag{118}$$

$$c_t, k_t > 0 \ \forall t. \tag{119}$$

where u is a continuous and increasing (utility) function with $u(0) \geq u_{\min} > -\infty$ and $0 \leq \beta < 1$. To use the theorems we need to make some assumptions. First, we need boundedness. For this purpose, we assume

$$f(0) = 0, \tag{120}$$

$$f'(k) \geq 0 \forall k \geq 0, \tag{121}$$

$$\exists \bar{k} > 0, \text{such that } f(k) \leq k \forall k \geq \bar{k}. \tag{122}$$

Now, define $\mathbf{S} = [0, \bar{k}]$ and note that if k_0 is in $\mathbf{S}$, so is all admissible k_t. Then, $u(c_t)$ is bounded since $u_{\min} \leq u(c) \leq u(\bar{k})$ implying that any value function must satisfy $\frac{u_{\min}}{1-\beta} \leq V(k) \leq \frac{u(\bar{k})}{1-\beta}$. By restricting the state space

$\mathbf{S} = [0, \bar{k}]$, we can therefore restrict our search for value functions that are bounded on our state space.

Now, consider the Bellman equation

$$V(k) \equiv \max_{c \geq 0} u(c) + \beta V(f(k-c)) \equiv T(V(k)) \tag{123}$$

Using the first result, we need for any two bounded functions $v(k), w(k), k \in S$, .

$$v(k) \geq w(k) \forall k \Rightarrow T(v(k)) \geq T(w(k)) \forall k. \tag{124}$$

which is satisfied. To see this, define

$$c^* = \arg\max_c u(c) + \beta w(f(k) - c) \tag{125}$$

then,

$$\begin{aligned} T(v(k)) &= \max_c u(c) + \beta v(f(k) - c) \geq u(c^*) + \beta v(f(k) - c^*) & (126) \\ &\geq u(c^*) + \beta w(f(k) - c^*) = T(w(k)). & (127) \end{aligned}$$

Regarding condition 2, we note

$$\begin{aligned} T(v(k) + \kappa) &= \max_{c \geq 0} u(c) + \beta(v(f(k-c)) + \kappa) & (128) \\ &= \max_{c \geq 0} u(c) + \beta v(f(k-c)) + \beta\kappa & (129) \\ &= T(v(k)) + \beta\kappa, & (130) \end{aligned}$$

so the second condition is satisfied if . So the Bellman equation is a contraction mapping and always has one and one only unique solution, $V(k)$.